Fun with Renormalization and Regularization

Dominik Stöckinger

TU Dresden

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Outline

- DREG, FDH and DRED: Consistency, UV and IR properties
  [Gnendiger, Signer, DS; Broggio, Gnendiger, Signer, DS, Visconti; DS, Unger (preliminary)]

- Renormalization of VEVs [Sperling, DS, Voigt]
Outline

1. Regularization: DREG, FDH, DRED: UV, IR
   - Precise definitions of all schemes possible
   - Puzzles can be resolved
   - Simple point of view for DRED — understand IR structure
   - Is DRED supersymmetric??

2. Renormalization of VEVs

3. Conclusions
Motivation

Regularization necessary to define QFT at the quantum level

\[ \int_{|p|<\Lambda} d^4p \]

\[ \mu^{4-D} \int d^D p \]
Motivation

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\[ \int_{|p|<\Lambda} d^4p \]

Choice of regularization is unphysical

DREG

\[ \mu^{4-D} \int d^D p \]
Motivation

Regularization necessary to define QFT at the quantum level

\[ \int_{|p| < \Lambda} d^4p \]

- Choice of regularization is unphysical
- Unitarity/causality determine physics
- e.g. anomaly/breaking of scale invariance
Motivation

Regularization necessary to define QFT at the quantum level

cutoff-scale $\Lambda$

$$\int_{|p|<\Lambda} d^4p$$

DREG

$$\mu^{4-D} \int d^Dp$$

- Choice of regularization is unphysical
- Unitarity/causality determine physics
- e.g. anomaly/breaking of scale invariance
- regularization must be consistent with unitarity/causality
  (proven for DREG: [Speer '74][Breitenlohner, Maison '77], equivalence of DRED, no inconsistency: [Jack, Jones, Roberts '94][DS '05])
Motivation

Regularization necessary to define QFT at the quantum level

\[ \int_{|p|<\Lambda} d^4 p \]

\[ \mu^{4-D} \int d^D p \]

History: puzzles, problems

- DREG breaks SUSY
- “DRED is mathematically inconsistent” [Siegel '80]
- “DRED has IR factorization problem” [van Neerven, Smith, et al '88 and '05][Zerwas et al]
- “No DRED IR factorization problem found” [Kunszt, Signer, Trocsanyi '94; Catani et al ’97]
- “DRED violates unitarity” ['t Hooft, van Damme '84]
- “Some published results therefore wrong” [Harlander, Kant, Mihaila, Steinhauser '06; Kilgore '11]
Motivation

Regularization necessary to define QFT at the quantum level

\[ \int_{|p| < \Lambda} d^4p \]

Aims:

- resolve inconsistencies, study symmetry properties of schemes
- \textbf{today}: consistent definitions of DREG, DRED, FDH; IR relations; SUSY breaking of DRED(?)

DREG

\[ \mu^{4-D} \int d^Dp \]
Definitions/explicit construction

4S: ordinary 4-dimensional Minkowski/momentum space, metric $\tilde{g}^{\mu\nu}$

QDS: “$D$-dimensional space” [Wilson’73],[Collins] :=
   truly $\infty$-dimensional space with some $D$-dim characteristics:
   - $D$-dimensional Integral = linear mapping
     $$\int d^D k e^{-k^2} = \pi^{D/2}$$
   - $g_{(D)}^{\mu\nu}$: bilinear form
   - $\gamma$-matrices similar

Q2$\epsilon$S: “$2\epsilon$-dimensional space” analogous
Q4S: “quasi-4-dimensional space” Q4S := QDS $\oplus$ Q2$\epsilon$S

hierarchy of spaces 4S $\subset$ QDS $\subset$ Q4S [DS ’05]

although $\mu = 0, 1, 2, \ldots \infty$, useful relations hold, e.g.

$$g^{\mu\nu} = g_{(D)}^{\mu\nu} + g_{(2\epsilon)}^{\mu\nu}, \quad g^{\mu\nu}_\mu = 4, \quad g^{\mu\nu} k_{(D)\nu} = k_{(D)}^{\mu}$$
How does Q4S avoid Siegel’s inconsistency?

Siegel: “With

\[ \varepsilon_{\mu_1\mu_2\mu_3\mu_4}^{(4)} \varepsilon_{\nu_1\nu_2\nu_3\nu_4}^{(4)} \propto \det((g^{(4)}_{\mu_i\nu_j})) \]

calculate

\[ \varepsilon^{(D)}_{\mu\nu\rho\sigma} \varepsilon^{(\varepsilon)}_{\alpha\beta\gamma\delta} \varepsilon^{(D)}_{\mu\nu\rho\sigma} \varepsilon^{(\varepsilon)}_{\alpha\beta\gamma\delta} \]

in two different ways

\[ \Rightarrow 0 = D(D - 1)^2(D - 2)^2(D - 3)^2(D - 4) \]

different calculational steps lead to different results,

mathematical inconsistency!!!”

[Siegel'80]

Don’t allow explicit index counting (step one) any more, because

\[ g^{(4)}_{\mu\nu} \in \text{quasi-4-dim space!} \]
Define four commonly used schemes [Signer, DS '08]

\[ Q4S = QDS \oplus Q2\epsilon S, \]

\[ g^{\mu\nu} = \hat{g}^{\mu\nu} + \tilde{g}^{\mu\nu} \]

- difference: internal and/or external gauge fields
Define four commonly used schemes \[ \text{[Signer, DS '08]} \]

$$Q4S=QDS \oplus Q2\epsilon S,$$

$$g^{\mu\nu} = \hat{g}^{\mu\nu} + \tilde{g}^{\mu\nu}$$

- **difference**: internal and/or external gauge fields

- **Note**: van Neerven, Smith, Zerwas et al used DRED, Kunszt, Signer, Trocsanyi, Catani et al used DR=FDH ⇒

  resolves half of factorization problem
Define four commonly used schemes \[\text{[Signer, DS '08]}\]

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- \textbf{Note:} van Neerven, Smith, Zerwas et al used DRED, Kunszt, Signer, Trocsanyi, Catani et al used DR=FDH \[\Rightarrow\]
  resolves half of factorization problem
- focus first on UV: CDR=HV \[\equiv\] DREG, FDH=DRED
Dim. Regularization (DREG)

\[ D \text{ dimensions} \]
\[ D \text{ Gluon/photon-components} \]
\[ 4 \text{ Gluino/photino-components} \]
\[ \hat{\gamma}^\mu \hat{\gamma}^\rho p_\rho \hat{\gamma}_\mu = (2 - D) \hat{\gamma}^\rho p_\rho \]

**Dim. Regularization (DREG)**

- **D dimensions**
- **D** Gluon/photon-components
- **4** Gluino/photino-components
\[ \hat{\gamma}^\mu \hat{\gamma}^\rho p_\rho \hat{\gamma}_\mu = (2 - D) \hat{\gamma}^\rho p_\rho \]

