Loop functions in thermal QCD

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(2) M. Berwein, N. Brambilla, P. Petreczky and A. Vairo
Polyakov loop at next-to-next-to-leading order
Phys. Rev. D93 (2016) 034010 `arXiv:1512.08443`

(3) M. Berwein, N. Brambilla and A. Vairo
Renormalization of Loop Functions in QCD
Phys. Part. Nucl. 45 (2014) 656 `arXiv:1312.6651`

(4) M. Berwein, N. Brambilla, J. Ghiglieri and A. Vairo
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(5) N. Brambilla, J. Ghiglieri, P. Petreczky and A. Vairo
The Polyakov loop and correlator of Polyakov loops at next-to-next-to-leading order
Phys. Rev. D82 (2010) 074019 `arXiv:1007.5172`

Loop functions

Loop functions

These are gauge invariant quantities measurable by lattice QCD and relevant for the dynamics of static sources in a thermal bath at temperature T.

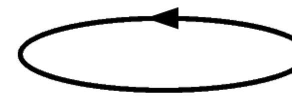
- e.g. Petreczky EPJC 43 (2005) 51

Despite their relevance, not too much has been known about loop functions in perturbation theory until recently.

Loop functions

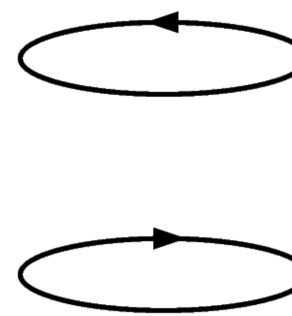
- Polyakov loop average in a thermal ensemble at a temperature T

$$P(T)|_R \equiv \frac{1}{d_R}\langle \mathrm{Tr}\ L_R \rangle \qquad (\mathrm{R} \equiv \text{color representation})$$



$d_A = N^2 - 1,\, d_F = N$ and $L_R(\mathbf{x}) = \mathcal{P}\exp\left(ig\int_0^{1/T} d\tau A^0(\mathbf{x},\tau)\right)$

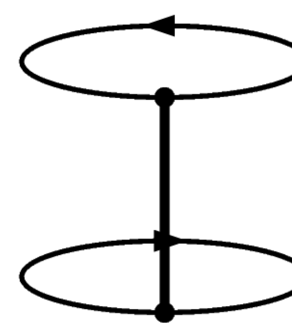
- Polyakov loop correlator



$$P_c(r,T) \equiv \frac{1}{N^2}\langle \mathrm{Tr}L_F^\dagger(\mathbf{0})\mathrm{Tr}L_F(\mathbf{r})\rangle = \frac{1}{N^2}\sum e^{-E_n/T}$$

  - `Lüscher Weisz JHEP 0207 (2002) 049`
    `Jahn Philipsen PR D70 (2004) 074504`

- Cyclic Wilson loop



$$W_c(r,T) \equiv \frac{1}{N}\langle \mathrm{Tr}L_F^\dagger(\mathbf{0})U^\dagger(1/T)L_F(\mathbf{r})U(0)\rangle$$

where $U(1/T) = \mathcal{P}\exp\left(ig\int_0^1 ds\,\mathbf{r}\cdot\mathbf{A}(s\mathbf{r},1/T)\right) = U(0)$.

Divergences

- Ultraviolet divergences come from regions where two or more vertices are contracted to one point.
- In the case of internal vertices divergences are removed through renormalization.
- For loop functions one also gets divergences from the contraction of line vertices along the contour.

The superficial degree of divergence is

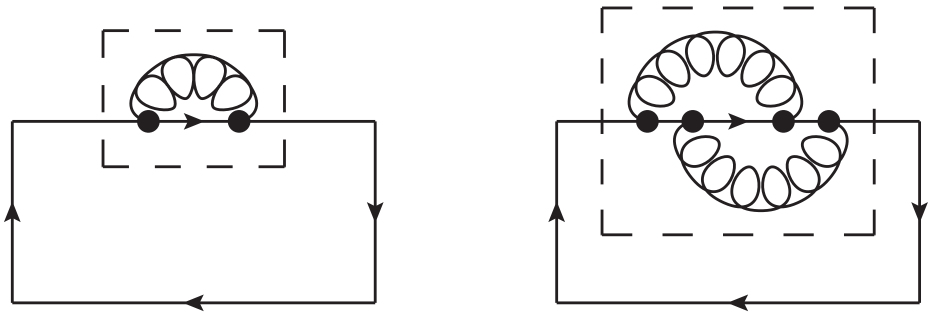
$$\omega = 1 - N_{\rm ex} \quad \text{smooth point}$$
$$\omega = -N_{\rm ex} \quad \text{cusp or intersection}$$

$N_{\rm ex} =$ number of propagators connecting the contraction point to uncontracted vertices.

Divergences

Three possible line vertex divergences.

(1) All vertices are contracted to a smooth point, which leads to a linear divergence;

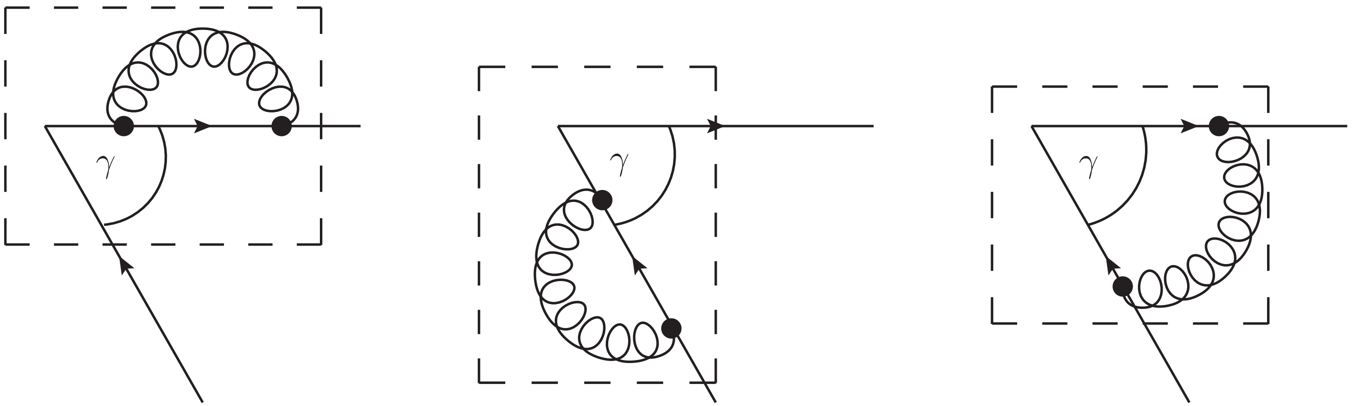


Linear divergences are proportional to the length of the contour and can be removed by a mass term.

(2) The contraction of vertices to a smooth point leaves an external propagator connecting a contracted to an uncontracted vertex: this leads to a logarithmic divergence that can be removed by using renormalized fields and couplings.

(3) All vertices are contracted to a singular point, which gives a logarithmically divergent contribution; these are either cusp or intersection divergences.

Cusps



The renormalization constant for a non-cyclic (time extension smaller than $1/T$) rectangular Wilson loop is determined by four right-angled cusps. In the $\overline{\mathrm{MS}}$-scheme:

$$Z = \exp\left[-2C_F\alpha_s\mu^{-2\varepsilon}/(\pi\bar{\varepsilon})\right]; \qquad 1/\bar{\varepsilon} \equiv 1/\varepsilon - \gamma_E + \ln 4\pi$$

Cusp divergences are absent in a cyclic Wilson loop.

