Julius-Maximilians-
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RECOLA

a generator for tree-level and one-loop electroweak matrix elements

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in collaboration with S. Actis, L. Hofer, J.-N. Lang, M. Pellen, A. Scharf and S. Uccirati

Seminar, Vienna, January 10, 2017

Outline

1. Introduction
2. RECOLA: a generator for electroweak one-loop amplitudes
3. COLLIER: a Fortran library for tensor integrals
4. Application: Top–antitop production including decays
5. Conclusion

Relevance of electroweak corrections @ LHC

- LHC running @ $13\,\mathrm{TeV}$
  $\hookrightarrow$ electroweak corrections (EWC) $\sim$ some 10%
  enhanced by Sudakov logarithms $\log \frac{E}{M_\mathrm{W}}$
- integrated LHC luminosity will reach some $100\,\mathrm{fb}^{-1}$ (2016: $40\,\mathrm{fb}^{-1}$)
  $\hookrightarrow$ many measurements at several-per-cent level
  $\hookrightarrow$ typical size of EWC
- high-precision measurements: cross-section ratios, M_W, $\sin^2\theta^\mathrm{lept}_\mathrm{eff}$
  $\hookrightarrow$ EWC are crucial
  EW 2-loop corrections needed
- Les Houches wishlist 2013 and update 2015:
  LHC processes where NLO EWC (and NNLO QCD) are needed

$\Rightarrow$ automation of calculation of EWC strongly desireable

NLO QCD automation established: several (public) tools exist

tool	method	collaboration
ROCKET	generalized unitarity	Ellis et al.
BLACKHAT	generalized unitarity	Berger et al.
NJET	generalized unitarity	Badger et al.
HELAC-NLO	4-dimensional (OPP) integrand reduction	Bevilacqua et al.
MADLOOP	4-dimensional (OPP) integrand reduction	Hirschi et al.
GOSAM	d-dimensional integrand reduction	Cullen et al.
FORMCALC	d-dimensional integrand reduction	Hahn et al.
OPENLOOPS	recursion relations for “open loops”	Cascioli et al.
RECOLA	recursion relations for “open loops”	Actis et al.

crucial ingredients for reduction

- recursive calculation of amplitudes
- generalized unitaritiy
- reduction at integrand level
- improved reduction methods for tensor integrals

Methods for QCD can be transferred to full SM!

Complications mainly in calculation of loops:

- more contributions (diagrams, off-shell currents)
- more and very different mass scales
  $\hookrightarrow$ numerical stability more problematic
- more complicated renormalization (more parameters)
- mixing of QCD and EW contributions
  (expansion in two couplings)
- chiral structure of weak interactions (treatment of γ_5)
- more complicated treatment of unstable particles
  (decay width = EW one-loop effect
  $\Rightarrow$ gauge invariance non trivial)

Solutions exist!

Automation of EW NLO is just happening!

Tools:

tool	collaboration	published applications
GoSam	Chiesa et al.	$\mathrm{pp} \to \mathrm{W} + 2$ jets
MadGraph5_aMC@NLO	Frixione et al.	$\mathrm{pp} \to \mathrm{t\bar{t}} + \{\mathrm{H}, \mathrm{Z}, \mathrm{W}\}$
		$\mathrm{pp} \to \mathrm{jj}$
OpenLoops	Pozzorini et al.	$\mathrm{pp} \to \mathrm{W} + \{2\ \text{jets}, 3\ \text{jets}\}$
Recola	Actis et al.	$\mathrm{pp} \to \mathrm{jj}\ell^+\ell^-$
		$\mathrm{pp} \to 4\ell$
		$\mathrm{pp} \to \mathrm{e}^+\nu_\mathrm{e}\mu^-\bar{\nu}_\mu\mathrm{b\bar{b}}(+\mathrm{H})$
		$\mathrm{pp} \to \mathrm{e}^+\nu_\mathrm{e}\mu^+\nu_\mu\mathrm{jj}$ (vector-boson scattering)

This talk:

- Introduction of Recola
- Discussion of application of Recola: top–antitop production including decays

Outline

General form of one-loop amplitudes (free of unphysical singularities)

$$\delta\mathcal{M}^{\text{1-loop}} = \sum_j \sum_{R_j} c^{(j,R_j,N_j)}_{\mu_1\cdots\mu_{R_j}} T^{\mu_1\cdots\mu_{R_j}}_{(j,R_j,N_j)} = \sum_j d^{(j,N_j)} T_{(j,0,N_j)}$$

tensor integrals

$$T^{\mu_1\cdots\mu_{R_j}}_{(j,R_j,N_j)} = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2}\int \mathrm{d}^D q \, \frac{q^{\mu_1}\cdots q^{\mu_{R_j}}}{D_{j,0}\cdots D_{j,N_j-1}}, \qquad D_{j,a} = (q+p_{j,a})^2 - m^2_{j,a}$$

tensor coefficients

$c^{(j,R_j,N_j)}_{\mu_1\cdots\mu_{R_j}}$ free of unphysical singularities, $d^{(j,N_j)}$ involve unphysical singularities

proposal of van Hameren ’09:
calculate $c^{(j,R_j,N_j)}_{\mu_1\cdots\mu_{R_j}}$ numerically in a recursive way

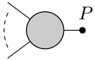
implemented for full Standard Model in RECOLA Actis, Denner, Hofer, Scharf, Uccirati ’16
(Recursive computation of one-loop amplitudes)

evaluation of tensor integrals by COLLIER Denner, Dittmaier, Hofer ’16
(Complex one loop library in extended regularizations)

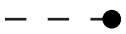
Basic building blocks of tree-level recursion:

off-shell current of particle P with n external legs

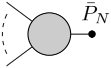
$$w(P, \mathcal{C}, \{l_1, \ldots, l_n\}) \quad = \quad n$$



- w is a scalar, spinor or vector corresponding to P (set of numbers)
- $\mathcal{C}$ represents the colour
- $\{l_1, \ldots, l_n\}$ list of primary external legs
- off-shell currents for external legs ($n = 1$) are wave functions

 $= u_\lambda(p)$,  $= \bar{u}_\lambda(p)$,  $= \epsilon_\lambda(p)$,  $= 1$

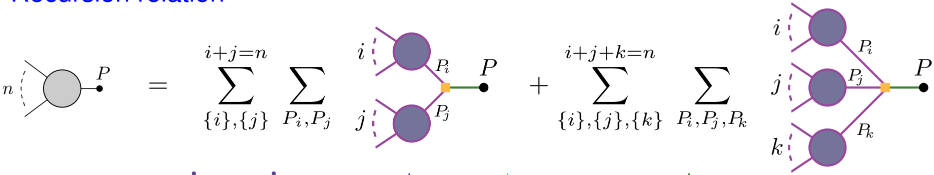
Amplitude for process with N external particles:

$\mathcal{M} \; = \; N-1$ 

$\times$ (propagator of $\bar{P}_N)^{-1} \times$  $\bar{P}_N$

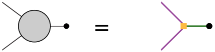
amputate off-shell line and multiply with wave function

Recursion relation

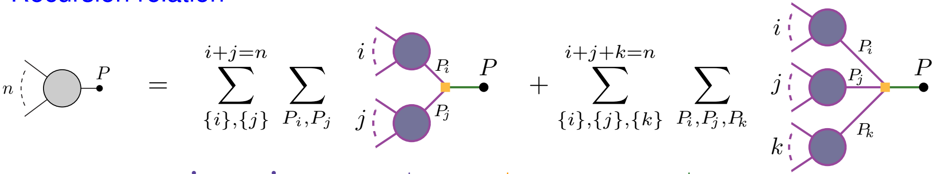


incoming currents × **vertex** × **propagator**

2-leg currents:

