ON-SHELL CONDITIONS
IN THEORIES WITH FLAVOR MIXING

Maximilian Löschner
Advisor: Walter Grimus

Seminar on Particle Physics
18 October 2016
On-shell conditions in theories with flavor mixing are already an integral part of the Standard Model.

In extensions of the SM, mixing for fermions and scalars are likely to occur.

- Want a proper foundation for the definition of these conditions.
INTRODUCTION


‣ one-loop effects of the quark mixing matrix are practically negligible

<table>
<thead>
<tr>
<th>Table 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comparison of different approximations for the W-decay width (in GeV) with a top-quark mass $m_t = 150$ GeV and the corresponding W-mass $M_W = 80.199$ GeV</td>
</tr>
</tbody>
</table>

| $\Gamma(W \to ud)$ | $0.644297$ | $0.666486$ | $0.666497$ | $0.666497$ | $0.666496$ |
| $\Gamma(W \to us)$ | $0.327699 \times 10^{-1}$ | $0.338985 \times 10^{-1}$ | $0.338992 \times 10^{-1}$ | $0.338992 \times 10^{-1}$ |
| $\Gamma(W \to ub)$ | $0.330212 \times 10^{-4}$ | $0.341585 \times 10^{-4}$ | $0.342405 \times 10^{-4}$ | $0.342403 \times 10^{-4}$ |
| $\Gamma(W \to cd)$ | $0.327787 \times 10^{-1}$ | $0.339076 \times 10^{-1}$ | $0.339201 \times 10^{-1}$ | $0.339201 \times 10^{-1}$ |
| $\Gamma(W \to cs)$ | $0.642531$ | $0.664660$ | $0.664909$ | $0.664909$ |
| $\Gamma(W \to cb)$ | $0.142516 \times 10^{-2}$ | $0.147424 \times 10^{-2}$ | $0.147830 \times 10^{-2}$ | $0.147830 \times 10^{-2}$ |
| $\Gamma(W \to \text{hadrons})$ | $0.135384 \times 10^{+1}$ | $0.140046 \times 10^{+1}$ | $0.140074 \times 10^{+1}$ | $0.140074 \times 10^{+1}$ |
| $\Gamma(W \to \text{leptons})$ | $0.676933$ | $0.674715$ | $0.674715$ | $0.674715$ |
| $\Gamma(W \to \text{all})$ | $0.203077 \times 10^{+1}$ | $0.207516 \times 10^{+1}$ | $0.207545 \times 10^{+1}$ | $0.207545 \times 10^{+1}$ |

$\delta V_{\text{CKM}}$
• Still, the derivation of on-shell conditions in theories with mixing remained a bit vague for the general reader of the relevant literature

• An interesting theoretical problem in itself

› **Review on the derivation** and use of on-shell conditions in theories with flavor mixing
PREREQUISITES

• All masses are different

• Conditions only usable in regions where absorptive parts are negligible, otherwise only use dispersive part:

\[
\frac{1}{p^2 - \mu^2 + i\epsilon} = \text{P} \frac{1}{p^2 - \mu^2} - i\pi\delta(p^2 - \mu^2)
\]

i.e. decompose propagator via principle value and delta function (origin: Sokhotski-Plemelj theorem for real line)

› Corresponds to commonly used definitions renormalization conditions only using the real part of the self-energies

› Hermitian counterterms in Lagrangian (alternatively e.g. complex-mass scheme)
REAL SCALAR PROPAGATOR

- Commonly used on-shell condition for real scalar propagator:

\[ \Delta(p^2) \bigg|_{p^2 \to m^2} = \frac{1}{p^2 - m^2}, \quad m = m_{\text{phys}} \]

- Inspires the form of the condition in multi-particle case

\[ \Delta_{ij}(p^2) \bigg|_{p^2 \to m_n^2} = \frac{\delta_{in}\delta_{nj}}{p^2 - m_n^2} + \Delta_{ij}^{(0)} + \mathcal{O}(p^2 - m_n^2), \quad \epsilon_n \equiv p^2 - m_n^2 \]