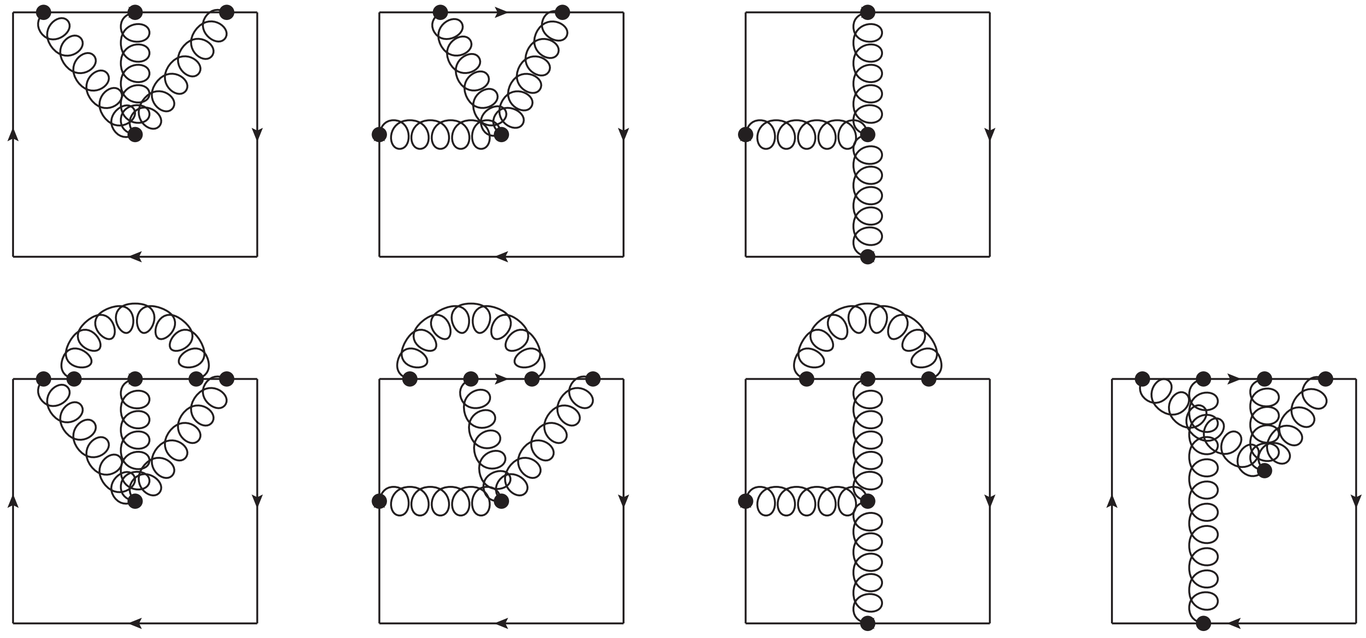
- `Korchemsky Radyushkin NPB 283 (1987) 342`

Intersections

Divergences appear when all vertices are contracted to an intersection point.

- When one vertex is on the string, if every vertex can be contracted to the intersection, then the contribution of the diagram cancels because of cyclicity.
- If all vertices are on a quark line, then the diagram contributes equally to the Polyakov loop, which is finite after charge renormalization.

Hence a connected diagram cannot give rise to an intersection divergence, because either we are in one of the situations above, or it has at least one uncontracted vertex and therefore it is finite.



Polyakov loop

Polyakov loop and free energy

Polyakov loop, P, and free energy of a static quark, F_Q, in a general representation R of SU(N) of dimension d_R:

$$P = \frac{1}{d_R}\,\mathrm{Tr}\left\langle \mathcal{P}\exp\left[\mathrm{ig}\int_0^{1/\mathrm{T}} \mathrm{d}\tau\, \mathrm{A}_0(\tau,\mathrm{x})\right]\right\rangle \equiv \mathrm{e}^{-\mathrm{F_Q}/\mathrm{T}}$$

- For $n_f = 0$ is an order parameter for deconfinement.
- For $n_f > 0$ at high temperatures it is a measure of the screening properties of the deconfined medium.

The Polyakov loop at $\mathcal{O}(g^3)$

$$P = 1 + \frac{C_R \alpha_s m_D}{2T}, \qquad F_Q = -\frac{C_R \alpha_s m_D}{2}$$

where C_R is the quadratic Casimir of the representation R.
The Debye mass m_D is given by

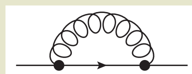
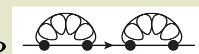
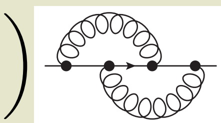
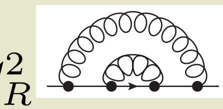
$$\Pi_{00}(|\mathbf{k}| \ll T) \approx m_D^2 = \frac{C_A + T_F n_f}{3} g^2 T^2$$

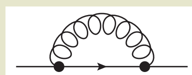
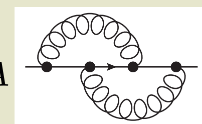
In the weak coupling one assumes the hierarchy of scales

$$T \gg m_D \sim gT \gg m_M \sim g^2 T$$

where m_M is the magnetic mass.

Exponentiation

$P = 1 + C_R$  $+ C_R^2$  $+ C_R\left(C_R - \frac{C_A}{2}\right)$  $+ C_R^2$  $+ \ldots$

$= \exp\Big[C_R$  $- \frac{1}{2} C_R C_A$  $+ \ldots\Big] = \exp\left(D_1 + D_2 + \ldots\right)$

- Dots stand for order g^6 contributions.
- The free energy is proportional to C_R: Casimir scaling.

$$D_1 = -\frac{C_R g^2}{2T}\int_k D_{00}(0,\boldsymbol{k})$$

$$D_2 = -\frac{C_R C_A g^4}{4T}\int_{k,\,q}\left[\frac{1}{12T}D_{00}(0,\boldsymbol{k})D_{00}(0,\boldsymbol{q}) - {\sum_{k_0}}' \frac{1}{k_0^2} D_{00}(k_0,\boldsymbol{k})\big(2D_{00}(0,\boldsymbol{q}) - D_{00}(k_0,\boldsymbol{q})\big)\right]$$

The Polyakov loop at $\mathcal{O}(g^4)$

$$P = 1 + \frac{C_R \alpha_s}{2} \frac{m_D}{T} + \frac{C_R \alpha_s^2}{2} \left[C_A \left(\ln \frac{m_D^2}{T^2} + \frac{1}{2} \right) - n_f \ln 2 \right]$$

- The logarithm, $\ln m_D^2/T^2$, signals that an infrared divergence at the scale T has canceled against an ultraviolet divergence at the scale m_D.

○ Burnier Laine Vepsäläinen JHEP 1001 (2010) 054
Brambilla Ghiglieri Petreczky Vairo PR D82 (2010) 074019

Literature

In 1981, Gava and Jengo obtained at $\mathcal{O}(g^4)$ (for $n_f = 0$):

$$P_{\rm GJ} = 1 + \frac{C_R \alpha_{\rm s}}{2} \frac{m_D}{T} + \frac{C_R C_A \alpha_{\rm s}^2}{2} \left(\ln \frac{m_D^2}{T^2} - 2 \ln 2 + \frac{3}{2} \right)$$

The result is incorrect. The origin of the error can be traced back to not having resummed the Debye mass in the temporal gluons contributing to the static gluon self energy.

- Gava Jengo PL B105 (1981) 285

Our result agrees with the determination of Burnier, Laine and Vepsäläinen, who use a dimensionally reduced EFT framework in a covariant or Coulomb gauge.

- Burnier Laine Vepsäläinen JHEP 1001 (2010) 054

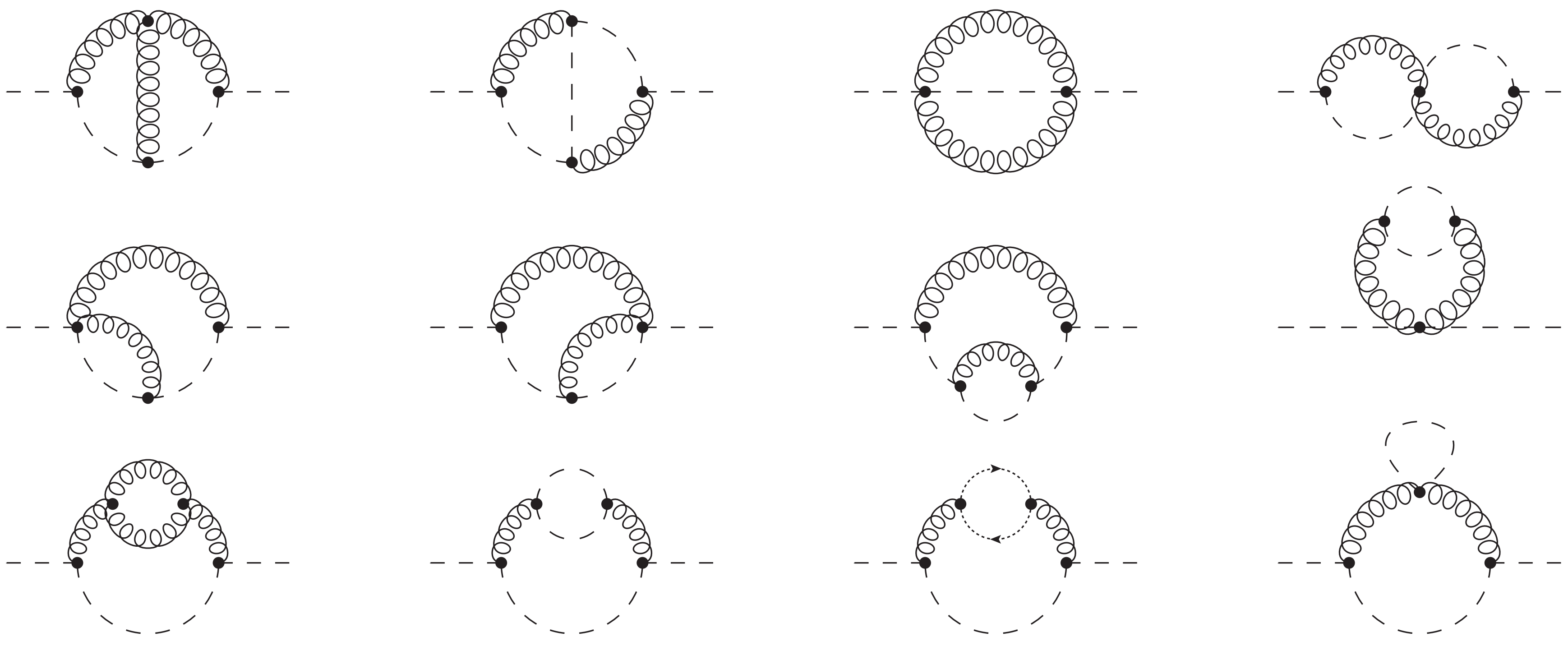
Contributions from the scales T and m_D at $\mathcal{O}(g^5)$