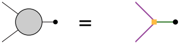


Recursion relation

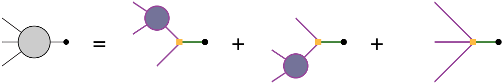


incoming currents × **vertex** × **propagator**

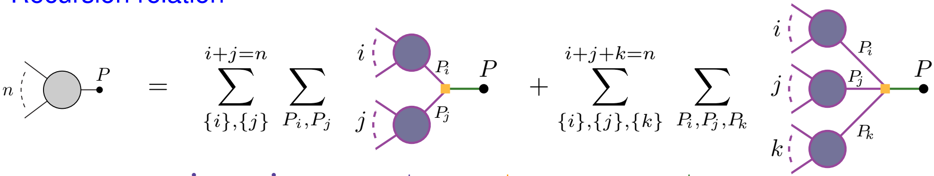
2-leg currents:



3-leg currents:

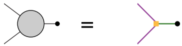


Recursion relation

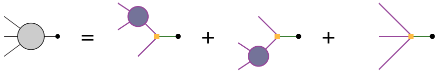


incoming currents × **vertex** × **propagator**

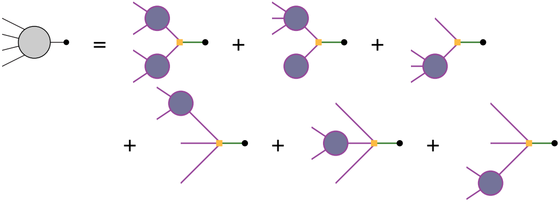
2-leg currents:



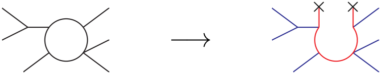
3-leg currents:



4-leg currents:

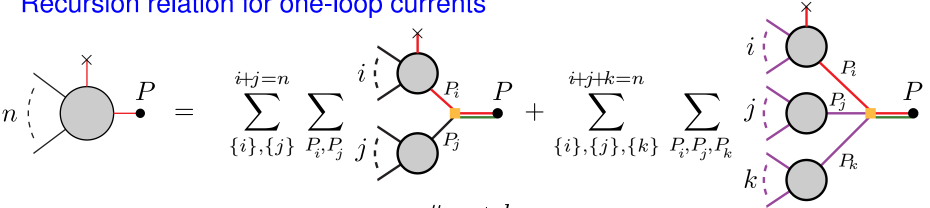


Cut loop line and consider tree diagrams with two more legs



relation can be defined uniquely

Recursion relation for one-loop currents



(**vertex**) $\times$ (**propagator**) $= \dfrac{a^\mu q_\mu + b}{(q+p)^2 - m^2}$ $\qquad q =$ loop momentum

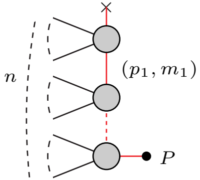
loop current $= \displaystyle\sum_{r=0}^{k} a^{(k,r)}_{\mu_1\cdots\mu_r} \frac{q^{\mu_1}\cdots q^{\mu_r}}{\prod_{h=1}^{k}[(q+p_h)^2 - m_h^2]}$

tensor coefficients computed recursively

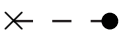
tensor integrals

Basic building blocks for one-loop recursion:
one-loop off-shell currents

$$w_{i_k}(P, \mathcal{C}, \{l_1, \ldots, l_n\}, \{p_1, \ldots, p_k\}, \{m_1, \ldots, m_k\}) =$$



- $\{p_1, \ldots, p_k\}, \{m_1, \ldots, m_k\}$: sequence of momenta and masses in loop propagators
- i_k multi-index representing k, r and $\mu_1, \ldots, \mu_r$: $w_{i_k} = a^{(k,r)}_{\mu_1 \cdots \mu_r}$
- currents for the tree lines are the same as at tree level
- suitable wave functions for first and last loop line:

i  $= \psi_i$, i  $= \bar{\psi}_i$, i  $= \epsilon_i$, i  $= 1$

cutted lines are reconnected via polarization sums

$$\sum_{i=1}^{4} (\bar{\psi}_i)_\alpha (\psi_i)_\beta = \delta_{\alpha\beta}, \qquad \sum_{i=1}^{4} \epsilon_i^\mu \epsilon_i^\nu = \delta^{\mu\nu},$$

- coefficients $a^{(k,r)}_{\mu_1 \ldots \mu_r}$ of the last current equal tensor coefficients $c^{(j,R_j,N_j)}_{\mu_1 \cdots \mu_{R_j}}$

Colour-flow representation

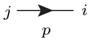
Maltoni, Paul, Stelzer, Willenbrock '02

Gluon field : $\sqrt{2}\, A^a_\mu \, (\lambda^a)^i_j = (\mathcal{A}_\mu)^i_j$

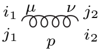
"usual" gluon with colour index $a = 1, \dots, 8$

gluon with colour-flow ${}^i_j$ $i, j = 1, 2, 3$

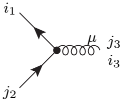
Feynman rules

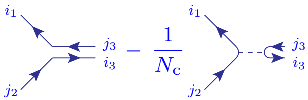


$$= \delta^i_j \times \frac{\mathrm{i}(\not{p} + m)}{p^2 - m^2}$$



$$= \times \frac{-\mathrm{i}\, g_{\mu\nu}}{p^2} = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1} \times \frac{-\mathrm{i}\, g_{\mu\nu}}{p^2}$$





$$= \left(\quad - \frac{1}{N_\mathrm{c}} \quad \right) \times \frac{\mathrm{i}\, g_\mathrm{s}}{\sqrt{2}} \gamma^\mu$$

$$= \left(\delta^{i_1}_{j_3}\delta^{i_3}_{j_2} - \frac{1}{N_\mathrm{c}} \delta^{i_1}_{j_2}\delta^{i_3}_{j_3} \right) \times \frac{\mathrm{i}\, g_\mathrm{s}}{\sqrt{2}} \gamma^\mu$$

colour matrices are just products of Kronecker deltas (nontrivial structure shifted to vertices)

Structure of the amplitude

$$\mathcal{A}^{i_1,\dots,i_n}_{j_1,\dots,j_n} = \sum_{P(j_1,\dots,j_n)} \delta^{i_1}_{j_1} \cdots \delta^{i_n}_{j_n} \, \mathcal{A}_P,$$

- Colour-dressed amplitudes:
  $\Rightarrow$ compute $\mathcal{A}^{i_1,\dots,i_n}_{j_1,\dots,j_n}$ for all possible colours (N_c^{2n})

  squared amplitude: $$\mathcal{M}^2 = \sum_{i_1\dots i_n, j_1,\dots,j_n} (\mathcal{A}^{i_1\cdots i_n}_{j_1\cdots j_n})^* \mathcal{A}^{i_1\cdots i_n}_{j_1\cdots j_n}$$

  requires colour-dressed off-shell currents

- Structure-dressed amplitudes:
  $\Rightarrow$ compute $\mathcal{A}_P$ for all possible P $(n!)$

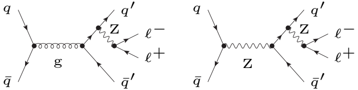
  squared amplitude: $$\mathcal{M}^2 = \sum_{P,P'} \mathcal{A}^*_P \, C_{PP'} \, \mathcal{A}_{P'},$$

  $C_{PP'}$ are trivial polynomials in N_c

  requires structure-dressed off-shell currents
  efficiently obtained in recursive procedure