- On-shell condition for propagator

\[
\Delta(p^2) \bigg|_{\epsilon \to 0} = \begin{pmatrix}
\mathcal{O}(1) & \cdots & \mathcal{O}(1) \\
\vdots & \ddots & \vdots \\
\mathcal{O}(1) & \cdots & \mathcal{O}(1)
\end{pmatrix} + \frac{1}{\epsilon_n} + \mathcal{O}(1)
\]
REAL SCALAR PROPAGATOR

• Problem: need conditions for the inverse propagator

• Reason: conditions should apply to the renormalized self energy (i.e. the counterterms therein)

\[
\int d(x - y) \langle \Omega | T \psi(x) \bar{\psi}(y) | \Omega \rangle e^{ip \cdot (x-y)} = \quad -\quad + \quad 1PI \quad + \quad 1PI \quad 1PI \quad + \ldots
\]

\[
= \frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2} \left( -i \Sigma(p^2) \right) \frac{i}{p^2 - m^2} + \ldots
\]

\[
= \frac{i}{p^2 - m^2} \left( 1 + \frac{\Sigma(p^2)}{p^2 - m^2} + \left( \frac{\Sigma(p^2)}{p^2 - m^2} \right)^2 + \ldots \right)
\]

\[
= \frac{i}{p^2 - m^2 - \Sigma(p^2)}.
\]

› Self-energy appears in the denominator of the two-point correlation function
• Simple to translate on-shell conditions to the self-energy in the case without mixing:

\[ \Delta(p^2) = \frac{1}{p^2 - m^2 - \Sigma(p^2)} \]

\[ \Rightarrow \Sigma(p^2)\big|_{p^2=m^2} = 0, \quad \frac{d}{dp^2} \Sigma(p^2)\big|_{p^2=m^2} = 0. \]

• In order to define similar conditions in the multi-particle case:

\[ (\Delta^{-1})_{ij} =: A_{ij} = A_{ij}^{(0)} + \epsilon_n A_{ij}^{(1)} + O(\epsilon_n^2), \quad \epsilon_n = p^2 - m_n^2 \]

and use the inversion condition to the propagator:

\[ \Delta_{ik} A_{kj} = A_{ik} \Delta_{kj} = \delta_{ij} \]
• Yields **conditions for the inverse propagator**:

\[ A_{in}^{(0)} = 0 \ \forall \ i = 1, \ldots, N \ \text{and} \ A_{nj}^{(0)} = 0 \ \forall \ j = 1, \ldots, N. \]

and moreover for the entries on the diagonal:

\[ A_{nn}^{(1)} = 1 \ \forall \ n \]

• Note that one can get even more conditions from the orthogonality, but these have nothing to do with the singularity structure ➤ not part of on-shell conditions

• Due to the choice of the inverse propagator, can equivalently use:

\[ A_{in}(m_n^2) = A_{nj}(m_n^2) = 0 \ \forall \ i, j = 1, \ldots, N \ \text{and} \ \frac{dA_{nn}(p^2)}{dp^2} \bigg|_{p^2=m_n^2} = 1, \]
NUMBER OF CONDITIONS

- Rows and columns in principle get independent conditions:

\[
A_{in}(m_n^2) = A_{nj}(m_n^2) = 0 \quad \forall \ i, j = 1, \ldots, N \quad \text{and} \quad \frac{dA_{nn}(p^2)}{dp^2}\bigg|_{p^2=m_n^2} = 1,
\]

\[
\begin{pmatrix}
A_{11} & \cdots & A_{1N} \\
\vdots & \ddots & \vdots \\
A_{N1} & \cdots & A_{NN}
\end{pmatrix}
\]

⇒ total number of conditions: \(2N^2 + N\)
• Rows and columns in principle get independent conditions:

\[ A_{in}(m_n^2) = A_{nj}(m_n^2) = 0 \ \forall \ i, j = 1, \ldots, N \ \text{and} \ \frac{dA_{nn}(p^2)}{dp^2} \bigg|_{p^2=m_n^2} = 1, \]

\[ \begin{pmatrix}
A_{11} & \cdots & \cdots & A_{1N} \\
\vdots & \ddots & \cdots & \vdots \\
A_{N1} & \cdots & \cdots & A_{NN}
\end{pmatrix} \]

\[ \Rightarrow \text{total number of conditions: } 2N^2 + N \]
# independent conditions vs. # counterterms