After charge renormalization ($\overline{\text{MS}}$ scheme):

$$(D_1 + D_2)\Big|_{g^5,\,T} = \frac{3C_R\alpha_s^2 m_D}{16\pi T}\left[3C_A + \frac{4}{3}T_F n_f\,(1 - 4\ln 2) + 2\beta_0\left(\gamma_E + \ln\frac{\mu}{4\pi T}\right)\right] - \frac{C_R C_F T_F n_f \alpha_s^3 T}{2m_D}$$

$$D_1\Big|_{g^5,\,m_D} = -\frac{C_R C_A^2 \alpha_s^3 T}{m_D}\left[\frac{89}{48} + \frac{\pi^2}{12} - \frac{11}{12}\ln 2\right]$$

$$\Pi^{(2)}_{m_D}(0, k \sim m_D)$$



Automatic reduction to master integrals

All integrals are of the form

$$B_M(i_1,i_2,i_3,i_4,i_5,i_6) = \int_{k,p,q} \frac{1}{\boldsymbol{p}^{2i_1}(\boldsymbol{p}-\boldsymbol{q})^{2i_2}\boldsymbol{q}^{2i_3}P(\boldsymbol{k}+\boldsymbol{p})^{i_4}P(\boldsymbol{k}+\boldsymbol{q})^{i_5}P(\boldsymbol{k})^{i_6}}$$

$$B_N(i_1,i_2,i_3,i_4,i_5,i_6) = \int_{k,p,q} \frac{1}{\boldsymbol{p}^{2i_1}\boldsymbol{q}^{2i_2}P(\boldsymbol{k}+\boldsymbol{p})^{i_3}P(\boldsymbol{k}+\boldsymbol{q})^{i_4}P(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})^{i_5}P(\boldsymbol{k})^{i_6}}$$

with $P(\boldsymbol{k}) = k^2 + m_D^2$.

Using algebraic and integration by parts identities the integrals reduce to $B_N(0,0,1,1,1,1)$, $B_M(0,0,0,1,1,1)$ and B_M integrals where at least one of $i_4,\ldots,i_6$ is zero or negative. Eventually all the few remaining master integrals are known.

- Broadhurst ZP C54 (1992) 599

  Gray Broadhurst Grafe Schilcher ZP C48 (1990) 673

  Braaten Nieto PR D53 (1996) 3421

The Polyakov loop at $\mathcal{O}(g^5)$

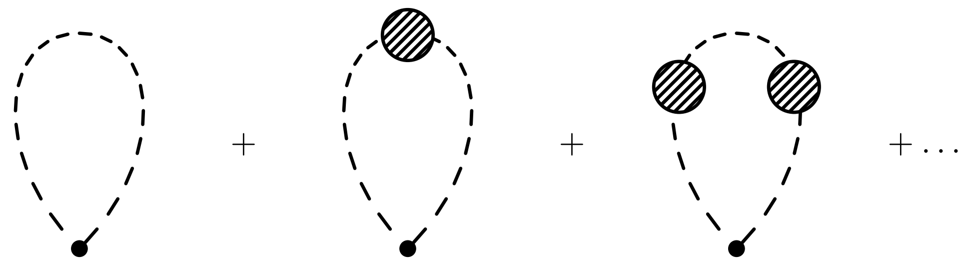
$$
\begin{aligned}
-\frac{F_Q}{T} = \ln P = \frac{C_R \alpha_s(\mu)\, m_D}{2T} &+ \frac{C_R \alpha_s^2}{2}\left[C_A\left(\frac{1}{2} + \ln\frac{m_D^2}{T^2}\right) - 2T_F n_f \ln 2\right] \\
&+ \frac{3C_R \alpha_s^2 m_D}{16\pi T}\left[3C_A + \frac{4}{3}T_F n_f\,(1 - 4\ln 2) + 2\beta_0\left(\gamma_E + \ln\frac{\mu}{4\pi T}\right)\right] \\
&- \frac{C_R C_F T_F n_f \alpha_s^3 T}{2m_D} \\
&- \frac{C_R C_A^2 \alpha_s^3 T}{m_D}\left[\frac{89}{48} + \frac{\pi^2}{12} - \frac{11}{12}\ln 2\right]
\end{aligned}
$$

- Berwein Brambilla Petreczky Vairo PR D93 (2016) 034010

Checks

- Feynman gauge and Coulomb gauge.
- Static gauge: $\partial_0 A_0 = 0$:

$$P = \frac{1}{d_R} \mathrm{Tr} \langle \exp\left(\frac{igA^0(\mathbf{x})}{T}\right)\rangle =$$



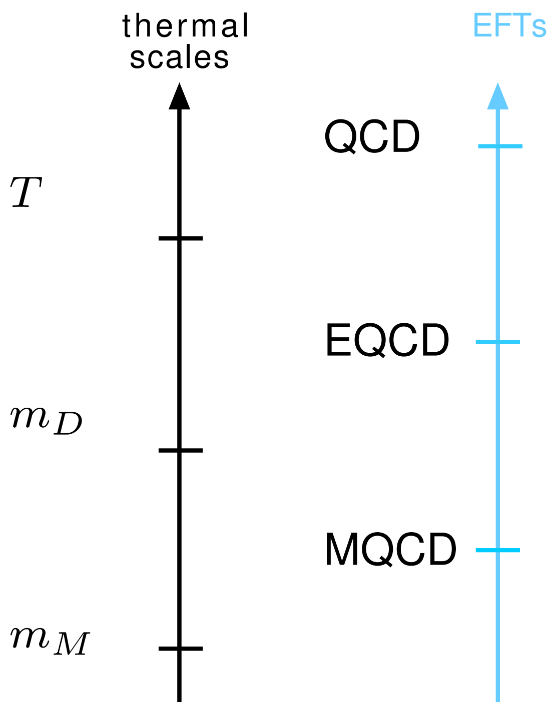
- Phase-space Coulomb gauge:

$$e^{-S} = \exp\left[-\int_0^{1/T} d\tau \int d^3x \left(\frac{1}{4} F^a_{ij} F^a_{ij} + \frac{1}{2} F^a_{0i} F^a_{0i}\right)\right]$$
$$= \mathcal{N}^{-1} \int \mathcal{D}E_i \exp\left[-\int_0^{1/T} d\tau \int d^3x \left(\frac{1}{4} F^a_{ij} F^a_{ij} + iE^a_i F^a_{0i} + \frac{1}{2} E^a_i E^a_i\right)\right]$$

  - Andrasi EPJ C37 (2004) 307

- Dimensionally reduced effective field theories.

Dimensionally reduced EFTs



Polyakov loop

$$P_{\rm EQCD} = Z_0^E - Z_2^E \frac{g^2}{2d_R T} {\rm Tr}\, \langle \tilde{\rm A}_0^2 \rangle + \dots$$

$$P_{\rm MQCD} = Z_0^M + \frac{Z_1^M}{2m_D^3} \langle \tilde{F}_{ij}^a \tilde{F}_{ij}^a \rangle + \dots$$

- The Polyakov loop may be calculated relying mostly on known results.
  - `Braaten Nieto PR D53 (1996) 3421`
    `Kajantie Laine Rummukainen Shaposhnikov NP B503 (1997) 357, ...`
- Non-perturbative contributions carried by m_M are of order g^7 ($Z_1^M \sim \alpha_s^2$).

Magnetic mass contributions

MQCD shows that magnetic mass contributions appear at $\mathcal{O}(g^7)$.