Features of RECOLA 1.0

- Full Standard Model (QCD + EW)
  - tree level and one-loop amplitudes
  - Feynman rules for counter terms — Denner '93
  - Feynman rules for rational terms (R_2) — Garzelli, Malamos, Pittau '10
- complex-mass scheme for unstable particles
- mass- and dimensional regularization supported for IR singularities
- renormalization
  - EW sector: on-shell renormalization
    different options for renormalization of α $(\alpha_{G_\mu}, \alpha(0), \alpha(M_\mathrm{Z}))$
  - α_s: $\overline{\mathrm{MS}}$ renormalization (variable and fixed flavour schemes)
- selection of resonances, e.g. $q\mathrm{g} \to q\mathrm{g}\mathrm{Z} \to q\mathrm{g}\ell^+\ell^-$
- selection of powers of α_s at matrix-element or cross-section level
  e.g. matrix element for $q\bar{q} \to q'\bar{q}'\ell^+\ell^-$:
  LO: $\mathcal{O}(\alpha_\mathrm{s}\alpha)$ and $\mathcal{O}(\alpha^2)$
  NLO: $\mathcal{O}(\alpha_\mathrm{s}^2\alpha)$, $\mathcal{O}(\alpha_\mathrm{s}\alpha^2)$ and $\mathcal{O}(\alpha^2)$



Features of RECOLA 1.0 (cont.)

- running of α_{s}, dynamical scale choice supported
- NLO amplitudes for specific helicities and colour structures
- colour- and spin-correlated amplitudes for dipole subtraction
- numerical check of cancellation of UV divergences possible
- fast, purely numerical `Fortran` code, low memory usage
- optimisations
  - calculation of colour structures
  - recalculation of currents for different helicity configurations avoided
  - helicity conservation for massless fermions used
- external library for tensor integrals needed $\Rightarrow$ COLLIER

RECOLA 1.0 has been published:

- paper/manual: arXiv:1605.01090 to appear in Comput.Phys.Commun.
- code: https://recola.hepforge.org/

Matrix-element generator for theories Beyond the Standard Model
generalization of RECOLA 1.0, under development J.N. Lang

Input: RECOLA model file with Feynman rules for

- usual Feynman rules
- counter terms in specific renormalization scheme(s)
- rational terms
- rules for recursive construction of off-shell currents

Features in addition to those of RECOLA 1.0:

- Background-Field gauge
- R_ξ-gauge

First application: Two-Higgs-Doublet model, Higgs-singlet extension of SM

$\mathrm{pp} \to \mathrm{H} f f'$ and $\mathrm{H} \to 4f$

REnormalization in Python aT 1 Loop   J.N. Lang

Toolchain: PYTHON, FORM, RECOLA
to generate a RECOLA model file

- Input: usual Feynman rules in UFO format   Degrande et al. '12
  (Universal FeynRules Output)
- Output:
  complete RECOLA model file with optimised FORTRAN code for recursive rules
  (vectorized, symmetrized, common subexpressions)
  FORM expressions that allow RECOLA 2.0 to generate FORM output for amplitudes
- applicable to renormalizable theories and effective theories
- supported renormalization schemes:
  - consistent renormalization of tadpoles   Denner et al. '16
  - on-shell, $\overline{\mathrm{MS}}$, MOM for 2-point functions
  - $\overline{\mathrm{MS}}$ renormalization for n-point functions $(n > 2)$
  - fixed-flavour scheme for strong coupling
  - $\alpha(0)$, G_μ scheme for EW coupling
  - specific renormalization schemes for 2-Higgs-Doublet Model

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General form of one-loop amplitudes (free of unphysical singularities)

$$\delta\mathcal{M} \quad = \quad \sum_j \sum_{R_j} c^{(j,R_j,N_j)}_{\mu_1\cdots\mu_{R_j}}\, T^{\mu_1\cdots\mu_{R_j}}_{(j,R_j,N_j)} = \sum_j d^{(j,N_j)}\, T_{(j,0,N_j)}$$

tensor integrals

$$T^{\mu_1\cdots\mu_{R_j}}_{(j,R_j,N_j)} = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2}\int \mathrm{d}^D q\, \frac{q^{\mu_1}\cdots q^{\mu_{R_j}}}{D_{j,0}\cdots D_{j,N_j-1}}, \qquad D_{j,a} = (q+p_{j,a})^2 - m_{j,a}^2$$

tensor coefficients

$c^{(j,R_j,N_j)}_{\mu_1\cdots\mu_{R_j}}$ free of unphysical singularities, $d^{(j,N_j)}$ involve unphys. sing.

proposal of van Hameren '09:
calculate $c^{(j,R_j,N_j)}_{\mu_1\cdots\mu_{R_j}}$ numerically in a recursive way

implemented for full Standard Model in RECOLA Actis, Denner, Hofer, Scharf, Uccirati (Recursive computation of one-loop amplitudes)

evaluation of tensor integrals by COLLIER Denner, Dittmaier, Hofer, in preparation (Complex one loop library in extended regularizations)

Different methods used depending on number N of propagators

- $N = 1, 2$: explicit analytical expressions (numerically stable)
- $N = 3, 4$: exploit Lorentz covariance
  different methods used depending on kinematics
  - standard Passarino–Veltman (PV) reduction — Passarino, Veltman '79
  - stable expansions in exceptional phase-space regions
    (small Gram determinants)
    Denner, Dittmaier '05
    (see also R.K.Ellis et al. '05; Binoth et al. '05; Ferroglia et al. '02)

  $\Rightarrow$ reduction $T^{N,R} \to T^{N,0}, T^{N-1,R'} \to T^{0\dots N,0}$
- $N \geq 5$: exploit 4-dimensionality of space-time
  $\Rightarrow$ direct reduction of $T^{N,R} \to T^{N-1,R-1}$ (free of inverse Gram dets.)
  Melrose '65; Denner, Dittmaier '02,'05; Binoth et al. '05; Diakonidis et al. '08,'09

$\Rightarrow$ fast and stable numerical reduction algorithm

Basic scalar integrals A_0, B_0, C_0, D_0 from explicit analytical expressions
't Hooft, Veltman '79; Beenakker, Denner '90; Denner, Nierste, Scharf '91; Ellis, Zanderighi '08; Denner, Dittmaier '11

- tensor integrals for arbitrary number of external momenta N (tested in physical processes up to $N = 9$)
- various expansion methods for exceptional phase-space points (to arbitrary order in expansion parameter)
- mass- and dimensional regularization supported for IR singularities
- complex masses supported (unstable particles)
- cache-system to avoid recalculation of identical integrals
- output: coefficients $T^{N}_{0\cdots 0 i_1 \cdots i_k}$ or tensors $T^{N,\mu_1\cdots\mu_R}$
- two independent implementations $\Rightarrow$ checks during run possible
- error estimates for tensor coefficients and tensor integrals
- complete set of one-loop scalar integrals for scattering processes

COLLIER used in matrix-element generators

- RECOLA — Actis et al. '12, '16
- OPENLOOPS — Cascioli, Maierhöfer, Pozzorini '11
- MADLOOP — Hirschi et al. '11

COLLIER 1.0 has been published:

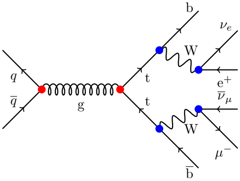
- paper/manual:
  arXiv:1604.06792, Comput.Phys.Commun. 212 (2017) 220
- code: https://collier.hepforge.org/

Top–antitop production including decays at LO

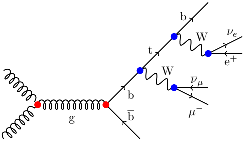
Study of top-pair production very important at LHC