- First note that the propagator, as well as its inverse, are symmetric

\[
(\Delta^{-1}(p^2))_{\text{disp}}^T = \Delta^{-1}(p^2)_{\text{disp}}
\]

- Number of independent conditions reduced to \( N^2 + N \)

\[
i \neq j: \ A_{ij}(m_j^2) = 0, \quad i = j: \ A_{ii}(m_i^2) = 0, \quad \frac{dA_{ii}(p^2)}{dp^2} \bigg|_{p^2=m_i^2} = 1.
\]

(note that this way of counting makes more sense for the fermions)

- Field strength renormalization constants form a general real matrix

\[ Z^{1/2} : \ N^2 \text{ degrees of freedom} \]

- Mass counterterms using a diagonal mass matrix:

\[ \delta \hat{m} : \ N \text{ degrees of freedom} \]

\( \Rightarrow \) #renormalization condition coincides with #counterterms
PROPOGATOR SYMMETRY

\( \left( \Delta^{-1}(p^2) \right)^T_{\text{disp}} = \Delta^{-1}(p^2)_{\text{disp}} \)

- Use the Källén-Lehmann representation of the renormalized propagator to show that it is real and symmetric: (origin: \( i \left( \Delta(x - y) \right)_{ij} = \langle 0 | T \varphi_i(x) \varphi_j(y) | 0 \rangle \))

\[
\Delta_{ij}(p^2) = \int_0^\infty d\mu^2 \rho_{ij}(\mu^2) \frac{1}{p^2 - \mu^2 + i\epsilon}.
\]

\[
\rho_{ij}(q^2)\Theta(q^0) = (2\pi)^3 \sum_n \delta^{(4)}(q - p_n) \langle 0 | \varphi_i(0) | n \rangle \langle n | \varphi_j(0) | 0 \rangle
\]

- Next invoke CPT invariance, which holds in any local, Lorentz-invariant theory:

\[
\langle (CPT)x | (CPT)y \rangle = \langle x | y \rangle^* = \langle y | x \rangle
\]

\( (CPT) \varphi_i(x) (CPT)^{-1} = \varphi_i(-x) \)

\[
\Rightarrow \langle 0 | \varphi_i(0) | n \rangle = \langle (CPT)0 | (CPT)\varphi_i(0)(CPT)^{-1} | (CPT)n \rangle^*
\]

\[
= \langle 0 | \varphi_i(0) | (CPT)n \rangle^*
\]

\[
= \langle 0 | \varphi_i(0) | n' \rangle^*
\]
• Inserting this into the spectral density, we find:

\[
\rho_{ij}(q^2) \Theta(q^0) \equiv (2\pi)^3 \sum_n \delta^{(4)}(q-p_n) \langle 0| \varphi_i(0)|n \rangle \langle n| \varphi_j(0)|0 \rangle
\]

\[
= (2\pi)^3 \sum_{n'} \delta^{(4)}(q-p_n) \langle 0| \varphi_i(0)|n' \rangle^* \langle n'| \varphi_j(0)|0 \rangle^*
\]

\[
= (\rho_{ij}(q^2))^* = \rho_{ji}(q^2)
\]

• With the spectral density being real and symmetric, we see that the same holds for the propagator (also the inverse):

\[
\Delta_{ij}(p^2) = \Delta_{ij}^*(p^2) = \Delta_{ji}(p^2)
\]
• Choose condition for propagator similar to scalar case:

\[ S_{ij} \xrightarrow{\varepsilon_n \to 0} \frac{\delta_{in} \delta_{nj}}{\not{p} - m_n} + \tilde{S}_{ij}, \quad \varepsilon_n = p^2 - m_n^2 \]

• Ansatz for propagator:

\[ S_{ij}(p) = C_{ij}(p^2)\not{p} - D_{ij}(p^2) \]

\[ (\not{p} - m_n) S_{ij} = \delta_{in} \delta_{nj} + (\not{p} - m_n) \tilde{S}_{ij} \]

\[ = \varepsilon_n C_{ij} - (\not{p} - m_n) (D_{ij} + m_n C_{ij}) \]