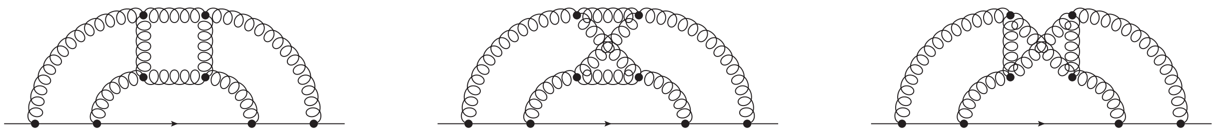
- At order g^5 the following two diagrams cancel when the spatial gluon carries a momentum of order m_M:

- The explicit cancellation of the magnetic mass contributions at order g^6 has also been checked.

Casimir scaling

Casimir scaling holds up to $\mathcal{O}(g^7)$ (including m_M contributions).

Possible Casimir scaling violations may happen at $\mathcal{O}(g^8)$, through diagrams like



+ 4 gluon vertex diagrams + light quark loop diagrams.

These are proportional to

$$C_R^{(4)} = f^{i_1 a_1 i_2} \cdots f^{i_4 a_4 i_1} \frac{1}{d_R} \mathrm{Tr}\left[T_R^{a_1} \cdots T_R^{a_4}\right], \qquad \text{with} \quad \frac{C_F^{(4)}}{C_A^{(4)}} = \frac{C_F}{C_A} \frac{N^2+2}{N^2+12}$$

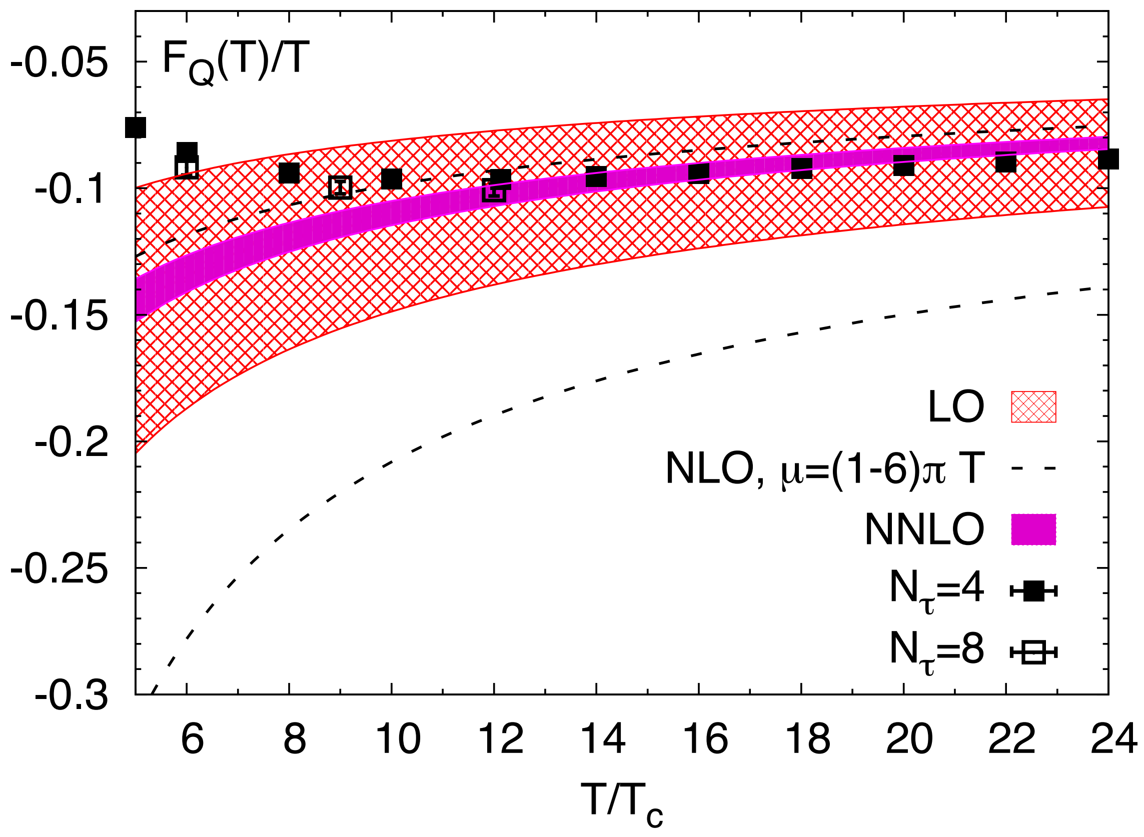
∘ Berwein Brambilla Petreczky Vairo PR D93 (2016) 034010

Outlook

- The computation at $\mathcal{O}(g^6)$ is feasible in perturbation theory. It is the last missing piece of the perturbative expansion of the Polyakov loop. Non-perturbative contributions from the magnetic mass are of $\mathcal{O}(g^7)$. It may conclusively bring the weak-coupling expansion in agreement with the lattice data at high temperatures.
- It may be important to have a definitive weak-coupling estimate of the Casimir scaling violation. This could happen at $\mathcal{O}(g^8)$.