- large cross section $\Rightarrow$ precise measurements possible
- heaviest particle of SM $\Rightarrow$ window to new physics

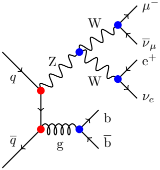
LO cross section for $\mathrm{pp} \to \mathrm{e}^+\nu_\mathrm{e}\mu\bar{\nu}_\mu\mathrm{b}\bar{\mathrm{b}}$: $\quad \mathcal{O}\left(\alpha_\mathrm{s}^2\alpha^4\right)$



two resonant tops



one resonant top



no resonant top

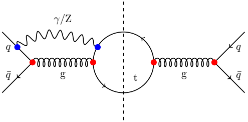
- EW contribution of $\mathcal{O}\left(\alpha^6\right)$ for $q\bar{q}$ channel neglected.
- photon-induced process $q\gamma \to \mathrm{e}^+\nu_\mathrm{e}\mu\bar{\nu}_\mu\mathrm{b}\bar{\mathrm{b}}$ of $\mathcal{O}\left(\alpha_\mathrm{s}\alpha^5\right)$ included

Existing results for top-pair production

- NLO QCD for off-shell top quarks: Melnikov, Schulze '09; Bevilacqua et al. '10;
  Denner et al. '10, '12; Frederix '13; Campbell et al. '12, '16
- NLO EW for on shell top quarks: Beenakker et al. '94,
  Bernreuther et al. '06, '08, '10, '12; Kühn et al. '05, '06, '13;
  Hollik, Kollar '07; Hollik, Pagani '11; Pagani et al. '16
- NNLO QCD: (on-shell top quarks) Czakon et al. '13, '16
- Resummation: (soft and small-mass logarithms to NNLL)
  Beneke et al. '10, '11; Czakon et al. '09; Ahrens et al. '10,
  Kidonakis '09, '10; Pecjak et al. '16
- NLO QCD matched to parton showers: Frixione et al. '03, 07;
  Kardos et al '11, '13; Alioli et al. '11; Cascioli et al. '13
  Höche et al. '14; Garzelli et al. '14; Campbell et al. '14; Ježo et al. '16

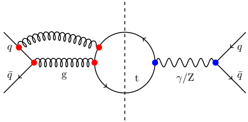
New: NLO EW for off-shell top quarks

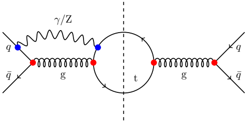
EW corrections to LO “QCD cross section”: $\mathcal{O}\left(\alpha_s^2\alpha^5\right)$



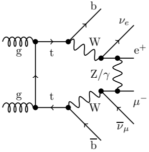
(decay products of top quarks suppressed in diagrams)

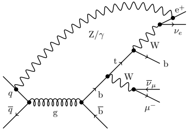
QCD corrections to LO “QCD–EW interference”: $\mathcal{O}\left(\alpha_s^2\alpha^5\right)$ not separable





octagon and heptagon diagrams





Details on virtual EW corrections

- all terms of $\mathcal{O}\left(\alpha_s^2\alpha^5\right)$ included
- terms of $\mathcal{O}\left(\alpha_s\alpha^6\right)$, $\mathcal{O}\left(\alpha^7\right)$ neglected
- on-shell renormalization scheme
- G_μ scheme for electromagnetic coupling:

$$\alpha_{G_\mu} = \frac{\sqrt{2}G_\mu M_W^2}{\pi}\left(1 - \frac{M_W^2}{M_Z^2}\right)$$

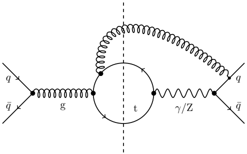
  absorbs running of α to EW scale and some universal corrections $\propto m_t^2$
- complex-mass scheme for top-quark and gauge-boson resonances

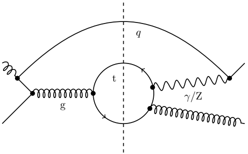
  Denner, Dittmaier, Roth, Wackeroth, Wieders '99, '05

  complex poles: $\mu_t = \sqrt{m_t^2 - i m_t \Gamma_t}$, $\mu_W^2 = M_W^2 - i M_W \Gamma_W$
  $\Rightarrow$ complex EW mixing angle
- matrix elements calculated with RECOLA and COLLIER
- 't Hooft–Feynman gauge

Contributions to σ in $\mathcal{O}(\alpha_s^2\alpha^5)$

- real photon emission from LO QCD contributions
- real gluon emission in QCD–EW interferences





(decay products of top quarks suppressed in diagrams)

colour structure $\Rightarrow$ only initial–final-state interference

soft and collinear singularities

- Catani–Seymour dipole subtraction Catani, Seymour '96; Dittmaier '99
- initial-state collinear singularities cancelled by $\overline{\mathrm{MS}}$ redefinition of PDFs
- recombination of collinear parton–photon and lepton–photon pairs (jet clustering)
  $\Rightarrow$ cancellation of singularities from collinear photon emission

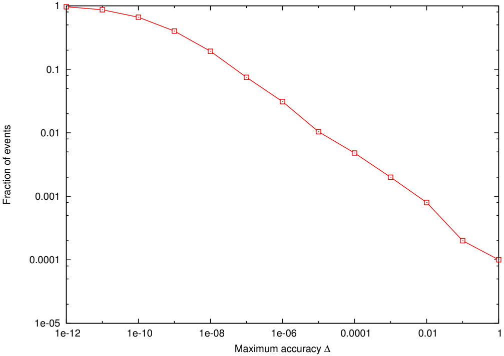
Phase-space integration with multi-channel Monte Carlo MOCANLO Feger

- Tree-level matrix elements and LO hadronic cross section successfully compared with MG5@NLO Alwall et al. '14
- IR-singularities, Monte Carlo integration
  - variation of α parameter in subtraction terms Nagy, Trócsányi '98
  - variation of technical cuts
  - variation of IR scale
- One-loop matrix elements
  - two independent libraries within COLLIER
  - comparison against double-pole approximation
  - check of Ward identity for matrix elements with external gluons

Check of Ward identity

Numerical check of Ward identity for $\mathrm{gg} \to \mathrm{e}^+\nu_\mathrm{e}\mu^-\bar{\nu}_\mu \mathrm{b}\bar{\mathrm{b}}$

(polarization vector of gluon replaced by normalized momentum p^μ/p_0)



$$\Delta = \frac{\mathrm{Re}\,\mathcal{M}_1(\epsilon \to p/p_0)\mathcal{M}_0^*}{\mathrm{Re}\,\mathcal{M}_0^*\mathcal{M}_1}$$

- typical accuracy: $10^{-8} - 10^{-10}$
- agreement worse than 10^{-3} for less than 0.02% of points