Dim. Regularization (DREG)

- *D* dimensions
- *D* Gluon/photon-components
- 4 Gluino/photino-components

SUSY broken, need SUSY-restoring counterterms
$\hat{g}$ $\hat{g}$ 
CDR

$\hat{g}$ $\bar{g}$
HV

$\hat{g} + \bar{g}$ $\bar{g}$
FDH ($\Longleftarrow$ DR)

$\hat{g} + \bar{g}$ $\hat{g} + \bar{g}$
DRED

4-dim. scheme (try 4S $\subset$ QDS)

$D$ dimensions
4 Gluon/photon-components
4 Gluino/photino-components
\[ \bar{\gamma}^\mu \hat{\gamma}^\rho p_\rho \bar{\gamma}_\mu = 2\bar{\gamma}^\rho p_\rho - 4\hat{\gamma}^\rho p_\rho \]

4-dim. scheme (try \(4S \subset QDS\))

- **D** dimensions
- 4 Gluon/photon-components
- 4 Gluino/photino-components
\[ \gamma^\mu \gamma^\rho p_\rho \gamma_\mu = 2\gamma^\rho p_\rho - 4\gamma^\rho p_\rho \]

complicated; no $D$-dim. gauge covariant derivative possible

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4 Gluino/photino-components
\[ \bar{\gamma}^\mu \hat{\gamma}^\rho p_\rho \bar{\gamma}_\mu = 2\bar{\gamma}^\rho p_\rho - 4\hat{\gamma}^\rho p_\rho \]

complicated; no \( D \)-dim. gauge covariant derivative possible
sometimes used in literature, e.g. for chiral anomaly [Anselmi '14]
FDH/DRED (use Q4S)

- $D$ dimensions
- 4 Gluon/photon-components
- 4 Gluino/photino-components
\[ \gamma^\mu \gamma^\rho p_\rho \gamma_\mu = (2 - 4) \gamma^\rho p_\rho \]

**FDH/DRED** (use Q4S)

- **D dimensions**
- 4 Gluon/photon-components
- 4 Gluino/photino-components
\[ \gamma^\mu \gamma^\rho p_\rho \gamma_\mu = (2 - 4) \gamma^\rho p_\rho \]

simple; $D$-dim gauge invariance holds
\[ \gamma^\mu \gamma^\rho p_\rho \gamma_\mu = (2 - 4) \gamma^\rho p_\rho \]

simple; \textit{D}-dim gauge invariance holds

Lorentz invariance only in \textit{D}-dim; not in full Q4S

\textbf{FDH/DRED} (use Q4S)

\textit{D} dimensions

4 Gluon/photon-components

4 Gluino/photino-components
Summary regarding UV regularization:

- **CDR** and **HV** only need QDS, simpler but break SUSY
- **FDH** and **DRED** require Q4S to preserve gauge invariance

Common formulation of **FDH** (Bern, Dixon, Freitas; Kilgore, . . .):

- “4 < D < N_s, internal gluons are N_s-dim.; at the end N_s = 4”
Important complication: Renormalization in FDH/DRED

- $\epsilon$-scalars not “protected” by gauge invariance

\[
D^\mu = \hat{\partial}^\mu + igA^\mu = \hat{\partial}^\mu + ig\hat{A}^\mu + ig\tilde{A}^\mu
\]

4-component Gluon in DRED $g$ $\equiv$ D-component gauge field $\hat{g}$ $+$ $\epsilon$-scalars $\tilde{g}$

- Different couplings, especially $\delta\alpha_s \neq \delta\alpha_e, \beta^s \neq \beta^e, \ldots$

- Distinction required, otherwise divergent/non-unitary results

[Jack, Jones, Roberts '94][Harlander, Kant, Mihaila, Steinhauser '06][Kilgore '11]
FDH or DRED

HV or CDR of theory with new, $\epsilon$-scalar of multiplicity $N_\epsilon = 2\epsilon$ with independent couplings

This point of view also allows to understand IR structure in these schemes
soft/soft-collinear:

$$\propto \mathbf{T}_i \cdot \mathbf{T}_j \left( \frac{\gamma_{cusp}^i}{4\epsilon^2} + \frac{\gamma_{cusp}^j}{2\epsilon} \frac{1}{2} \ln \frac{\mu^2}{-s_{ij}} \right)$$
soft/soft-collinear:

$$\propto \mathbf{T}_i \cdot \mathbf{T}_j \left( \frac{\gamma_{\text{cusp}}}{4\epsilon^2} + \frac{\gamma_{\text{cusp}}}{2\epsilon} \frac{1}{2} \ln \frac{\mu^2}{-s_{ij}} \right)$$

collinear:

$$\propto \frac{\gamma^i}{2\epsilon}$$
soft/soft-collinear:

\[ \propto T_i \cdot T_j \left( \frac{\gamma_{\text{cusp}}}{4\epsilon^2} + \frac{\gamma_{\text{cusp}}}{2\epsilon} \frac{1}{2} \ln \frac{\mu^2}{-s_{ij}} \right) \]

collinear:

\[ \propto \frac{\gamma_i}{2\epsilon} \]

**CDR: Three constants describe all IR divergences**

\[ \gamma_{\text{cusp}}^{\text{CDR}} = \frac{\alpha_s}{4\pi} (4) \]

\[ \gamma_q^{\text{CDR}} = \frac{\alpha_s}{4\pi} (-3C_F) \]

\[ \gamma_g^{\text{CDR}} = \frac{\alpha_s}{4\pi} \left[ -\frac{11}{3}C_A + \frac{2}{3}N_F \right] \]
Side remark about real corrections

\[ \propto \frac{P_{i \to jk}}{p_j \cdot p_k} \]

Unitarity:

\[ \int P_{i \to \text{anything}} = \gamma(i) \]

real \leftrightarrow \text{virtual coll. sing.}
1-loop CDR | 1-loop FDH, DRED
2-loop CDR | 2-loop FDH, DRED

\[ \gamma_g^{CDR} = \frac{N_c}{\pi} \left[ -\frac{11}{3} C_A + \frac{2}{3} N_F \right] \]

\[ \gamma_g^{FDH} = \ldots \]
\[ \gamma_g^{DRED} = \ldots \]

\[ \gamma_g^{DRED} = \frac{N_c}{\pi} (-4 C_A) + \frac{N_c}{\pi} (2 N_F T_R) \]
$\gamma_{g}^{\text{CDR}} = \frac{2}{\pi} \left[ -\frac{11}{3} C_A + \frac{2}{3} N_F \right]$