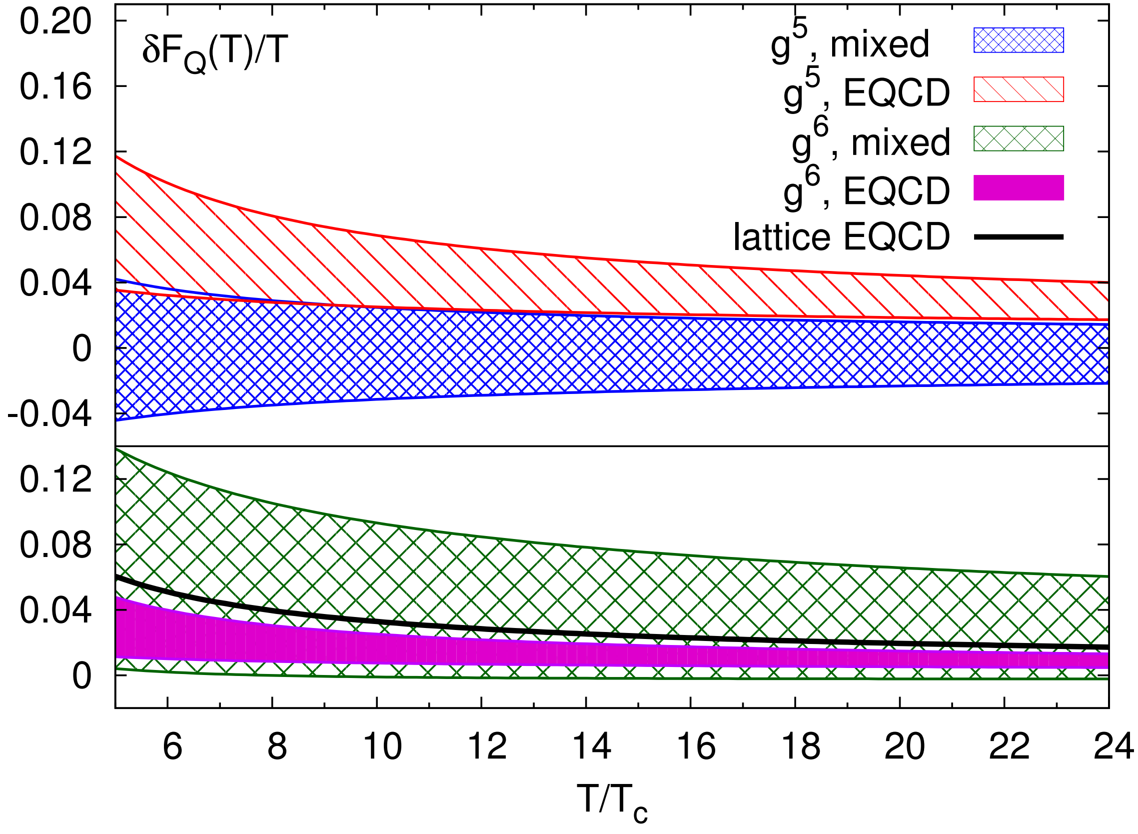
Polyakov loop: lattice

Free energy vs quenched lattice data



- Berwein Brambilla Petreczky Vairo PR D93 (2016) 034010

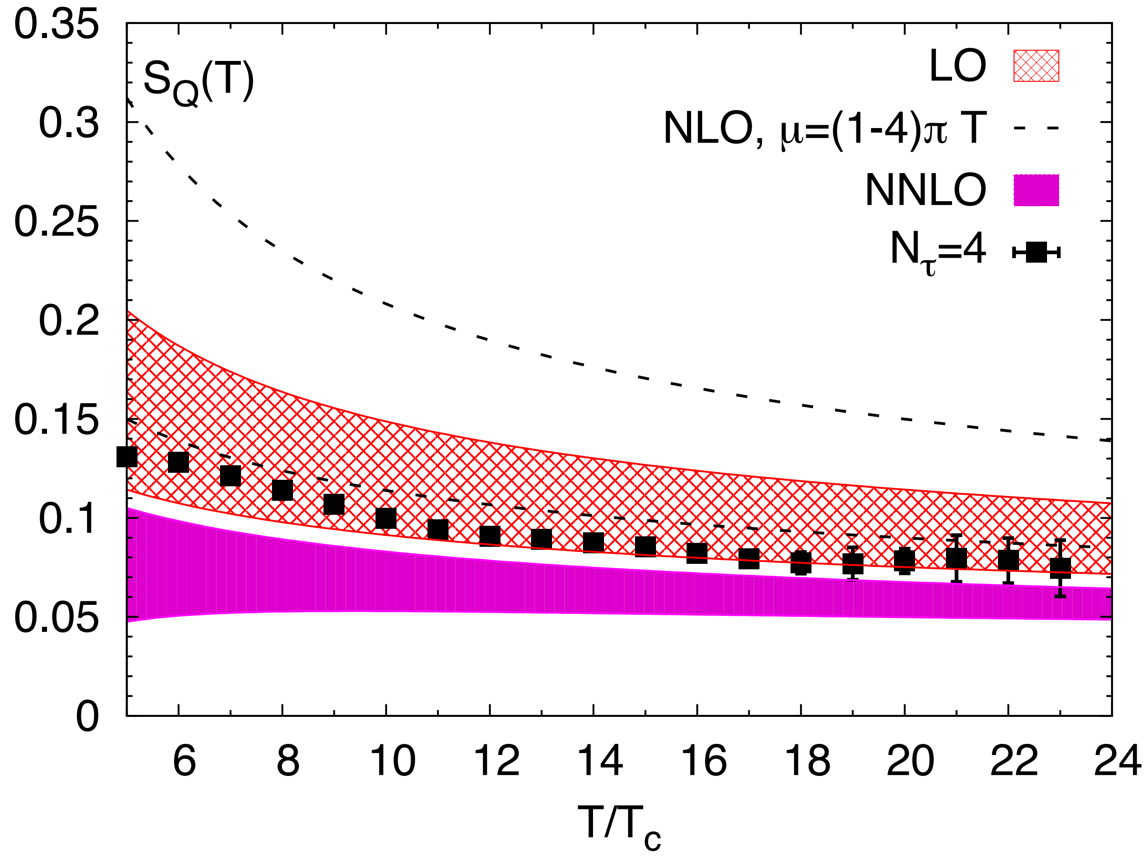
$\delta F_Q/T$ at $\mathcal{O}(g^5)$ and $\mathcal{O}(g^6)$



- Berwein Brambilla Petreczky Vairo PR D93 (2016) 034010

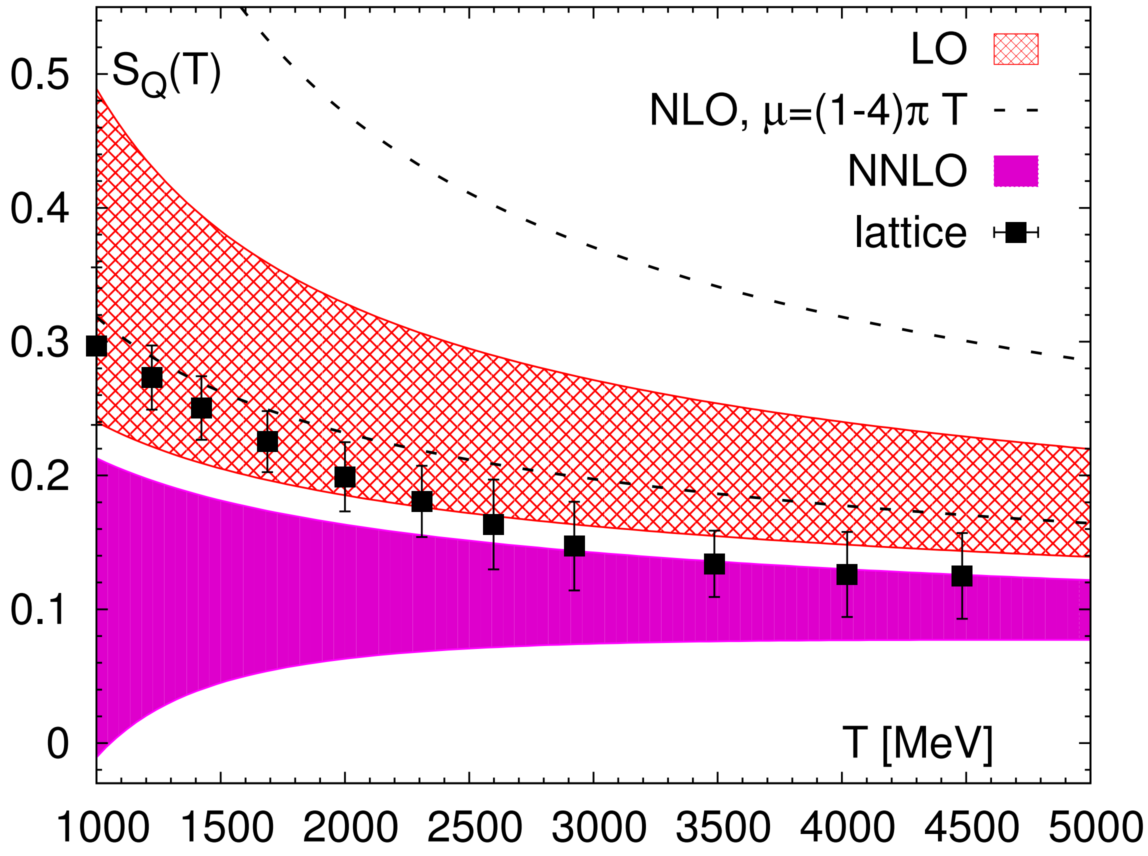
Entropy vs quenched lattice data

The entropy does not depend on the normalization shift: $S_Q = -\dfrac{\partial F_Q(T)}{\partial T}$.



○ Berwein Brambilla Petreczky Vairo PR D93 (2016) 034010

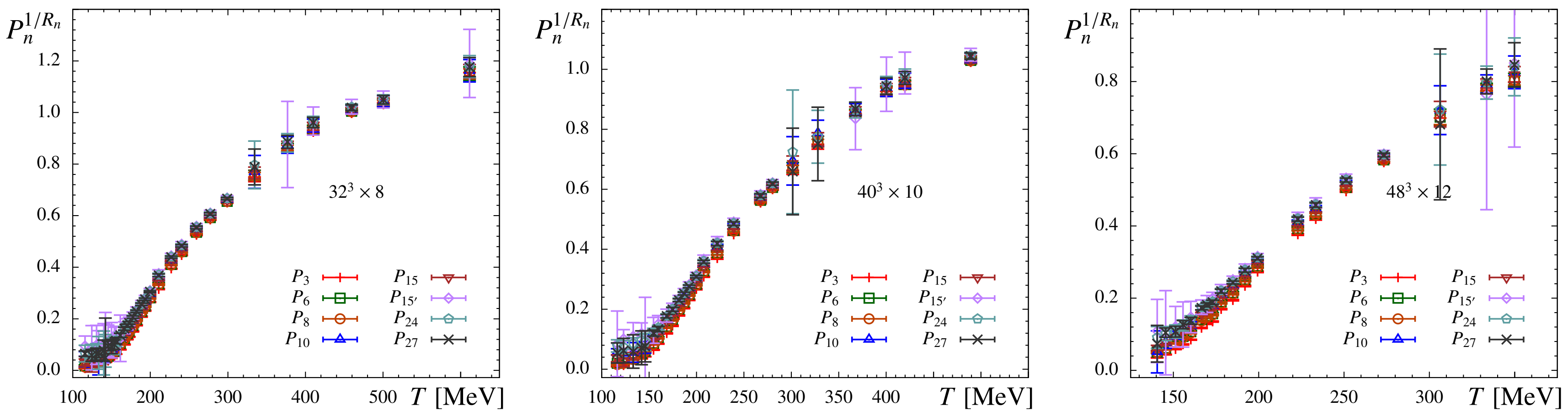
Entropy vs 2+1 flavor lattice data



Position of the entropy peak: $T_S = 153^{6.5}_{-5}$ MeV.