$\Rightarrow$ successful check of 8-point functions in COLLIER

Setup for phenomenological analysis

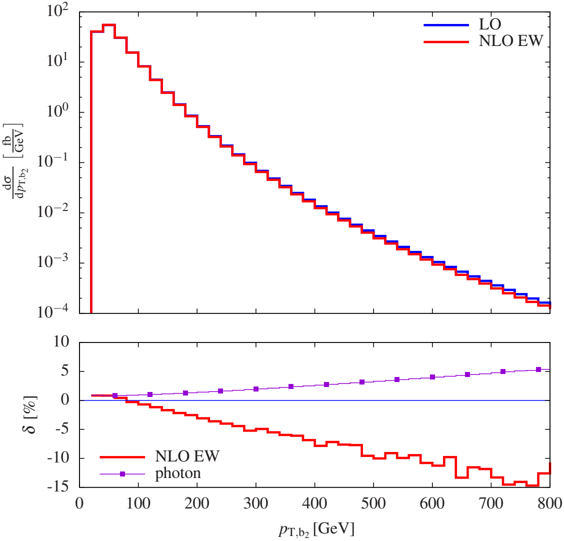
- massless bottom quarks (0.8% effect in our setup)
  $\hookrightarrow$ cuts needed to avoid collinear bottom quarks
- diagonal quark mixing matrix
- PDFs: NNPDF23_nlo_as_0119_qed Ball et al. '13
- bottom PDFs neglected (0.01% contribution)
- renormalization and factorization scales: $\mu_\mathrm{R} = m_\mathrm{t} = \mu_\mathrm{F}$
- jet clustering: anti-k_T algorithm with $\Delta R = 0.4$
  Cacciari, Salam, Soyez '08
- cuts:

$$
\begin{aligned}
\text{b jets:} \quad & p_{\mathrm{T,b}} > 25\,\mathrm{GeV}, \quad |y_\mathrm{b}| < 2.5 \\
\text{charged leptons:} \quad & p_{\mathrm{T},\ell} > 20\,\mathrm{GeV}, \quad |y_\ell| < 2.5 \\
\text{missing transverse momentum:} \quad & p_{\mathrm{T,miss}} > 20\,\mathrm{GeV} \\
\text{b-jet–b-jet distance:} \quad & \Delta R_\mathrm{bb} > 0.4
\end{aligned}
$$

Ch.	σ_{LO} [fb]	$\sigma_{\mathrm{NLO\ EW}}$ [fb]	δ [%]
gg	$2824.2(2)$	$2834.2(3)$	0.35
$q\bar{q}$	$375.29(1)$	$377.18(6)$	0.50
$gq(/\bar{q})$		$0.259(4)$	
γg	$27.930(1)$		
pp	$3199.5(2)$	$3211.7(3)$	0.38

- Cross section dominated by gg channel
- γg channel $\lesssim 1\%$
- small positive EW corrections
  (due to the choice of the top width including EW corrections in LO)
  must subtract twice the EW corrections to the top width (1.3%)
  to compare with on-shell calculations
  $\Rightarrow$ negative EW corrections for on-shell top quarks: -2.2%
  on-shell results: $-1\% \ldots -2\%$ Pagani et al. $\Rightarrow$ agreement within 1%

Transverse-momentum distribution of subleading b quark



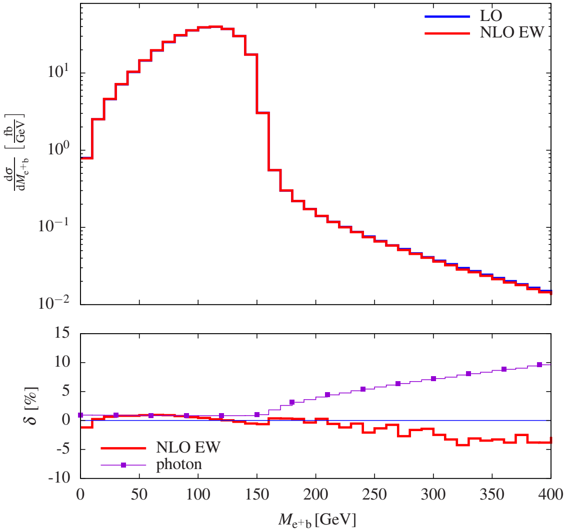
Sudakov logarithms $\rightarrow$ -15% EW corrections

Sizeable photon contributions $\rightarrow$ $+6\%$

Pagani '16

(more recent γ PDFs yield smaller effects)

Positron–bottom invariant-mass distribution



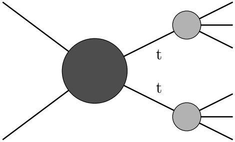
EW corrections of -5% in off-shell region

Sizeable photon contributions $\rightarrow +10\%$
(more recent γ PDFs yield smaller effects)

Leading order:

$$\mathcal{M}_{\mathrm{LO,DPA}}^{\mathrm{gg}\to\mathrm{t}\bar{\mathrm{t}}\to 6f} = \sum_{\lambda_\mathrm{t},\lambda_{\bar{\mathrm{t}}}} \frac{[\mathcal{M}_{\mathrm{LO}}^{\mathrm{gg}\to\mathrm{t}\bar{\mathrm{t}}}(\lambda_\mathrm{t},\lambda_{\bar{\mathrm{t}}})\mathcal{M}_{\mathrm{LO}}^{\mathrm{t}\to 3f}(\lambda_\mathrm{t})\mathcal{M}_{\mathrm{LO}}^{\bar{\mathrm{t}}\to 3f}(\lambda_{\bar{\mathrm{t}}})]_{\text{on-shell}}}{(p_\mathrm{t}^2 - m_\mathrm{t}^2 + \mathrm{i}m_\mathrm{t}\Gamma_\mathrm{t})(p_{\bar{\mathrm{t}}}^2 - m_\mathrm{t}^2 + \mathrm{i}m_\mathrm{t}\Gamma_\mathrm{t})}$$

- only contributions with two resonant tops
  ⇒ dominant contribution
- momenta in numerator projected on shell
  ⇒ gauge invariance



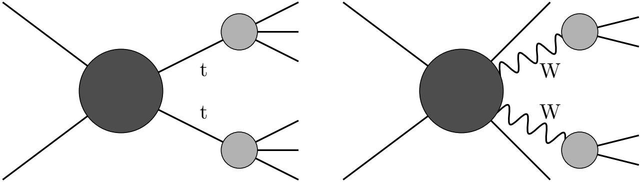
NLO:

- factorizable corrections: corrections to production or decay matrix elements
- non-factorizable corrections: IR-singular corrections connecting production and decay ⇒ universal correction factors

  Denner et al. ’00; Accomando et al. ’04 ; Dittmaier, Schwan ’15

Double-pole approximation for $\mathrm{pp} \to \mathrm{e}^+\nu_\mathrm{e}\mu\bar{\nu}_\mu \mathrm{b}\bar{\mathrm{b}}$

- DPA applied only to squared matrix element for (subtracted) virtual corrections
- leading order and real corrections treated exactly
- phase-space integration treated exactly
- two different DPAs for resonant $\mathrm{t}\bar{\mathrm{t}}$ or $\mathrm{W}^+\mathrm{W}^-$



two resonant W bosons
$\Rightarrow$ all contributions with two resonant tops and most other included!

- naive error estimate: $\mathcal{O}(\Gamma/M) \times (\alpha/s_\mathrm{w}^2) \times \log(\ldots) \sim \mathcal{O}(0.1\%)$

LO: WW DPA (-0.6%) better than tt DPA (-2.9%) $\quad (\Gamma_t/m_t \sim 0.8\%)$

Ch.	$\sigma_{\rm LO}^{\rm WW\ DPA}$ [fb]	$\delta_{\rm LO}^{\rm WW\ DPA}$ [%]	$\sigma_{\rm LO}^{\rm tt\ DPA}$ [fb]	$\delta_{\rm LO}^{\rm tt\ DPA}$ [%]
gg	$2808.4(6)$	-0.56	$2738.8(2)$	-3.0
$q\bar{q}$	$372.90(1)$	-0.64	$368.82(1)$	-2.2
pp	$3181.3(5)$	-0.57	$3107.6(2)$	-2.9

$$\delta^{DPA} = \sigma^{\rm DPA}/\sigma^{\rm full} - 1$$

NLO: WW DPA (-0.04%) equally good as tt DPA (0.07%)

$$((\alpha/s_{\rm w}^2) \times \Gamma_t/m_t \sim 0.03\%)$$

Ch.	$\sigma_{\rm NLO\ EW}^{\rm WW\ DPA}$ [fb]	$\delta_{\rm NLO\ EW}^{\rm WW\ DPA}$ [%]	$\sigma_{\rm NLO\ EW}^{\rm tt\ DPA}$ [fb]	$\delta_{\rm NLO\ EW}^{\rm tt\ DPA}$ [%]
gg	$2832.9(2)$	-0.046	$2836.5(2)$	$+0.082$
$q\bar{q}$	$377.36(8)$	$+0.047$	$377.23(5)$	$+0.013$
pp	$3210.5(2)$	-0.037	$3214.0(2)$	$+0.072$