⇒ Leads to general form of the propagator:

\[ C_{ij} = \frac{\delta_{in} \delta_{nj}}{\varepsilon} + C_{ij}^{(0)} + \mathcal{O}(\varepsilon), \]

\[ D_{ij} = -\frac{m_n \delta_{in} \delta_{nj}}{\varepsilon} + D_{ij}^{(0)} + \mathcal{O}(\varepsilon). \]
## CONDITIONS FOR FERMIONS

- Choice for the inverse propagator: \((S^{-1})_{ij}(p) = A_{ij}(p^2)\phi - B_{ij}(p^2)\)

- Choice for expansion of inverse propagator non-singular again:

\[
A_{ij} = A_{ij}^{(0)} + \varepsilon_n A_{ij}^{(1)} + O(\varepsilon_n^2), \quad B_{ij} = B_{ij}^{(0)} + \varepsilon_n B_{ij}^{(1)} + O(\varepsilon_n^2)
\]

- Use inversion relation to find:

\[
(S S^{-1})_{ij} = \delta_{ij} \Rightarrow C_{ik} A_{kj} p^2 + D_{ik} B_{kj} = \delta_{ij}, \quad C_{ik} B_{kj} + D_{ik} A_{kj} = 0,
\]

\[
(S^{-1} S)_{ij} = \delta_{ij} \Rightarrow A_{ik} C_{kj} p^2 + B_{ik} D_{kj} = \delta_{ij}, \quad B_{ik} C_{kj} + A_{ik} D_{kj} = 0.
\]

- Inserting expansions for prop. and inverse prop. yields final conditions:

\[
B_{in}(m_n^2) = m_n A_{in}(m_n^2) \quad \forall \; i = 1, \ldots, N;
\]

\[
B_{nj}(m_n^2) = m_n A_{nj}(m_n^2) \quad \forall \; j = 1, \ldots, N;
\]

\[
A_{nn}(m_n^2) + 2m_n^2 \left. \frac{dA_{nn}(p^2)}{dp^2} \right|_{p^2 = m_n^2} - 2m_n \left. \frac{dB_{nn}(p^2)}{dp^2} \right|_{p^2 = m_n^2} = 1.
\]
Symmetry relation for the fermionic propagator:
\[ \gamma_0 \left( S^{-1}(p) \right)^\dagger_{\text{disp}} \gamma_0 = S^{-1}(p)_{\text{disp}} \Rightarrow A^\dagger = A, \quad B^\dagger = B \]

- Can again be derived from Källén-Lehmann representation

Using this reduces the #independent renormalization conditions again:

\[ i \neq j : B_{ij}(m_j^2) = m_j A_{ij}(m_j^2), \]
\[ i = j : B_{ii}(m_i^2) = m_i A_{ii}(m_i^2), \]

\[ A_{ii}(m_i^2) + 2m_i^2 \frac{dA_{ii}(p^2)}{dp^2} \bigg|_{p^2=m_i^2} - 2m_i \frac{dB_{ii}(p^2)}{dp^2} \bigg|_{p^2=m_i^2} = 1 \]

- First relation is complex and not symmetric in i,j

**#independent conditions:**
\[ 2 \left( N^2 - N \right) + 2N = 2N^2 \]
Renormalized self-energy can be written as:

\[ \Sigma^{(r)}(p) = \Sigma(p) + \left( \mathbb{1} - \left( Z^{(1/2)} \right)^\dagger Z^{(1/2)} \right) \slashed{p} + \left( Z^{(1/2)} \right)^\dagger (\hat{m} + \delta \hat{m}) Z^{(1/2)} - \hat{m} \]

- Phase freedom in the complex field strength renormalization constants reduces \#parameters to fulfill renorm. conditions:

\[ Z^{(1/2)} \to e^{i\hat{\alpha}} Z^{(1/2)}, \quad e^{i\hat{\alpha}} = \text{diag}(e^{i\alpha_1}, \ldots, e^{i\alpha_N}) \]

- Then, \#independent parameters is:

\[ 2N^2 - N + N = 2N^2 \]