- Bazavov Brambilla Ding Petreczky Schadler Vairo Weber
  PR D93 (2016) 114502

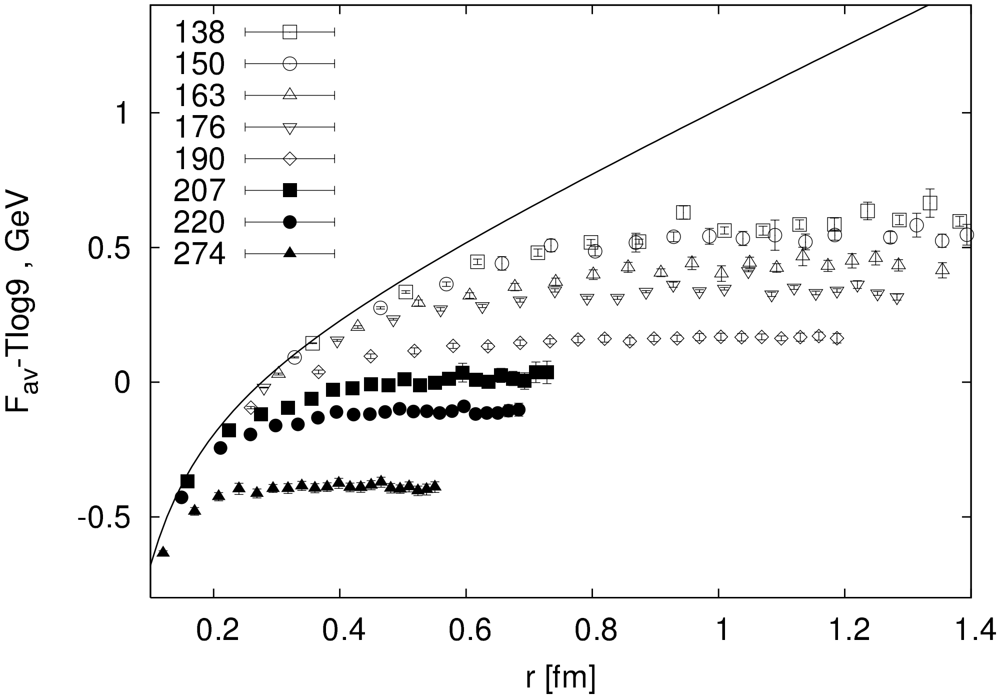
Casimir scaling



- Petreczky Schadler PR D92 (2015) 094517

Polyakov loop correlator

The correlator of two Polyakov loops



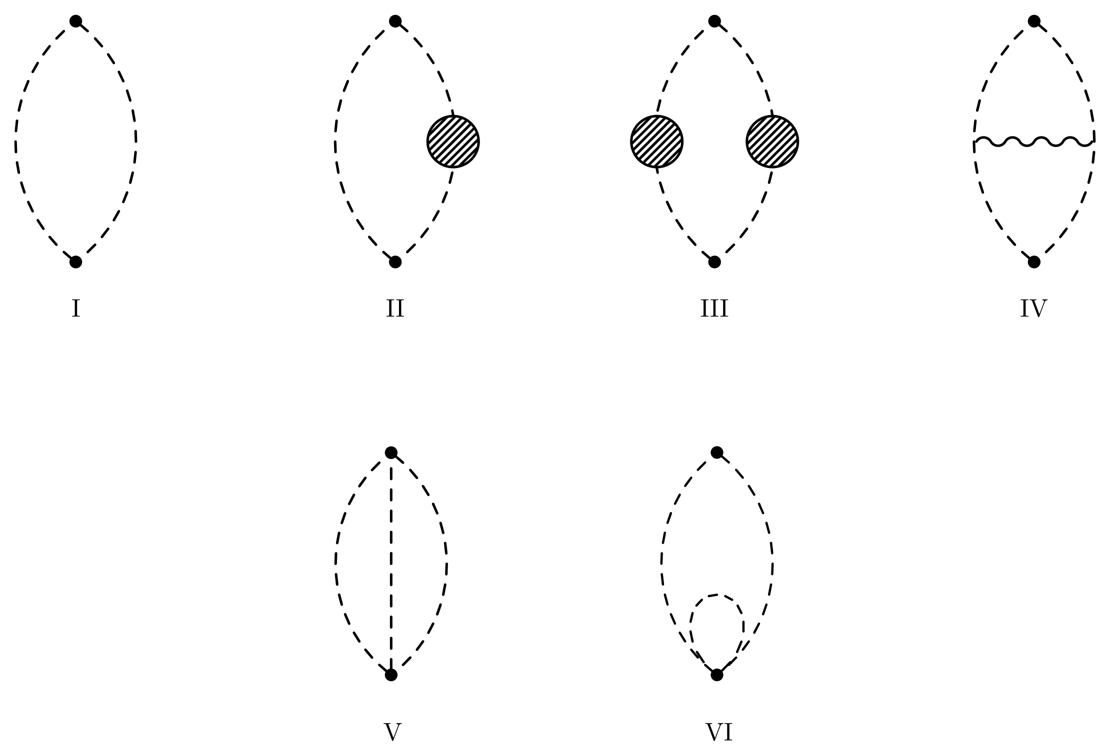
- Petreczky Petrov PR D70 (2004) 054503

The Polyakov loop correlator at NNLO

We assume the following hierarchy of scales:

$$\frac{1}{r} \gg T \gg m_D \gg \frac{g^2}{r}$$

and calculate the Polyakov loop correlator up to order $g^6(rT)^0$ in static gauge:



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and calculate the Polyakov loop correlator up to order $g^6(rT)^0$ in static gauge:

$$\begin{aligned} P_c(r,T) \quad = \quad & P(T)^2|_F + \frac{N^2-1}{8N^2}\left\{\frac{\alpha_s(1/r)^2}{(rT)^2} - 2\frac{\alpha_s^2}{rT}\frac{m_D}{T}\right. \\ & + \frac{\alpha_s^3}{(rT)^3}\frac{N^2-2}{6N} + \frac{1}{2\pi}\frac{\alpha_s^3}{(rT)^2}\left(\frac{31}{9}C_A - \frac{10}{9}n_f + 2\gamma_E\beta_0\right) \\ & + \frac{\alpha_s^3}{rT}\left[C_A\left(-2\ln\frac{m_D^2}{T^2} + 2 - \frac{\pi^2}{4}\right) + 2n_f\ln 2\right] \\ & \left. + \alpha_s^2\frac{m_D^2}{T^2} - \frac{2}{9}\pi\alpha_s^3 C_A\right\} + \mathcal{O}\left(g^6(rT), \frac{g^7}{(rT)^2}\right) \end{aligned}$$

○ Brambilla Ghiglieri Petreczky Vairo PR D78 (2008) 014017

Literature I

In 1986, Nadkarni calculated the Polyakov loop correlator at NNLO assuming the hierarchy:

$$T \gg 1/r \sim m_D$$

Whenever the previous calculations do not involve the hierarchy $rT \ll 1$, they agree with Nadkarni's ones, expanded for $m_D r \ll 1$.

○ Nadkarni PR D33 (1986) 3738

Literature II

EFT approaches for the calculation of the correlator of Polyakov loops for the situation $m_D \gtrsim 1/r$ and $T \gg 1/r$ were developed in the past. In that situation, the scale $1/r$ was not integrated out, and the Polyakov-loop correlator was described in terms of dimensionally reduced effective field theories of QCD, while the complexity of the bound-state dynamics remained implicit in the correlator.

Those descriptions are valid for largely separated Polyakov loops when the correlator is either screened by the Debye mass, for $m_D r \sim 1$, or the mass of the lowest-lying glueball, for $m_D r \gg 1$.

- `Braaten Nieto PRL 74 (1995) 3530`
  `Nadkarni PR D33 (1986) 3738`

Literature III

In an EFT/pNRQCD framework $P_c(r,T)$ can be put in the form

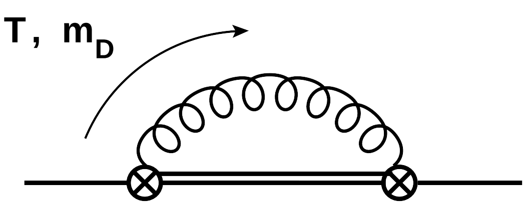
$$P_c(r,T) = \frac{1}{N^2}\left[e^{-f_s(r,T,m_D)/T} + (N^2-1)e^{-f_o(r,T,m_D)/T} + \mathcal{O}\left(\alpha_s^3 (rT)^4\right)\right]$$

$f_s = Q\bar{Q}$-color singlet free energy, $f_o = Q\bar{Q}$-color octet free energy
to be matched from the singlet and octet pNRQCD propagators

$$\frac{\langle S(\mathbf{r},\mathbf{0},1/T)S^\dagger(\mathbf{r},\mathbf{0},0)\rangle}{\mathcal{N}} = e^{-V_s(r)/T}(1+\delta_s) \equiv e^{-f_s(r,T,m_D)/T}$$

$$\frac{\langle O^a(\mathbf{r},\mathbf{0},1/T)O^{a\,\dagger}(\mathbf{r},\mathbf{0},0)\rangle}{\mathcal{N}} = e^{-V_o(r)/T}\left[(N^2-1)\langle P_A\rangle + \delta_o\right] \equiv (N^2-1)e^{-f_o(r,T,m_D)/T}$$

where δ_s and δ_o stand for thermal loop corrections to the singlet/octet propagators:

$\delta_s =$ 

∘ Brambilla Ghiglieri Petreczky Vairo PR D78 (2008) 014017

f_s and f_o are both finite and gauge invariant. They also do not depend on some special choice of Wilson lines connecting the initial and final quark and antiquark states.