$\gamma_{g}^{\text{FDH}} = \ldots$

$\gamma_{g}^{\text{DRED}} = \ldots$

$\gamma_{\tilde{g}}^{\text{DRED}} = \frac{\alpha_s}{4\pi} (-4 C_A) + \frac{\alpha_s}{4\pi} (2N_F T_R)$

**FDH: Three constants** $\gamma_{\text{cusp}}, \gamma_{q}, \gamma_{g}$ describe all IR divergences

additional virtual state $\tilde{g}$: values of $\gamma$’s change

[Signer, DS ’08]

[Kunszt, Signer, Trocsanyi ’94]

[Catani, Seymour, Trocsanyi ’97]
| 1-loop CDR | 1-loop FDH, DRED |
| 2-loop CDR | 2-loop FDH, DRED |

[Signer, DS '08]

\[
\gamma_{CDR}^g = \frac{\alpha_s}{\pi} \left[ -\frac{11}{3} C_A + \frac{2}{3} N_F \right]
\]

\[
\gamma_{FDH}^g = \ldots
\]

\[
\gamma_{DRED}^g = \ldots
\]

\[
\gamma_{DRED}^{\tilde{g}} = \frac{\alpha_s}{16} (-4 C_A) + \frac{\alpha_s}{24} (2 N_F T_R)
\]

---

**DRED: Four constants needed:** \( \gamma_{cusp}, \gamma_q, \gamma_{\hat{g}}, \gamma_{\tilde{g}} \)

additional external \( \tilde{g} \): additional \( \gamma_{\tilde{g}} \)

[Signer, DS '08; Broggi, Gnendiger, Signer, DS, Visconti]

split \( g = \hat{g} + \tilde{g} \) required to understand factorization

(solves “problem” of \([\text{Beenakker, Kuijf, v Neerven, Smith '88}]\)[v Neerven, Smith '04])
Apply in practice to calculation of $M^{1\text{-loop}}$

Direct computation of $M^{1\text{-loop}}$ understand IR structure:

\[ M_{\text{HV}}^{1\text{-loop}}(\ldots g \ldots) = \]

\[ M_{\text{FDH}}^{1\text{-loop}}(\ldots g \ldots) = \]

\[ M_{\text{DRED}}^{1\text{-loop}}(\ldots g \ldots) = \]
Apply in practice to calculation of $\mathcal{M}^{1\text{-loop}}$

Direct computation of $\mathcal{M}^{1\text{-loop}}$

\[
\mathcal{M}_{1\text{-loop}}^\text{HV}(\ldots g \ldots) = \\
\mathcal{M}_{1\text{-loop}}^\text{FDH}(\ldots g \ldots) = \\
\mathcal{M}_{1\text{-loop}}^\text{DRED}(\ldots g \ldots) =
\]

understand IR structure:
Apply in practice to calculation of $\mathcal{M}^{1\text{-loop}}$

Direct computation of $\mathcal{M}^{1\text{-loop}}$

\[ \mathcal{M}_{1\text{-loop}}^{\text{HV}}(\ldots g \ldots) = \ldots + \frac{\gamma_{g}^{\text{HV}}}{2\epsilon} \mathcal{M}_{\text{tree}}^{\text{HV}}(\ldots g \ldots) \]

\[ \mathcal{M}_{1\text{-loop}}^{\text{FDH}}(\ldots g \ldots) = \ldots + \frac{\gamma_{g}^{\text{FDH}}}{2\epsilon} \mathcal{M}_{\text{tree}}^{\text{FDH}}(\ldots g \ldots) \]

\[ \mathcal{M}_{1\text{-loop}}^{\text{DRED}}(\ldots g \ldots) = \ldots + \frac{\gamma_{\hat{g}}^{\text{DRED}}}{2\epsilon} \mathcal{M}_{\text{tree}}^{\text{DRED}}(\ldots \hat{g} \ldots) + \frac{\gamma_{\tilde{g}}^{\text{DRED}}}{2\epsilon} \mathcal{M}_{\text{tree}}^{\text{DRED}}(\ldots \tilde{g} \ldots) \]
Apply in practice to calculation of $\mathcal{M}^{1\text{-loop}}$

Direct computation of $\mathcal{M}^{1\text{-loop}}$

\[ \mathcal{M}_{1\text{-loop}}^{\text{HV}}(\ldots g \ldots) = \ldots + \frac{\gamma g}{2\epsilon} \mathcal{M}_{\text{tree}}^{\text{HV}}(\ldots g \ldots) \]

\[ \mathcal{M}_{1\text{-loop}}^{\text{FDH}}(\ldots g \ldots) = \ldots + \frac{\gamma g}{2\epsilon} \mathcal{M}_{\text{tree}}^{\text{FDH}}(\ldots g \ldots) \]

\[ \mathcal{M}_{1\text{-loop}}^{\text{DRED}}(\ldots g \ldots) = \ldots + \frac{\gamma \hat{g}}{2\epsilon} \mathcal{M}_{\text{tree}}^{\text{DRED}}(\ldots \hat{g} \ldots) + \frac{\gamma \tilde{g}}{2\epsilon} \mathcal{M}_{\text{tree}}^{\text{DRED}}(\ldots \tilde{g} \ldots) \]

“Factorization problem” of van Neerven, Smith, et al was:

expected r.h.s. = \ldots + \frac{\gamma g}{2\epsilon} \mathcal{M}_{\text{tree}}^{\text{DRED}}(\ldots g \ldots)
Apply in practice to calculation of $\mathcal{M}^{1\text{-loop}}$

Direct computation of $\mathcal{M}^{1\text{-loop}}$

\[
\mathcal{M}^{HV}_{1\text{-loop}}(\ldots g \ldots) = \ldots + \frac{\gamma^H}{2\epsilon} \mathcal{M}^{HV}_{\text{tree}}(\ldots g \ldots)
\]

\[
\mathcal{M}^{FDH}_{1\text{-loop}}(\ldots g \ldots) = \ldots + \frac{\gamma^F}{2\epsilon} \mathcal{M}^{FDH}_{\text{tree}}(\ldots g \ldots)
\]

\[
\mathcal{M}^{DRED}_{1\text{-loop}}(\ldots g \ldots) = \ldots + \frac{\hat{\gamma}^D}{2\epsilon} \mathcal{M}^{DRED}_{\text{tree}}(\ldots \hat{g} \ldots) + \frac{\hat{\tilde{\gamma}}^D}{2\epsilon} \mathcal{M}^{DRED}_{\text{tree}}(\ldots \tilde{g} \ldots)
\]

\[\text{Remainder: finite, regularization-independent } \mathcal{M}_{\text{fin}} \]

⇒ can simply translate between schemes
<table>
<thead>
<tr>
<th>1-loop CDR</th>
<th>1-loop FDH, DRED</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-loop CDR</td>
<td>2-loop FDH, DRED</td>
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</table>