LO: WW DPA (-0.6%) better than tt DPA (-2.9%) $\quad(\Gamma_\mathrm{t}/m_\mathrm{t} \sim 0.8\%)$

Ch.	$\sigma^{\mathrm{WW\ DPA}}_{\mathrm{LO}}$ [fb]	$\delta^{\mathrm{WW\ DPA}}_{\mathrm{LO}}$ [%]	$\sigma^{\mathrm{tt\ DPA}}_{\mathrm{LO}}$ [fb]	$\delta^{\mathrm{tt\ DPA}}_{\mathrm{LO}}$ [%]
gg	2808.4(6)	−0.56	2738.8(2)	−3.0
$q\bar{q}$	372.90(1)	−0.64	368.82(1)	−2.2
pp	3181.3(5)	−0.57	3107.6(2)	−2.9

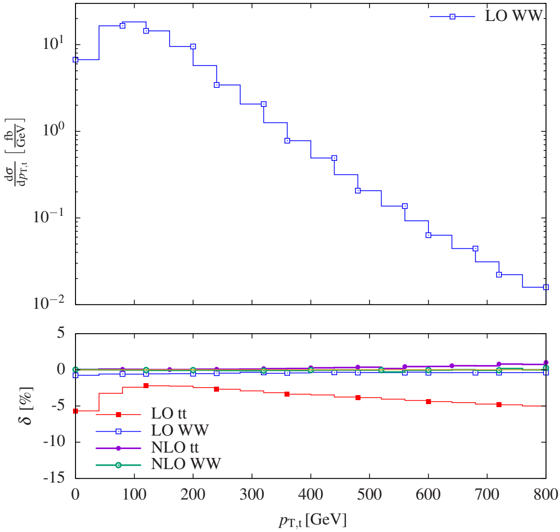
$$\delta^{DPA} = \sigma^{\mathrm{DPA}}/\sigma^{\mathrm{full}} - 1$$

NLO: WW DPA (-0.04%) equally good as tt DPA (0.07%)

$$((\alpha/s_\mathrm{w}^2) \times \Gamma_\mathrm{t}/m_\mathrm{t} \sim 0.03\%)$$

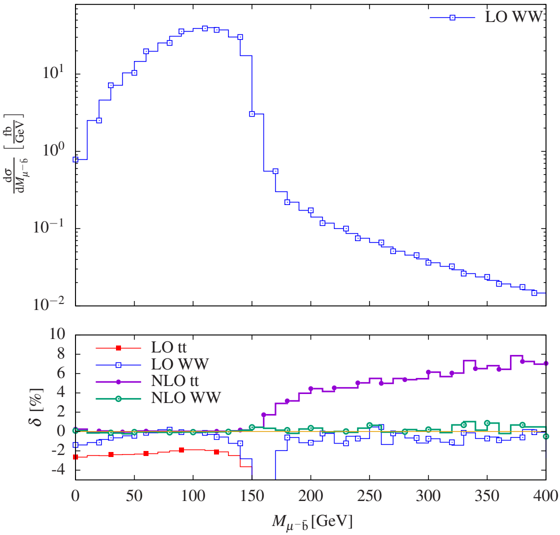
Ch.	$\sigma^{\mathrm{WW\ DPA}}_{\mathrm{NLO\ EW}}$ [fb]	$\delta^{\mathrm{WW\ DPA}}_{\mathrm{NLO\ EW}}$ [%]	$\sigma^{\mathrm{tt\ DPA}}_{\mathrm{NLO\ EW}}$ [fb]	$\delta^{\mathrm{tt\ DPA}}_{\mathrm{NLO\ EW}}$ [%]
gg	2832.9(2)	−0.046	2836.5(2)	+0.082
$q\bar{q}$	377.36(8)	+0.047	377.23(5)	+0.013
pp	3210.5(2)	−0.037	3214.0(2)	+0.072

Transverse-momentum distribution of top quark



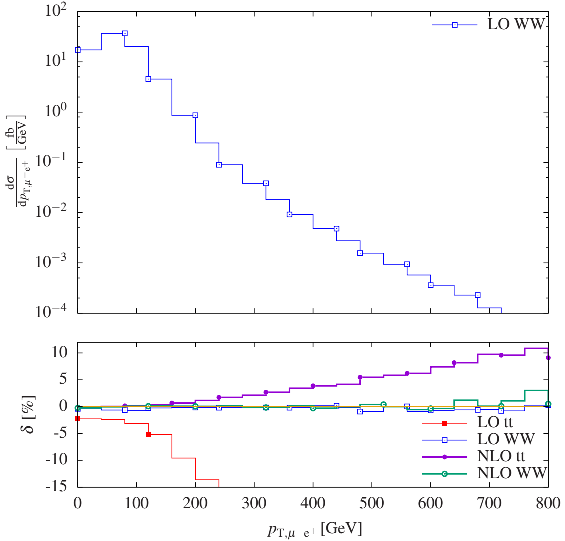
Both DPAs work well for top-dominated observables

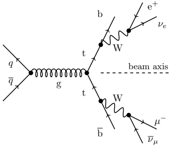
deviation similar as for integrated cross section

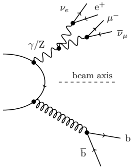


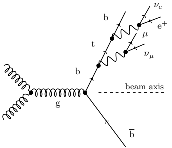
tt DPA is failing for off-shell top quarks

large deviation of WW DPA above threshold: $\Gamma_W = 0$ in DPA
$\Rightarrow$ faster decrease









tt DPA is failing where 0/1-top resonance contributions become sizeable

Conclusion

- Electroweak corrections relevant for many LHC processes
  importance increases with energy and luminosity
- General tools for their calculation:
  - COLLIER: fast and numerically stable calculation of one-loop tensor integrals arXiv:1604.06792
  - RECOLA: recursive generator for tree-level and one-loop amplitudes in the full Standard Model arXiv:1605.01090
- Electroweak corrections to off-shell top-antitop production
  arXiv:1607.05571
  - full NLO EW corrections calculated for $\mathrm{pp} \to \mathrm{e}^+\nu_\mathrm{e}\mu\bar{\nu}_\mu\mathrm{b}\bar{\mathrm{b}}$
  - DPA for WW provides very good approximation
    DPA for tt fails where on-shell top quarks do not dominate
  - EW corrections below one per cent for integrated cross section
  - EW corrections can reach -15% in distributions
- Recent extension: Electroweak corrections to off-shell top-antitop production in association with a Higgs boson arXiv:1612.07138

Outline

6 Backup

General form of one-loop amplitudes

$$\delta\mathcal{M}^{\text{1-loop}} = \sum_j \sum_{R_j} c^{(j,R_j,N_j)}_{\mu_1\cdots\mu_{R_j}} T^{\mu_1\cdots\mu_{R_j}}_{(j,R_j,N_j)} + \delta\mathcal{M}^{\text{1-loop}}_{\text{CT}} + \delta\mathcal{M}^{\text{1-loop}}_{\text{R2}}$$