The calculation provides an independent determination of $V_o - V_s$ at two loops.
○ Kniehl Penin Schröder Smirnov Steinhauser PL B607 (2005) 96

The color-singlet quark-antiquark potential has been calculated in real-time formalism in the same thermodynamical situation considered here.

- The real part of the real-time potential differs from $f_s(r, T, m_D)$ by

$$\frac{1}{9}\pi N C_F \alpha_s^2 r T^2 - \frac{\pi}{36} N^2 C_F \alpha_s^3 T$$

  The difference matters when using free-energy lattice data for the quarkonium in media phenomenology.

- The real-time potential has also an imaginary part that is absent in the free energy.

Literature IV

Jahn and Philipsen have analyzed the gauge structure of the allowed intermediate states in the correlator of Polyakov loops: the quark-antiquark component, φ, of an intermediate state made of a quark located in $\mathbf{x}_1$ and an antiquark located in $\mathbf{x}_2$ should transform as

$$\varphi(\mathbf{x}_1, \mathbf{x}_2) \to g(\mathbf{x}_1)\varphi(\mathbf{x}_1, \mathbf{x}_2)g^\dagger(\mathbf{x}_2)$$

under a gauge transformation g.

- The decomposition of the Polyakov loop correlator in terms of a color singlet and a color octet correlator is in accordance with that result, for both a $Q\bar{Q}$ singlet and octet field transform in that way.
- We remark, however, a difference in language: singlet and octet in f_s and f_o refer to the gauge transformation properties of the quark-antiquark fields, while, in Jahn and Philipsen, they refer to the gauge transformation properties of the physical states. In that last sense, of course, octet states cannot exist as intermediate states in the correlator of Polyakov loops.

○ Jahn Philipsen PR D70 (2004) 074504

Literature V

Burnier, Laine and Vepsäläinen have performed a weak-coupling calculation of the untraced Polyakov-loop correlator in Coulomb gauge and of the cyclic Wilson loop up to order g^4.

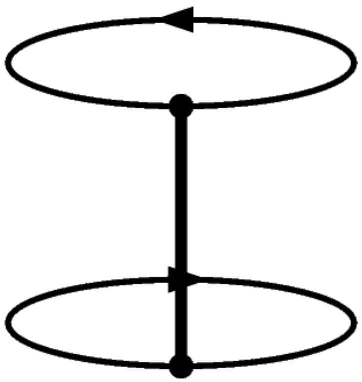
Both these objects may be seen as contributing to the correlator of two Polyakov loops. The first quantity is gauge dependent. We will discuss the relation of the cyclic Wilson loop with the Polyakov-loop correlator.

- Burnier Laine Vepsäläinen JHEP 1001 (2010) 054

Cyclic Wilson loop

Divergences of the cyclic Wilson loop

Differently from $P(T)$ and $P_c(r,T)$, $W_c(r,T)$ is divergent after charge and field renormalization. This divergence is due to intersection points.



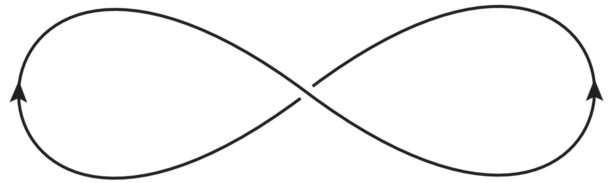
Although it may seem that the cyclic Wilson loop has a continuously infinite number of intersection points, one needs to care only about the two endpoints, for the Wilson loop contour does not lead to divergences in the other ones.

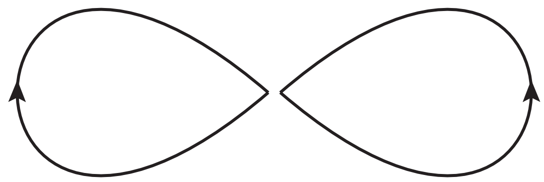
How to renormalize intersection divergences

For intersection points connected by 2 Wilson lines (angles θ_k) and cusps (angles φ_l):

$$W^{(R)}_{i_1 i_2 \ldots i_r} = Z_{i_1 j_1}(\theta_1) Z_{i_2 j_2}(\theta_2) \cdots Z_{i_r j_r}(\theta_r) Z(\varphi_1) Z(\varphi_2) \cdots Z(\varphi_s) W_{j_1 j_2 \ldots j_r}$$

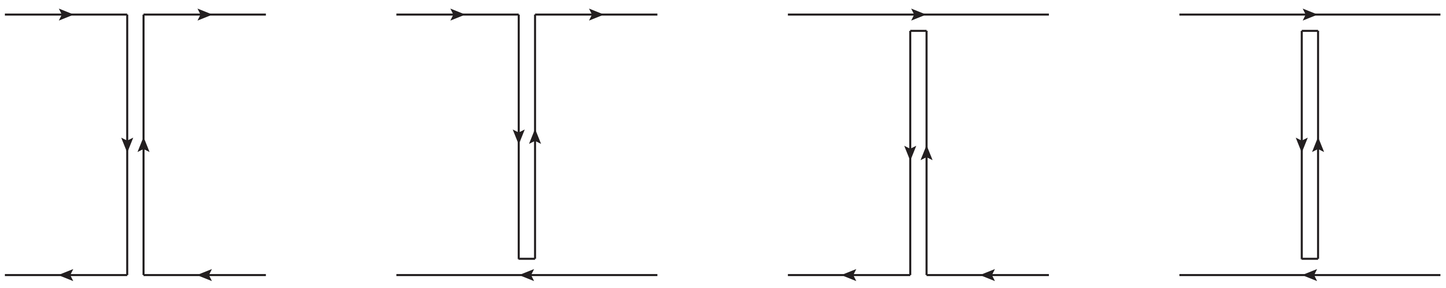
- The indices i_k and j_k label the different possible path-ordering prescriptions.
- The loop functions are color-traced and normalized by the number of colours.
- This ensures that all loop functions are gauge invariant.
- The coupling in $W^{(R)}_{i_1 i_2 \ldots i_r}$ is the renormalized coupling.
- The matrices Z are the renormalization matrices.





○ Brandt Neri Sato PR D24 (1981) 879

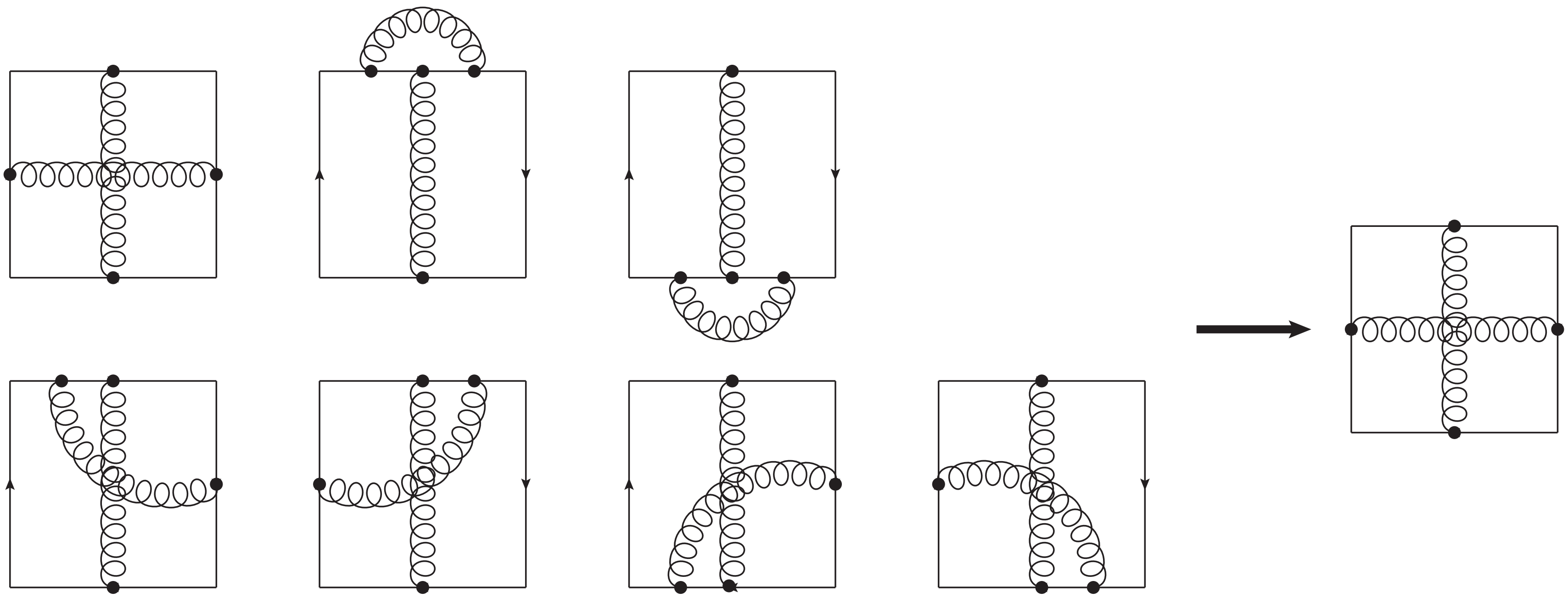
How to renormalize the cyclic Wilson loop



$$\begin{pmatrix} W_c^{(R)} \\ P_c \end{pmatrix} = \begin{pmatrix} Z & 1-Z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} W_c \\ P_c \end{pmatrix}$$