- It works similarly
  - PhD theses C. Gnendiger, A. Visconti
  - [Gnendiger, Signer, DS '14][Broggio, Gnendiger, Signer, DS, Visconti PLB '15, JHEP '15]
Key insight: All IR divs $\rightarrow$ factor $Z, Z^{-1} |\mathcal{A}\rangle = \text{fin.}$

$$\frac{d}{d \ln \mu} Z = -\Gamma Z$$

$2\text{nd insight}$ [valid at least up to 2-loop order]

$$\Gamma = \sum_{(ij)} \frac{T_i \cdot T_j}{2} \gamma_{\text{cusp}} \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i$$

Changes in FDH and DRED: Same structure . . . but the $\gamma$'s change and everything depends on $\alpha_s, \alpha_e, \alpha_4\epsilon$
Is DRED supersymmetric???
Slavnov-Taylor id. and Quantum Action Principle

Slavnov-Taylor identities: desired identities for Green functions

\[ 0 = \delta_{\text{sym}} \langle T \phi_1 \ldots \phi_n \rangle = \langle T (\delta_{\text{sym}} \phi_1) \ldots \phi_n \rangle + \ldots \]

Regularized Quantum Action Principle

\[ i \delta_{\text{sym}} \langle T \phi_1 \ldots \phi_n \rangle_{\text{reg}} = \langle T \phi_1 \ldots \phi_n \Delta \rangle_{\text{reg}}, \quad \Delta = \int d^D x \delta_{\text{sym}} \mathcal{L}_{\text{reg}} \]

Proofs for particular regularizations:

BPHZ [Lowenstein et al '71]
DREG [Breitenlohner, Maison '77]
DRED [DS '05]

\[ \rightsquigarrow \text{can check whether STI is valid on regularized level} \]
Further explicit checks of SUSY in DRED?

Status: many SUSY identities checked in DRED:

- [Capper, Jones, van Nieuwenhuizen’80]
- [Martin, Vaughn ’93]
- [Jack, Jones, North ’96]
- [Beenakker, Höpker, Zerwas’96]
- [Hollik, Kraus, DS’99]
- [Hollik, DS’01]
- [Fischer, Hollik, Roth, DS’03]
- [Harlander, Kant, Mihaila, Steinhauser’07]

- sufficient for many SUSY processes
  ⇒ multiplicative renormalization o.k.
  ⇒ no SUSY-restoring counterterms

- but not all identities have been checked
e.g. two-loop Higgs mass calculations:
  ⇒ SUSY-restoring counterterms required?
Higgs boson mass and quartic coupling

Higgs mass

- $M_h$ governed by quartic Higgs self coupling $\lambda$
- $\lambda \propto g^2$ in MSSM
Quartic coupling and SUSY

\[ \frac{\tilde{H}\tilde{H}}{\propto g h} \]

\[ \lambda \]

Slavnov-Taylor identity

- expresses \( \lambda \propto g^2 \)
- Needs to be verified

\[ 0 \overset{?}{=} \delta_{\text{SUSY}} \langle hhh\tilde{H} \rangle \]
Quartic coupling and SUSY

Slavnov-Taylor identity
- expresses $\lambda \propto g^2$
- Needs to be verified

explicit: $0 = \frac{\delta^5 S(\Gamma)}{\delta \phi_a \delta \phi_b \delta \phi_c \delta \tilde{H} \delta \bar{\epsilon}} = \Gamma_{\tilde{H} Y_i} \Gamma_{\phi_a \phi_b \phi_c \phi_i} + \Gamma_{\phi_a \phi_b} Y_\lambda \epsilon \Gamma_{\phi_c \tilde{H} \lambda} + \ldots$
Quartic coupling and SUSY

\[ \lambda \propto g^2 \]

Slavnov-Taylor identity
- expresses \( \lambda \propto g^2 \)
- Needs to be verified

If verified:
- Usual, multiplicative renormalization o.k.
- otherwise, SUSY-restoring counterterms would have to be added

\[ 0 = \delta_{\text{SUSY}} \langle hhHhH \rangle \]
Quartic coupling and SUSY

\[ \lambda \propto g^2 \]

Needs to be verified

Strategy:
- Use quantum action principle in DRED [DS '05]

\[ 0 \overset{?}{=} \delta_{\text{SUSY}} \langle hhh\Hbar \rangle \equiv \langle \Delta hhh\Hbar \rangle \]
Quartic coupling and SUSY

\[ \lambda \propto g^2 \]

Needs to be verified

Strategy:

- Use quantum action principle in DRED
  
\[ \delta_{\text{SUSY}} \langle hhh\tilde{H} \rangle = \langle \Delta hhh\tilde{H} \rangle \]

\[ \Delta = \delta_{\text{SUSY}} \int d^D x \mathcal{L} \]

0 \equiv \delta_{\text{SUSY}} \langle hhh\tilde{H} \rangle \equiv \langle \Delta hhh\tilde{H} \rangle
Quartic coupling and SUSY

Old check: STI at 2-loop
\( \mathcal{O}(\alpha_t^2, b, \alpha_t b \alpha_s) \) [Hollik, DS '05]

\[ \langle \Delta hhh\tilde{H} \rangle = 0 \iff \bar{\epsilon} \tilde{q}_q h \tilde{H} \tilde{q} + \ldots = 0 \]
Quartic coupling and SUSY

New check: 3-loop STI at $\mathcal{O}(\alpha_t \alpha_s^2)$ [DS, Unger, preliminary]

\[ \langle \Delta hhh\tilde{H} \rangle = 0 \Leftrightarrow \]

Ingredients:
- structure of possible SUSY-restoring c.t. $\Rightarrow$ can set $p_{hi} = 0$
- can simplify closed fermion loop to at most three $\gamma$-matrices
- recipe of [DS '05] then proves result
Quartic coupling and SUSY

\[ \lambda \]

\[ \tilde{H} \]

\[ \tilde{W} \]

\[ h + \]

\[ \propto g \]

finite

Outlook:

- check complete two-loop, three-loop (?)
- check further, relevant STIs
Outline

1. Regularization: DREG, FDH, DRED: UV, IR

2. Renormalization of VEVs
   - Why interesting? Definitions of VEV
   - Essential point: relation to gauge fixing
   - Investigate in detail: extended BRS/STI
   - Results and insights

3. Conclusions
Renormalization: which counterterms are necessary?

Other questions: physical meaning of renormalized theory, unitarity/symmetries of physical S-matrix, absence of anomalies...
Higgs vacuum

What is the VEV $v$ and its physical properties?
Renormalization of VEVs

Higgs/spontaneously broken gauge invariance:

\[ \phi \rightarrow \phi + \nu \]

such that \( \langle \phi \rangle = 0 \), i.e. tadpoles vanish

Need to renormalize:

\[ \phi \rightarrow \sqrt{Z} \phi, \quad \nu \rightarrow \nu + \delta \nu \]
Motivation