- loop amplitudes have to be renormalized
  $\Rightarrow$ add counter terms $\delta\mathcal{M}^{\text{1-loop}}_{\text{CT}}$
  - calculated via tree-level matrix elements with special Feynman rules
    counter-term vertices appear only once in each diagram
- loop amplitude is calculated numerically in $D = 4$ dimensions
  indices μ_i are 4-dimensional!
  $\Rightarrow$ add rational part $\delta\mathcal{M}^{\text{1-loop}}_{\text{R2}}$ Ossola, Papadopoulos, Pittau '08
  - results from UV poles of loop amplitudes $\underbrace{c^{(j,R_j,N_j)}_{\mu_1\cdots\mu_{R_j}}}_{D-4\ \text{part}} \underbrace{T^{\mu_1\cdots\mu_{R_j}}_{(j,R_j,N_j)}}_{1/(D-4)\ \text{part}} = \frac{\mathcal{O}(D-4)}{(D-4)} =$ finite
  - result only from UV-divergent vertex functions
    accounted for by special Feynman rules similar to counter terms
    Draggiotis, Garzelli, Malamos, Papadopoulos, Pittau '09 –'10

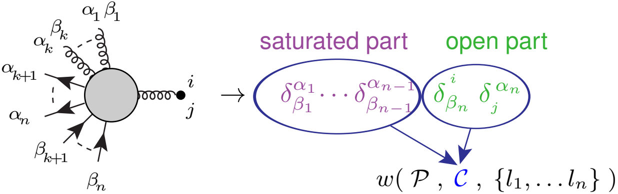
Colour structures: product of Kronecker δs

external currents:

$$\beta \longrightarrow\!\bullet\, i \;=\; u_\lambda(p)\,\delta^i_\beta \qquad \alpha \longleftarrow\!\bullet\, j \;=\; \bar{u}_\lambda(p)\,\delta^\alpha_j \qquad {}^{\alpha}_{\beta}\,\text{ooo}\!\bullet\,{}^{i}_{j} \;=\; \epsilon_\lambda(p)\left(\delta^i_\beta\delta^\alpha_j - \frac{1}{N_\mathrm{c}}\delta^i_j\delta^\alpha_\beta\right)$$

α, β: colour indices of external particles, i, j "open" colour indices

colour structure of off-shell current:



with all possible permutations of $\beta_1, \ldots \beta_n, j$

recursion procedure:

- saturated parts of incoming currents multiply
- open parts of incoming currents are contracted

optimization: compute currents differing just by colour structure only once
colour structures involving δ^i_j do not contribute for ${}^i_j$ gluon

- Memory for executables, object files and libraries: negligible
- RAM: less than 2 Gbyte also for complicated processes
- CPU time (processor Intel(R) Core(TM) i5-2450 CPU 2.50 GHz):
  - QCD corrections ($\mathrm{W}^+ \to \ell^+ \nu_\ell$, $\mathrm{W}^- \to \ell^- \bar{\nu}_\ell$, colour and helicity summed)

Process	t_{gen}	t_{TIs}	t_{TCs}
$\mathrm{u}\bar{\mathrm{d}} \to \mathrm{W}^+\mathrm{gg}$	2.4 s	4.0 ms	1.1 ms
$\mathrm{u}\bar{\mathrm{d}} \to \mathrm{W}^+\mathrm{ggg}$	15 s	67 ms	45 ms
$\mathrm{u}\bar{\mathrm{d}} \to \mathrm{W}^+\mathrm{W}^-\mathrm{gg}$	76 s	83 ms	16 ms

  - EW+QCD corrections (colour and helicity summed)

Process	t_{gen}	t_{TIs}	t_{TCs}
$\mathrm{u}\bar{\mathrm{d}} \to \ell^+\ell^-\mathrm{gg}$	3.2 s	27 ms	25 ms
$\mathrm{u}\bar{\mathrm{d}} \to \ell^+\ell^-\mathrm{u}\bar{\mathrm{u}}$	5.0 s	68 ms	35 ms
$\mathrm{u}\bar{\mathrm{d}} \to \ell^+\ell^-\mathrm{ggg}$	44 s	331 ms	684 ms
$\mathrm{u}\bar{\mathrm{d}} \to \ell^+\ell^-\mathrm{u}\bar{\mathrm{u}}\mathrm{g}$	50 s	835 ms	632 ms

checks

- many matrix elements at LO and NLO QCD+EW for $2 \to \{2, 3(, 4)\}$ processes checked against POLE Accomando, Denner, Meier '06
- many matrix elements at LO and NLO-QCD for $2 \to \{2, 3, 4\}$ processes checked against OPENLOOPS Cascioli, Maierhöfer, Pozzorini '11
- $\mathrm{pp} \to \nu_{\mathrm{e}} \mathrm{e}^+ \mu^- \bar{\nu}_\mu \mathrm{b}\bar{\mathrm{b}}$ at LHC at NLO QCD (off-shell $\mathrm{t}\bar{\mathrm{t}}$ production) cross-section and distributions checked against Denner, Dittmaier, Kallweit, Pozzorini '12

- $\mathrm{pp} \to \mathrm{Z} + 2\,\mathrm{jets}$ (partonic processes $gq \to \mathrm{Z}gq$)
  EW NLO corrections Actis, Denner, Hofer, Scharf, Uccirati '13
- $\mathrm{pp} \to \ell^+\ell^- + 2\,\mathrm{jets}$ ($gq \to \ell^+\ell^- gq$, $q\bar{q} \to \ell^+\ell^- q'\bar{q}'$)
  EW NLO, $\mathcal{O}(\alpha_\mathrm{s}^2\alpha^2 \times \alpha)$ contributions Denner, Hofer, Scharf, Uccirati '14
- $\mathrm{pp} \to \nu_\mu\mu^+\mathrm{e}^-\nu_\mathrm{e}, \mu^+\mu^-\mathrm{e}^-\mathrm{e}^+, \mu^+\mu^-\mu^+\mu^-$
  (off-shell WW and ZZ production)
  EW NLO, $\mathcal{O}(\alpha^4 \times \alpha)$ contributions Biedermann et al. '16
- $\mathrm{pp} \to \ell^+\nu_\ell\mathrm{jjb\bar{b}b\bar{b}}$ ($\mathrm{gg} \to \ell^+\nu_\ell q'\bar{q}''\mathrm{b\bar{b}b\bar{b}}$, $q\bar{q} \to \ell^+\nu_\ell q'\bar{q}''\mathrm{b\bar{b}b\bar{b}}$)
  full LO matrix element including all interferences ($2 \to 8$ process)
  $\mathcal{O}((\alpha_\mathrm{s}^3\alpha + \alpha_\mathrm{s}^2\alpha^2 + \alpha_\mathrm{s}\alpha^3 + \alpha^4)^2)$ (off-shell $\mathrm{t\bar{t}H}$ production)
  Denner, Feger, Scharf '14
- $\mathrm{pp} \to \mathrm{e}^+\nu_\mathrm{e}\mu^-\bar{\nu}_\mu\mathrm{b\bar{b}H}$ ($\mathrm{t\bar{t}H}$ production with off-shell top quarks)
  QCD NLO, $\mathcal{O}(\alpha_\mathrm{s}^2\alpha^5 \times \alpha_\mathrm{s})$, ($2 \to 7$ process) Denner, Feger '15
  EW NLO, $\mathcal{O}(\alpha_\mathrm{s}^2\alpha^5 \times \alpha)$ Denner, Lang, Pellen, Uccirati '16
- $\mathrm{pp} \to \ell^+\nu_\ell\mathrm{jjb\bar{b}}$ (off-shell $\mathrm{t\bar{t}}$ production)
  EW NLO, $\mathcal{O}(\alpha_\mathrm{s}^2\alpha^4 \times \alpha)$, ($2 \to 6$ process) Denner, Pellen '16
- $\mathrm{pp} \to \mathrm{e}^+\nu_\mathrm{e}\mu^+\nu_\mu\mathrm{jj}$ ($\mathrm{W^+W^-}$ scattering)
  EW NLO, $\mathcal{O}(\alpha^6 \times \alpha)$, ($2 \to 6$ process) Biedermann, Denner, Pellen '16