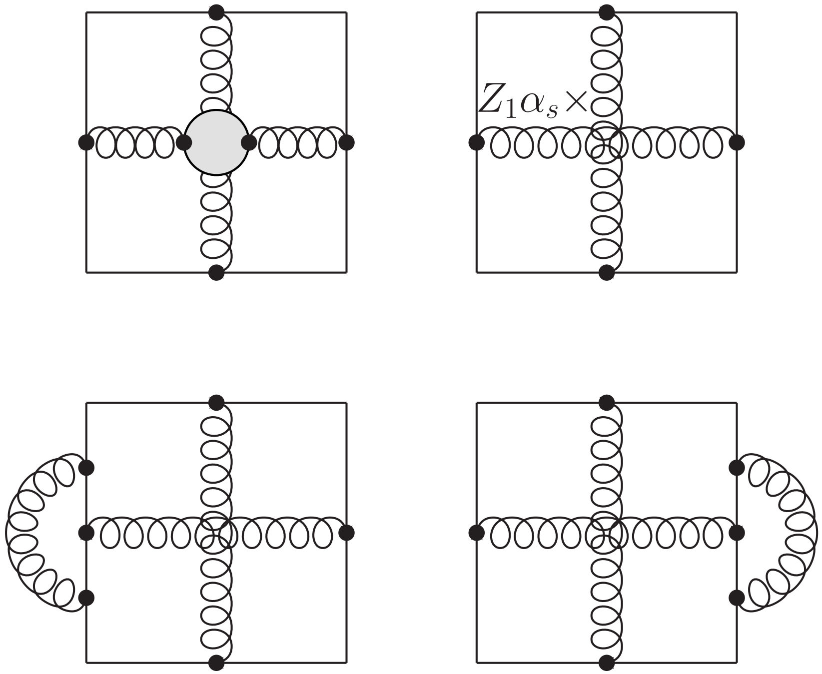
$$Z = 1 + Z_1 \alpha_{\rm s} \mu^{-2\varepsilon} + Z_2 \left(\alpha_{\rm s}\mu^{-2\varepsilon}\right)^2 + \mathcal{O}(\alpha_{\rm s}^3)$$

Z_1



$$Z_1 = -\frac{C_A}{\pi}\frac{1}{\bar{\varepsilon}}$$

Z_2



Z_2 reabsorbs all divergences of the type $\alpha_\mathrm{s}^3/(rT)$.

All other divergences at $\mathcal{O}(\alpha_\mathrm{s}^3)$ are reabsorbed by Z_1 (combined with $P_c(r,T)$ at $\mathcal{O}(\alpha_\mathrm{s}^2)$)!

Renormalization group equation at one loop

$$\begin{cases} \mu \dfrac{\mathrm{d}}{\mathrm{d}\mu} \left(W_c^{(R)} - P_c \right) = \gamma \left(W_c^{(R)} - P_c \right) \\ \mu \dfrac{\mathrm{d}}{\mathrm{d}\mu} \alpha_\mathrm{s} = -\dfrac{\alpha_\mathrm{s}^2}{2\pi} \beta_0 \end{cases}$$

γ is the anomalous dimension of $W_c^{(R)} - P_c$:

$$\gamma \equiv \frac{1}{Z} \mu \frac{\mathrm{d}}{\mathrm{d}\mu} Z = 2C_A \frac{\alpha_\mathrm{s}}{\pi} + \mathcal{O}(\alpha_\mathrm{s}^2)$$

$$\left(W_c^{(R)} - P_c \right)(\mu) = \left(W_c^{(R)} - P_c \right)(1/r) \left(\frac{\alpha_\mathrm{s}(\mu)}{\alpha_\mathrm{s}(1/r)} \right)^{-4C_A/\beta_0}$$

W_c: final result

In $\overline{\mathrm{MS}}$ at NLO and LL accuracy (i.e. including all terms $\alpha_s/(rT) \times (\alpha_s \ln \mu r)^n$), assuming the hierarchy of scales $\frac{1}{r} \gg T \gg m_D \gg \frac{g^2}{r}$, we obtain

$$
\begin{aligned}
\ln W_c^{(R)} \quad = \quad & \frac{C_F \alpha_s(1/r)}{rT} \Bigg\{ 1 + \frac{\alpha_s}{4\pi} \left[\left(\frac{31}{9} C_A - \frac{10}{9} n_f \right) + 2\,\beta_0 \gamma_E \right] \\
& \qquad + \frac{\alpha_s C_A}{\pi} \left[1 + 2\gamma_E - 2\ln 2 + \sum_{n=1}^{\infty} \frac{2(-1)^n \zeta(2n)}{n(4n^2-1)} (rT)^{2n} \right] \Bigg\} \\
& + \frac{4\pi \alpha_s C_F}{T} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left(e^{i\mathbf{r}\cdot\mathbf{k}} - 1 \right) \left[\frac{1}{\mathbf{k}^2 + \Pi_{00}^{(T)}(0,\mathbf{k})} - \frac{1}{\mathbf{k}^2} \right] + C_F C_A \alpha_s^2 \\
& + \frac{C_F \alpha_s}{rT} \left[\left(\frac{\alpha_s(\mu)}{\alpha_s(1/r)} \right)^{-4C_A/\beta_0} - 1 \right] + \mathcal{O}\left(g^5\right)
\end{aligned}
$$

$\Pi_{00}^{(T)}(0,\mathbf{k})$ = (known) thermal part of the gluon self-energy in Coulomb gauge.

Long distance

We have computed W_c for $1/r \gg T \gg m_D \gg g^2/r$, but the renormalization of W_c is general and not bound to this hierarchy.
The renormalization equation must hold also at large distances, $rm_D \sim 1$. There

$$W_c = 1 + \frac{4\pi C_F \alpha_s(\mu)}{T} \frac{e^{-m_D r}}{4\pi r} + \frac{4 C_F C_A \alpha_s^2}{T} \frac{e^{-m_D r}}{4\pi r} \frac{1}{\varepsilon} + \ldots$$

The term $\exp(-m_D r)/(4\pi r)$ comes from the screened temporal gluon propagator, $D_{00}(0, \mathbf{k}) = 1/(\mathbf{k}^2 + m_D^2)$, and the dots stand for finite terms or for h.o. terms.

This expression is renormalized by the same renormalization equation with the same renormalization constant Z as computed at short distances.

Linear divergences

In general, loop functions have power divergences, which factorize and exponentiate to give a factor $\exp\left[\Lambda L(C)\right]$, where $L(C)$ is the length of the contour and Λ is some linearly divergent constant. In dimensional regularization such linear divergences are absent, but they would be present in other schemes such as lattice regularization.

∘ Polyakov NPB 164 (1980) 171

An efficient way to calculate the exponent of Wilson loops is the so-called replica trick:

$$\langle W_1 \cdot W_2 \cdots W_N \rangle = 1 + N \ln\langle W \rangle + \mathcal{O}(N^2)$$

$W_i =$ ith copy of W in a replicated theory of QCD not interacting with the others.

∘ Gardi Laenen Stavenga White JHEP 1011 (2010) 155
Gardi Smillie White JHEP 1306 (2013) 088

$$\exp\left[-2\Lambda_F/T - \Lambda_A r\right] \times Z \times \big(W_c(\mathbf{r}) - P_c(\mathbf{r})\big) \text{ is finite}$$

Z is in the same renormalization scheme as the linear divergences.

Outlook

The renormalization of W_c allows the proper calculation of this quantity on the lattice.

The right quantity to compute is the multiplicatively renormalizable combination

$$W_c - P_c$$

A finite quantity is

$$\frac{(W_c - P_c)(r)}{(W_c - P_c)(r_0)} \times \frac{(W_c - P_c)(2r_0 - r)}{(W_c - P_c)(r_0)}$$

where r_0 is a given fixed distance.