$\delta v$ appears in/is needed for:

- $\delta v$ in loop calculations:
  
  \[M_W = \frac{g v}{2},\]
  \[\rightarrow \delta M_W = \frac{\delta g v}{2} + \frac{g \delta v}{2},\]
  
  \[\tan \beta = \frac{v_u}{v_d},\]
  \[\rightarrow \delta \tan \beta = \left( \frac{\delta v_u}{v_u} - \frac{\delta v_d}{v_d} \right) \tan \beta\]

- $\beta_v$, $\beta_{\tan \beta}$ in spectrum generators (Softsusy, SPheno, FlexibleSUSY, Sarah)

Aim: better understanding of $\delta v$, general computation

Could get $\delta v$ from e.g. $\delta M_W \rightarrow$ no insight.
Details and questions

Most generic renormalization transformation:

$$(\phi + \nu) \rightarrow \sqrt{Z}\phi + \nu + \delta \nu$$

or $$\(\phi + \nu) \rightarrow \sqrt{Z}(\phi + \nu + \delta \nu)$$

Ultimately $\delta \nu$ is important for $\delta \tan \beta$, $\beta$ functions, etc.

$\delta \nu$ characterizes to what extent $\nu$ renormalizes differently from $\phi$. 
Most generic renormalization transformation:

\[(\phi + \nu) \rightarrow \sqrt{Z}\phi + \nu + \delta\nu\]

or

\[(\phi + \nu) \rightarrow \sqrt{Z}(\phi + \nu + \delta\bar{\nu})\]

Ultimately \(\delta\nu\) is important for \(\delta\tan\beta\), \(\beta\) functions, etc.

\(\delta\bar{\nu}\) characterizes to what extent \(\nu\) renormalizes differently from \(\phi\).

Questions:

1. When/why \(\delta\bar{\nu} \neq 0\)?
2. Properties of \(\delta\bar{\nu}\)?
3. \(\beta\nu\) and applications.
Details and questions

Most generic renormalization transformation:

\[(\phi + v) \rightarrow \sqrt{Z}\phi + v + \delta v\]

or \[(\phi + v) \rightarrow \sqrt{Z}(\phi + v + \delta \bar{v})\]

Ultimately \(\delta v\) is important for \(\delta \tan \beta\), \(\beta\) functions, etc.

\(\delta \bar{v}\) characterizes to what extent \(v\) renormalizes differently from \(\phi\).

Questions:

1. When/why \(\delta \bar{v} \neq 0\)?
2. Properties of \(\delta \bar{v}\)?
3. \(\beta v\) and applications.

Idea:

- \(\delta \bar{v} = v\delta \hat{Z}\)
- compute \(\delta \hat{Z}\) (using STI)
Current status for RGE coefficients

needed by SUSY spectrum generators (Spheno, Softsusy, SuseFlav, FlexibleSUSY, Sarah)

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<tr>
<th>Model</th>
<th>$\beta$(phys. parameter)</th>
<th>$\gamma$(fields)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall$ gauge theory</td>
<td>✓ [Machacek, Vaughn '83, Luo et al '03]</td>
<td>✓</td>
</tr>
<tr>
<td>$\forall$ SUSY model</td>
<td>✓ [Martin, Vaughn; Jack, Jones; Yamada '93]</td>
<td>✓ partially</td>
</tr>
</tbody>
</table>

Note in SUSY: $\gamma$(scalar in WZ gauge+Landau or $R_\xi$ gauge) $\neq \gamma$(superfield) $\equiv \gamma$(light cone gauge)

<table>
<thead>
<tr>
<th>Model</th>
<th>$\beta^{(1)}_V$</th>
<th>$\beta^{(2)}_V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSSM</td>
<td>✓ [Chankowski Nucl.Phys. B423]</td>
<td>✓ [Yamada 94] $O(g^2 Y^2)$</td>
</tr>
<tr>
<td>E$_6$SSM</td>
<td>✓ [Athron, DS, Voigt '12]</td>
<td>×</td>
</tr>
<tr>
<td>$\forall$ gauge theory</td>
<td>?</td>
<td>×</td>
</tr>
<tr>
<td>$\forall$ SUSY model</td>
<td>?</td>
<td>×</td>
</tr>
</tbody>
</table>

Here: fill the gaps
Meaning of running $v$, alternative treatment

- Fix renormalization scale $\mu$, renormalize in $\overline{\text{MS}}/\overline{\text{DR}}$-scheme
- adjust $v$ such that tadpoles $\langle \phi \rangle = 0$

$\Rightarrow v = \text{minimum of renormalized effective scalar potential at scale } \mu$

- Change $\mu$
- change parameters, including $v$, according to $\beta$ functions
- all Green functions unchanged, including $\langle \phi \rangle = 0$

$\Rightarrow \text{Minimum } v \text{ of renormalized effective scalar potential is } \mu \text{-dependent and gauge dependent } \Rightarrow \text{not an observable}$
Meaning of running $\nu$, alternative treatment

- Fix renormalization scale $\mu$, renormalize in $\overline{\text{MS}}/\overline{\text{DR}}$-scheme
- adjust $\nu$ such that tadpoles $\langle \phi \rangle = 0$

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- change parameters, including $\nu$, according to $\beta$ functions
- all Green functions unchanged, including $\langle \phi \rangle = 0$

$\Rightarrow \text{Minimum } \nu \text{ of renormalized effective scalar potential is } \mu\text{-dependent and gauge dependent } \Rightarrow \text{not an observable}$

Very different treatment of $\nu$ possible,

e.g. [Jegerlehner, Kalmykov, Kniehl '13][Bednyakov, Pikelner, Velizhanin '13]:

- always define $\nu_{\text{bare}} = \text{Minimum of bare scalar potential}$
- then $\nu_{\text{bare}} = \text{abbreviation of combination of bare parameters}$
- In this scheme, $\delta \nu, \delta M_W, \delta \tan \beta = \text{gauge independent, but tadpoles are divergent (physical quantities unchanged)}$
Motivation

Questions:

1. When/why $\delta \bar{\nu} \neq 0$?
2. Properties of $\delta \bar{\nu}$?
3. $\beta_\nu$ and applications.

Idea:

- $\delta \bar{\nu} = \nu \delta \hat{Z}$
- compute $\delta \hat{Z}$ (using STI)

Personal motivation/more detailed questions:
Motivation

Questions:
1. When/why $\delta \bar{v} \neq 0$?
2. Properties of $\delta \bar{v}$?
3. $\beta_v$ and applications.