Passarino–Veltman reduction

covariant decomposition of tensor integrals:

$$T^{N,\mu_1\cdots\mu_R} = \sum_{i_1,\ldots,i_k} T^{N,R}_{\underbrace{0\cdots0}_{R-k}\, i_1\cdots i_k} \{\underbrace{g\cdots g}_{(R-k)/2}\, p_{i_1}\cdots p_{i_k}\}^{\mu_1\cdots\mu_R} \sim \int \mathrm{d}^D q\, \frac{q^{\mu_1}\cdots q^{\mu_{R_j}}}{D_{j,0}\cdots D_{j,N_j-1}}$$

contract tensor integral with external momenta p_i^μ and metric tensor $g^{\mu\nu}$ and use

$$\begin{aligned} 2p_i^\mu q_\mu &= -f_i + D_i - D_0, \qquad f_i = p_i^2 - m_i^2 + m_0^2 \\ g^{\mu\nu} q_\mu q_\mu &= m_0^2 + D_0 \end{aligned}$$

cancel denominators and insert covariant decomposition
$\Rightarrow$ recursive solution for tensor coefficients:

$$T^{N,R} = \frac{1}{\Delta}\left[T^{N,R-1}, T^{N,R-2}, T^{N-1}\right]$$

reduction to lower-rank and lower-N integrals $\Rightarrow$ scalar integrals

Gram determinant: $\Delta = \det(Z)$ with $Z_{ij} = 2p_i p_j, \quad Z^{-1} = \tilde{Z}/\Delta$

Reduction of 3-point and 4-point integrals (cont.)

$$2(D+R-N-1)\underbrace{T^N_{00i_3\ldots i_R}}_{\text{rank } R} = 2m_0^2\underbrace{T^N_{i_3\ldots i_R}}_{\text{rank } R-2} + \sum_{n=1}^{N-1} f_n \underbrace{T^N_{ni_3\ldots i_R}}_{\text{rank } R-1} + (T^{N-1}\text{ terms})$$

$i_1 \neq 0$:

$$\Delta \underbrace{T^N_{i_1\ldots i_R}}_{\text{rank } R} = \sum_{n=1}^{N-1} \tilde{Z}_{i_1 n}\Bigg[-f_n \underbrace{T^N_{i_2\ldots i_R}}_{\text{rank } R-1} - 2\sum_{r=2}^{R} \delta_{ni_r} \underbrace{T^N_{00i_2\ldots\hat{i}_r\ldots i_R}}_{\text{rank } R} + (T^{N-1}\text{ terms})\Bigg]$$

$\hookrightarrow$ recursive calculation of $T^N_{i_1\ldots i_R}$ from scalar integral T^N_0 and $T^{N-1}_{i_2\ldots i_R}$:

T^N_0 = basis integral $\rightarrow T^N_{i_1} \rightarrow T^N_{i_1 i_2} \rightarrow T^N_{i_1 i_2 i_3} \rightarrow \ldots$

- explicit D requires expansion of $T^N_{00\ldots}$ around $D=4$
  - UV-poles produce finite polynomial terms (rational terms)
    [$\mathcal{O}(D-4)/(D-4)$]
    easily obtained from recursion relations
    (no Δ^{-1} since $T^N_{i_1\ldots i_R}$ finite for $i_j \neq 0$)
  - $T^N_{00\ldots}$ do not involve IR poles $\Rightarrow$ no IR rational terms
    reduction valid for any IR regularization
- appearance of inverse Gram determinant Δ
  $\Rightarrow$ potential instabilities for $\Delta \rightarrow 0$ in exceptional points

$$\text{PV:} \quad T^{N,R} = \frac{1}{\Delta}\left[T^{N,R-1}, T^{N,R-2}, T^{N-1}\right]$$

small Gram determinant: $\Delta \to 0$

- finite $T^{N,R}$ as sum of $1/\Delta$-singular terms
  - spurious singularities cancel to give $\mathcal{O}(\Delta)/\Delta$-result
  - numerical determination of $T^{N,R}$ becomes unstable

- $T^{N,R-1}, T^{N,R-2}, T^{N-1}$ become linearly dependent
  - ⇒ scalar integrals D_0, C_0, B_0, A_0 become linearly dependent
  - ⇒ $\mathcal{O}(\Delta)/\Delta$-instabilities intrinsic to all methods relying on the full set of basis integrals D_0, C_0, B_0, A_0

- solution: choose appropriate set of base functions depending on phase-space point

Expansion in Gram determinant

$$\Delta\, T^{N,R+1} = \left[T^{N,R}, T^{N,R-1}, T^{N-1}\right] \quad (1)$$

- exploit linear dependence of $T^{N,R}, T^{N,R-1}, T^{N-1}$ for $\Delta = 0$ to determine $T^{N,R}$ up to terms of $\mathcal{O}(\Delta)$
- calculate $T^{N,R+1}$ from $\Delta\, T^{N,R+2} = \left[T^{N,R+1}, T^{N,R}, T^{N-1}\right]$ in the same way
- use $T^{N,R+1}$ in (1) to compute $\mathcal{O}(\Delta)$ in $T^{N,R}$
- higher orders in Δ iteratively:
  $\mathcal{O}(\Delta^k)$ of $T^{N,R}$ requires $T^{N,R+k}$ and thus T^{N-1} up to rank $R+k$
- basis of scalar integrals effectively reduced
  (e.g. D_0 expressed by C_0's)

$\Rightarrow$ stable results for $T^{N,R}$ for small Δ

Expansion breaks down in certain regions of phase space
$\Rightarrow$ alternative expansions

Tensor integrals and coefficients

$$T^{N,\mu_1\cdots\mu_R} = \sum_{i_1,\dots,i_k} T^{N,R}_{\underbrace{0\cdots0}_{R-k}\, i_1\cdots i_k} \{ \underbrace{g\cdots g}_{(R-k)/2}\; p_{i_1}\cdots p_{i_k} \}^{\mu_1\cdots\mu_R}$$

COLLIER:

- output: tensor coefficients $T^{N,R}_{\underbrace{0\cdots0}_{R-k}\, i_1\cdots i_k}$ or tensor integrals $T^{N,\mu_1\cdots\mu_R}$
- efficient algorithm to construct tensors from coefficients for arbitrary N, P via recursive calculation of tensor structures
- for $N \geq 6$: direct reduction at tensor level

$$\begin{array}{ccccccccccc}
B_{i_1\dots i_P} & \longrightarrow & C_{i_1\dots i_P} & \longrightarrow & D_{i_1\dots i_P} & \longrightarrow & E_{i_1\dots i_P} & \longrightarrow & F_{i_1\dots i_P} & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
B^{\mu_1\cdots\mu_P} & & C^{\mu_1\cdots\mu_P} & & D^{\mu_1\cdots\mu_P} & & E^{\mu_1\cdots\mu_P} & \longrightarrow & F^{\mu_1\cdots\mu_P} & \longrightarrow
\end{array}$$

# of tensor coefficients $\geq$ # of tensor elements for $N \geq 5$!