- Coincides again with the \#independent conditions
ALTERNATIVE FORMAL DERIVATION

- Can alternatively define: 
  \((S^{-1})_{ij}(\phi) =: T_{ij}(\phi) = A_{ij}(p^2)\phi - B_{ij}(p^2)\)

- Note that \(\frac{dp^2}{d\phi} = 2\phi\)

- Then, one arrives at completely equivalent conditions:

\[
T_{in}(\phi = m_n) = T_{nj}(\phi = m_n) = 0 \quad \forall i, j
\]
\[
\left.\frac{dT_{nn}(\phi)}{d\phi}\right|_{\phi=m_n} = 1.
\]
FERMIONS W/O P-CONSERVATION

• Choices for propagator and inverse prop. similar to P-conserving case:

\[
S = \psi (C_L \gamma_L + C_R \gamma_R) - (D_L \gamma_L + D_R \gamma_R)
\]

\[
S^{-1} = \psi (A_L \gamma_L + A_R \gamma_R) - (B_L \gamma_L + B_R \gamma_R)
\]

• Expansions of propagator constituents works just as before

• Inversion relation looks similar too, but:

**left- and right-chiral parts mix:**

\[
(SS^{-1})_{ij} = \delta_{ij} \Rightarrow (C_R A_L p^2 + D_L B_L)_{ij} = \delta_{ij}, \quad (C_L B_L + D_R A_L)_{ij} = 0,
\]

\[
(C_L A_R p^2 + D_R B_R)_{ij} = \delta_{ij}, \quad (C_R B_R + D_L A_R)_{ij} = 0,
\]

\[
(S^{-1} S)_{ij} = \delta_{ij} \Rightarrow (A_R C_L p^2 + B_L D_L)_{ij} = \delta_{ij}, \quad (B_R C_L + A_L D_L)_{ij} = 0,
\]

\[
(A_L C_R p^2 + B_R D_R)_{ij} = \delta_{ij}, \quad (B_L C_R + A_R D_R)_{ij} = 0.
\]
• Final conditions after invoking propagator symmetry:

\[(B_L)_{ij}(m_j^2) = m_j(A_R)_{ij}(m_j^2),\]
\[(B_R)_{ij}(m_j^2) = m_j(A_L)_{ij}(m_j^2),\]
\[\left(\frac{A_L}{A_R}\right)_{ii}(m_i^2) + m_i^2 \frac{d}{dp^2} ((A_L)_{ii}(p^2) + (A_R)_{ii}(p^2)) \bigg|_{p^2=m_i^2} = 1.\]

• Counting more subtle here. Know from propagator symmetry:

\[A_L^\dagger = A_L, \quad A_R^\dagger = A_R, \quad B_L^\dagger = B_R.\]

\[\Rightarrow \text{Im} (A_L)_{ii} = \text{Im} (A_R)_{ii} = 0, \quad (B_R)_{ii}(m_i^2) = (B_L)_{ii}(m_i^2))^*\]

• Inserting this into conditions leaves as independent ones (for i=j):

\[(A_L)_{ii}(m_i^2) = (A_R)_{ii}(m_i^2), \quad \text{Im} (B_L)_{ii}(m_i^2) = 0, \quad \text{Re} (B_L)_{ii}(m_i^2) = m_i(A_R)_{ii}(m_i^2)\]

\[\#\text{independent conditions: } 4 \left( N^2 - N \right) + 3N + N = 4N^2\]
Self-energy for fermions w/o CP-conservation:

\[
\Sigma^{(r)}(p) = \Sigma(p) + \left( 1 - \left( Z_L^{(1/2)} \right) \dagger Z_L^{(1/2)} \right) \psi \gamma_L + \left( 1 - \left( Z_R^{(1/2)} \right) \dagger Z_R^{(1/2)} \right) \psi \gamma_R \\
+ \left( Z_R^{(1/2)} \right) \dagger (\hat{m} + \delta \hat{m}) Z_L^{(1/2)} \gamma_L + \left( Z_L^{(1/2)} \right) \dagger (\hat{m} + \delta \hat{m}) Z_R^{(1/2)} \gamma_R - \hat{m}
\]