Idea:
- $\delta \bar{v} = \nu \delta \hat{Z}$
- compute $\delta \hat{Z}$ (using STI)

Personal motivation/more detailed questions:

2001:
A: "MSSM: $\frac{\delta \bar{v}_u}{\nu_u} - \frac{\delta \bar{v}_d}{\nu_d} =$finite!"
B: "Why?"
A: ???
Motivation

Questions:

1. When/why $\delta \bar{V} \neq 0$?
2. Properties of $\delta \bar{V}$?
3. $\beta_V$ and applications.

Idea:

- $\delta \bar{V} = v \delta \hat{Z}$
- Compute $\delta \hat{Z}$ (using STI)

Personal motivation/more detailed questions:

2001: A: "MSSM: $\frac{\delta \bar{V}_u}{V_u} - \frac{\delta \bar{V}_d}{V_d} =$ finite!"
   B: "Why?"
   A: "???"

SUSY non-renormalization theorems? only at one-loop? Later: also true at $O(\alpha_s \alpha_{\text{top}})$ [Rzehak]
Motivation

Questions:
1. When/why $\delta \bar{v} \neq 0$?
2. Properties of $\delta \bar{v}$?
3. $\beta_v$ and applications.

Idea:
- $\delta \bar{v} = \nu \delta \hat{Z}$
- compute $\delta \hat{Z}$ (using STI)

Personal motivation/more detailed questions:

2011: A: “Also true in 2HDM!”
Motivation

Questions:

1. When/why $\delta \bar{v} \neq 0$?
2. Properties of $\delta \bar{v}$?
3. $\beta_v$ and applications.

Idea:

- $\delta \bar{v} = v \delta \hat{Z}$
- compute $\delta \hat{Z}$ (using STI)

Personal motivation/more detailed questions:

2011: C: "Why $\delta \bar{v} \neq 0$ at all?"
Motivation

Questions:
1. When/why $\delta \bar{v} \neq 0$?
2. Properties of $\delta \bar{v}$?
3. $\beta_v$ and applications.

Idea:
- $\delta \bar{v} = v \delta \hat{Z}$
- compute $\delta \hat{Z}$ (using STI)

Personal motivation/more detailed questions:

2011: C: "Why $\delta \bar{v} \neq 0$ at all?"
Answer: global gauge invariance broken by $R_\xi$ gauge

→ idea: take this seriously!
Idea as usual:

- get most general possible divergence from Slavnov-Taylor identity
- use BRS invariance

**Trick:** extend BRS invariance for additional insight
QCD running in 5 loops

Konstantin Chetyrkin (KIT)
in collaboration to

\[
\beta_{\text{QCD}} \text{ is expressed completely through } Z\text{-factors appearing in the (renormalized) QCD Lagrangian}
\]

\[
\mathcal{L}_{R}^{\text{QCD}} = \frac{1}{4} Z_{3} (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu})^{2} - \frac{1}{2} g Z_{1}^{3g} (\partial_{\mu} A_{\nu}^{a} - \partial_{\nu} A_{\mu}^{a}) (A_{\mu} \times A_{\nu})^{a} - \frac{1}{4} g^{2} Z_{1}^{4g} (A_{\mu} \times A_{\nu})^{2} \\
+ Z_{3}^{c} \partial_{\nu} \bar{c} (\partial_{\nu} c) + g Z_{1}^{ccg} \partial^{\mu} \bar{c} (A_{\mu} \times c) + Z_{2} \bar{\psi} i \gamma^{5} \psi - Z_{\psi \psi} m_{f} \bar{\psi} \psi + g Z_{1}^{\psi \psi \bar{c}} \bar{\psi} A_{\psi}
\]

Minimal (and simplest) set of \(Z\)-factors to compute \(\beta\): \(Z_{3}, Z_{3}^{c}, Z_{1}^{ccg}\)

Let us concentrate on \(Z_{1}^{ccg}\) and consider vertex function

\(\rightarrow\) used STI
Abelian Higgs model = $\phi^4 + \text{QED}$

$$L = |D_\mu \phi|^2 - m^2 |\phi + v|^2 - \lambda |\phi + v|^4 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + L_{\text{fix,gh}}$$

$$L_{\text{fix}} = -\frac{1}{2\xi} F^2$$

$$F = \partial_\mu A_\mu - \xi \text{ev}(2 \text{Im} \phi)$$

global transformation:

$$\delta_{\text{rigid}} \phi(x) = i \alpha (\phi(x) + v),$$

but

$$\delta_{\text{rigid}} L \neq 0$$

Symmetry broken by $L_{\text{fix}}$, **cannot conclude** $\delta_{\text{rigid}} L_{\text{bare}} = 0$, hence

$$\phi + v \rightarrow \sqrt{Z_\phi} (\phi + v + \delta \bar{v})$$

is allowed and expected to be required!
Influence of global gauge invariance in a nutshell

When does $\delta \bar{V}$ appear?

- global gauge invariance $\implies \delta \bar{V} = 0$
- no global gauge invariance $\implies \delta \bar{V} \neq 0$

$R_\xi$ gauge fixing:

$$F = \partial^\mu A_\mu - \xi ev(2 \text{Im}\phi)$$

$R_\xi$ breaks global gauge invariance for $\xi \neq 0$ $\implies \delta \bar{V} \neq 0$. 
Investigation of $\delta\nu$ in a nutshell

**Problem:** $R_\xi$ breaks global gauge invariance for $\xi \neq 0 \Rightarrow \delta\bar{\nu} \neq 0$.

**Trick:** Keep global gauge invariance in intermediate steps! [Kraus, Sibold 95]

also [Kraus ’97] [Hollik, Kraus, Roth, Rupp, Sibold, DS ’02]

Introduce background field $\hat{\phi}(x)$, only at the end: $\hat{\phi}(x) = \hat{\nu} = \text{const}$

$$\phi \rightarrow \phi_{\text{eff}} := \phi + \hat{\phi}$$

where $\hat{\phi}$ has same gauge transformation as $\phi$.

Modified $R_\xi$ gauge fixing:

$$F = \partial^\mu A_\mu + ie\xi(\hat{\phi}^\dagger \phi - \phi^\dagger \hat{\phi})$$

- Global gauge invariance! $\Rightarrow \delta\bar{\nu} = 0$
- Renormalization $\phi_{\text{eff}} \rightarrow \sqrt{Z}(\phi + \sqrt{\hat{Z}}\hat{\phi})$
- $\delta\bar{\nu}_{\text{eff}} = \hat{\nu}\delta\hat{Z}$, easy to compute using STI
Abelian Higgs model with background field \( \phi_{\text{eff}} = \phi + \hat{\phi} \)

\[
\mathcal{L} = \left| D^\mu \phi_{\text{eff}} \right|^2 - m^2 \left| \phi_{\text{eff}} \right|^2 - \lambda \left| \phi_{\text{eff}} \right|^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_{\text{fix,gh}} + \mathcal{L}_{\text{ext}}
\]

\[
\mathcal{L}_{\text{fix,gh}} = s \left[ \bar{c} \left( \frac{\xi}{2} B + F \right) \right], \quad s \bar{c} = B, sB = 0
\]

\[
F = \partial^\mu A_\mu + ie\xi (\hat{\phi}^\dagger \phi - \phi^\dagger \hat{\phi})
\]

\[
\mathcal{L}_{\text{ext}} = K_\phi s \phi + K_{\phi^\dagger} s \phi^\dagger,
\]

This \( \mathcal{L} \) reproduces standard-\( \mathcal{L} \) for \( \hat{\phi} = \hat{\nu} \), \( \hat{q} = 0 \) and is invariant under:
Abelian Higgs model with background field $\phi_{\text{eff}} = \phi + \hat{\phi}$

$$\mathcal{L} = |D_\mu \phi_{\text{eff}}|^2 - m^2 |\phi_{\text{eff}}|^2 - \lambda |\phi_{\text{eff}}|^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_{\text{fix,gh}} + \mathcal{L}_{\text{ext}}$$

$$\mathcal{L}_{\text{fix,gh}} = s \left[ \bar{c} \left( \frac{\xi}{2} B + F \right) \right], \quad s\bar{c} = B, sB = 0$$

$$F = \partial_\mu A_\mu + ie\xi (\hat{\phi}^\dagger \phi - \phi^\dagger \hat{\phi})$$

$$\mathcal{L}_{\text{ext}} = K_\phi s\phi + K_{\phi^\dagger} s\phi^\dagger,$$

This $\mathcal{L}$ reproduces standard-$\mathcal{L}$ for $\hat{\phi} = \hat{\nu}$, $\hat{\eta} = 0$ and is invariant under:

**global symmetry:**

$$\delta_{\text{rigid}} \phi = i \alpha \phi$$

$$\delta_{\text{rigid}} \hat{\phi} = i \alpha \hat{\phi}$$
Abelian Higgs model with background field $\phi_{\text{eff}} = \phi + \Phi$

\[
\mathcal{L} = |D^\mu \phi_{\text{eff}}|^2 - m^2 |\phi_{\text{eff}}|^2 - \lambda |\phi_{\text{eff}}|^4 - \frac{1}{4} F^{\mu \nu} F_{\mu \nu} + \mathcal{L}_{\text{fix,gh}} + \mathcal{L}_{\text{ext}}
\]

\[
\mathcal{L}_{\text{fix,gh}} = s \left[ \tilde{c} \left( \frac{\xi}{2} B + F \right) \right], \quad s\tilde{c} = B, sB = 0
\]

\[
F = \partial^\mu A_\mu + ie\xi (\hat{\phi}^\dagger \phi - \phi^\dagger \hat{\phi})
\]

\[
\mathcal{L}_{\text{ext}} = K_\phi s\phi + K_{\phi^\dagger} s\phi^\dagger,
\]

This $\mathcal{L}$ reproduces standard-$\mathcal{L}$ for $\hat{\phi} = \hat{v}$, $\hat{q} = 0$ and is invariant under:

**BRS invariance:**

\[
s\phi_{\text{eff}} = c\delta_{\text{gauge}} \phi_{\text{eff}} = -iec\phi_{\text{eff}}
\]

\[
s\text{Rest} = \text{standard}
\]

\[
s^2 = 0
\]
Abelian Higgs model with background field

**Secondary trick:** BRS transformation $\rightarrow$ control $\hat{\phi}$

\[
s\hat{\phi} = \hat{q}, \ s\hat{q} = 0
\]

this means that

\[
s\phi_{\text{eff}} = -iec\phi_{\text{eff}}
\]

requires:

\[
s\phi = -iec\phi_{\text{eff}} - \hat{q}
\]

hence:

\[
\mathcal{L}_{\text{ext}} = \ldots + K_\phi \left( -iec(\phi + \hat{\phi}) - \hat{q} \right)
\]
Abelian Higgs model with background field
As before: most general divergence structure, most general $\mathcal{L}_{\text{bare}}$ from

\[
\delta_{\text{rigid}} \mathcal{L} = 0 \quad \Rightarrow \quad \delta_{\text{rigid}} \mathcal{L}_{\text{bare}} = 0
\]

\[
s\mathcal{L} = 0 \quad \Rightarrow \quad \int \frac{\delta \Gamma_{\text{bare}}}{\delta K_{\phi}} \frac{\delta \Gamma_{\text{bare}}}{\delta \phi} = 0
\]

Result: $\mathcal{L}_{\text{bare}}$ generated by the renormalization transformation

\[
\phi_{\text{eff}} = \phi + \hat{\phi} \to \sqrt{Z} \left( \phi + \sqrt{\hat{Z}} \hat{\phi} \right)
\]

\[
\hat{q} \to \sqrt{Z} \sqrt{\hat{Z}} \hat{q}
\]

Crucial: $\mathcal{L}_{\text{bare}} = \ldots + K_{\phi} \left( -ie_{\text{bare}} c_{\text{bare}} (\phi + \sqrt{\hat{Z}} \hat{\phi}) - \sqrt{\hat{Z}} \hat{q} \right)$

\[
\hat{q}_a K_{\phi b} \quad = \quad -\frac{i}{2} \delta \hat{Z}
\]
Abelian Higgs model with background field

Compare with standard approach:

\[ \phi + v \rightarrow \sqrt{Z}(\phi + v + \delta v) \]

Here, for \( \hat{\phi} = \hat{v} \), \( \phi_{\text{eff}} = \phi + \hat{v} \) \( \Rightarrow \)

\[ \phi + \hat{v} \rightarrow \sqrt{Z}(\phi + \sqrt{\hat{Z}}\hat{v}) \]

hence

\[ v + \delta \hat{v}|_{\text{standard}} = \sqrt{\hat{Z}\hat{v}}|_{\text{here}} \]

What have we gained?