Again diagonal phase freedom, but the same for L/R:

\[
Z_L^{(1/2)} \rightarrow e^{i\hat{\alpha}} Z_L^{(1/2)}, \quad Z_R^{(1/2)} \rightarrow e^{i\hat{\alpha}} Z_R^{(1/2)}, \quad e^{i\hat{\alpha}} = \text{diag}(e^{i\alpha_1}, \ldots, e^{i\alpha_N})
\]

Of course, \#free parameters coincides with \#conditions:

\[(4N^2 - N) + N = 4N^2\]
• Majorana fields are equal to their charge conjugate:

\[ \psi_n(x) = C \gamma_0^T \psi_n^*(x) \]

\[ \Rightarrow \psi_j^T = -\bar{\psi}_j C \text{ and } \psi_i = C \bar{\psi}_i^T \]

• Use these equalities in the identity

\[ \langle 0 | T \psi_{ia}(x) \psi_{jb}(y) | 0 \rangle = -\langle 0 | T \psi_{jb}(y) \psi_{ia}(x) | 0 \rangle \]

• In the end this yields an additional propagator symmetry:

\[ S(p) = CS^T(-p)C^{-1} \]

› Inverse Majorana propagator has even less degrees of freedom:

\[ \gamma_0 \left( S^{-1}(p) \right)^\dagger \gamma_0 = S^{-1}(p) \]

\[ S^{-1}(p) = C \left( S^{-1}(-p) \right)^T C^{-1} \]

\[ \Rightarrow A_L^\dagger = A_L, \quad A_R^\dagger = A_R, \quad B_L^\dagger = B_R \]

\[ A_L^T = A_R, \quad B_L^T = B_L, \quad B_R^T = B_R. \]
• Remaining independent renormalization conditions:

\[ i \neq j : \quad (B_L)_{ij}(m_j^2) = m_j(A_R)_{ij}(m_j^2) \]
\[ (A_L)_{ii}(m_i^2) = (A_R)_{ii}(m_i^2), \quad \text{Im} (B_L)_{ii}(m_i^2) = 0, \quad \text{Re} (B_L)_{ii}(m_i^2) = m_i(A_R)_{ii}(m_i^2) \]

(+ condition containing the derivatives)

which means we have as the \# independent conditions:

\[
2(N^2 - N) + 2N + N = 2N^2 + N
\]

• Loose freedom of rephasing due to relation between L/R-parts

\[
\psi_i^{(b)} = \psi_{Li}^{(b)} + \left( \psi_{Li}^{(b)} \right)^c \Rightarrow Z_R^{(1/2)} = \left( Z_L^{(1/2)} \right)^* \\
Z_L^{(1/2)} \to e^{i\hat{\alpha}} Z_L^{(1/2)}, \quad Z_R^{(1/2)} \to e^{i\hat{\alpha}} Z_R^{(1/2)}, \quad e^{i\hat{\alpha}} = \text{diag}(e^{i\alpha_1}, \ldots, e^{i\alpha_N})
\]

(not to be confused with Majorana phases in mixing matrix)

• **Again equals \#counterterms** (one general complex FSRC + mass counterterms)
CONCLUSIONS

• On-shell renormalization conditions in theories with mixing already known, but still unclear in some specifics

• These conditions play an important role in extensions of the Standard Model in the fermion and scalar sector (with potentially strong mixing)

• Pains have been taken to dispel any unclear point in the derivation

• For more extras (e.g. explicit expressions for counterterms) and calculational details, see Int. J. Mod. Phys. A 31, 1630038 (2016)
THANKS!
BACKUP SLIDES
FLAVOR SYMMETRIES

• Attempt to describe/explain structure of $U_{\text{PMNS}}$ via symmetries of the mass matrix

• Use combination of discrete symmetries to approximate $U_{\text{PMNS}}$, e.g. $\mu$-$\tau$ symmetry


\[
S = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

\[
S M_\nu S = M_\nu^*
\]

$\Rightarrow |U_{\mu i}| = |U_{\tau i}| \quad \forall i$

$\Rightarrow \theta_{23} = 45^\circ, \quad \delta = \pm \frac{\pi}{2}$