- \( \delta \hat{v} \leftrightarrow \) dimensionless field renormalization constant

- \( \hat{Z} \) can be directly obtained from

\[ \hat{q}_a K_{\phi_b} \]
Understanding of \( \delta \bar{V} \) from

\[
\hat{q}_a \quad K_{\phi_b} = -\frac{i}{2} \delta \hat{Z}
\]

Very few Feynman rules for with well-defined origin

\[
\hat{q}_a \quad K_{\phi_b}
\]

\[
\hat{c}^A \quad \phi_b = \xi e
\]

[breaking of global invariance \( \propto \xi e(\hat{\phi}^\dagger \phi - \phi^\dagger \hat{\phi}) \)]

\[
K_{\phi_a} \quad c^A \quad \phi_b = e
\]

[BRST transform \( \propto e \)]
Understanding of $\delta \bar{V}$ from

$$\hat{q}_a \quad \square \quad K_{\phi_b} = -\frac{i}{2} \delta \hat{Z}$$

Very few Feynman rules for

$$\hat{q}_a \quad \square \quad K_{\phi_b}$$

with well-defined origin

$$\mathcal{L}_{\text{ext}} = K_{\phi} s_{\phi} + \cdots$$

$$= K_{\phi} (-i e c (\phi + \hat{\phi})) - K_{\phi} \hat{q} + \cdots$$
Understanding of $\delta \bar{V}$ from

\[
\hat{q}_a \cdot K_{\phi_b} = -\frac{i}{2} \delta \hat{Z}
\]

Very few Feynman rules for

\[
F = \partial^\mu A_\mu + ie \xi (\hat{\phi}^{\dagger} \phi - \phi^{\dagger} \hat{\phi})
\]

\[
\mathcal{L}_{\text{fix,gh}} = s [\bar{c} (F + \xi B/2)]
\]

\[
= -\bar{c} ie \xi (\hat{q}^{\dagger} \phi - \phi^{\dagger} \hat{q}) + \ldots
\]
Understanding of $\delta \bar{\nu}$ from $\hat{q}_a$ $K_{\phi_b} = -\frac{i}{2} \delta \hat{Z}$

Very few Feynman rules for $\hat{q}_a$ $K_{\phi_b}$ with well-defined origin

Feynman rules $\propto \hat{q}$ only from $\hat{\phi}$ in gauge fixing, vanish for $\xi = 0$, can depend only on gauge couplings (also in general models!)
**General model**

Scalar, spinor, vector fields $\phi_a, \psi_p, V^A$ with Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{inv}}|_{\phi \rightarrow \phi_{\text{eff}}} + \mathcal{L}_{\text{fix,gh}} + \mathcal{L}_{\text{ext}}$$

with

$$\mathcal{L}_{\text{inv}} = -\frac{1}{4} F^A_{\mu\nu} F^{A\mu\nu} + \frac{1}{2} (D_\mu \phi)_a (D^\mu \phi)_a + i \psi^\alpha_{\dot{\alpha}} \sigma^\mu_{\alpha\dot{\alpha}} \left( D^\dagger_\mu \bar{\psi}_{\dot{\alpha}} \right)_p$$

$$- \frac{1}{2!} m^2_{ab} \phi_a \phi_b - \frac{1}{3!} h_{abc} \phi_a \phi_b \phi_c - \frac{1}{4!} \lambda_{abcd} \phi_a \phi_b \phi_c \phi_d$$

$$- \frac{1}{2} \left[ (m_f)_{pq} \psi^\alpha_p \psi_q^\alpha + \text{h.c.} \right] - \frac{1}{2} \left[ Y^a_{pq} \psi^\alpha_p \psi_q^\alpha \phi_a + \text{h.c.} \right]$$

$$\mathcal{L}_{\text{fix,gh}} = s \left[ \bar{c}^A \left( F^A + \xi B^A / 2 \right) \right]$$

$$\mathcal{L}_{\text{ext}} = K_{\phi a} s \phi_a + K_{V^A_{\mu}} s V^A_{\mu} + K_{c^A} s c^A + \left[ K_{\psi_p s} \psi_p + \text{h.c.} \right]$$

General, modified $R_\xi$ gauge fixing:

$$F^A = \partial^\mu V^A_{\mu} + ig \xi (\hat{\phi})_a T^A_{ab} \phi_b$$

Dominik Stöckinger
Renormalization of VEVs, IR structure in FDH
Renormalization of VEVs
Calculation of $\delta \hat{Z} - 1$ Loop

\[ \hat{q}_a \quad \phi_b \quad \bar{c}^A = \xi g T^A_{ab}, \quad \phi_b \quad \hat{q}_a \quad c^A = g T^A_{ab} \]

\[ \hat{q}_a \quad K_{\phi_b} \quad + \quad \hat{q}_a \quad K_{\phi_b} \quad = \text{finite} \]

\[ (4\pi)^2 \delta \hat{Z}^{(1)} = 2g^2 \xi C^2(S) \frac{1}{\epsilon} \]

vanishes for $\xi = 0$, only depends on squared gauge couplings
Calculation of $\delta \hat{Z} - 2$ Loop

\[ (4\pi)^4 \delta \hat{Z}^{(2)} = g^2 \xi C^2(S) Y^2(S) \left( \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \right) + O(g^4) \]

no $Y^4$, no $\lambda$ terms (in contrast to $\delta Z$)
Calculation of $\delta \hat{Z} - 2$ Loop

$$\hat{q}_a \rightarrow K_{\phi_b}$$

$$\hat{q}_a \rightarrow K_{\phi_b}$$

$$\hat{q}_a \rightarrow K_{\phi_b}$$

$$\hat{q}_a \rightarrow K_{\phi_b}$$

$$\Rightarrow$$

$$(4\pi)^4 \delta \hat{Z}^{(2)} = g^2 \xi C^2(S) Y^2(S) \left( \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \right) + O(g^4)$$

no $Y^4$, no $\lambda$ terms (in contrast to $\delta Z$)

Note alternative:

$$\frac{\delta m_t}{m_t} = \frac{\delta t}{t} + \frac{\delta Z}{2} + \frac{\delta \hat{Z}}{2}$$
Results – $\beta_v$

\[
\begin{align*}
\nu + \delta\bar{\nu} &= \sqrt{\hat{Z}}\hat{\nu} &\rightarrow \hat{\gamma}(S) \\
\nu + \delta\nu &= \sqrt{Z}\sqrt{\hat{Z}}\hat{\nu} &\rightarrow \gamma(S) + \hat{\gamma}(S)
\end{align*}
\]

\[
\Rightarrow \\
\beta_v = [\gamma(S) + \hat{\gamma}(S)] \nu
\]

\[
\hat{\gamma}^{(1)}(S) = \frac{\xi}{(4\pi)^2}2g^2C^2(S)
\]

\[
\hat{\gamma}^{(2)}(S) = \frac{\xi}{(4\pi)^4} \left\{ g^4 \left[ 2(1 + \xi) C^2(S)C^2(S) + \frac{7 - \xi}{2} C_2(G)C^2(S) \right] \\
- 2g^2C^2(S) Y^2(S) \right\}
\]

can now be implemented into spectrum generator (generators) Sarah [Staub], FlexibleSUSY
Results – $\beta_{\tan \beta}$ in the MSSM

\[ \tan \beta = \frac{v_u}{v_d} \Rightarrow \frac{\beta_{\tan \beta}}{\tan \beta} = \gamma_u - \gamma_d + \hat{\gamma}_u - \hat{\gamma}_d \]

**MSSM:**

\[ \frac{\beta_{\tan \beta}}{\tan \beta}^{(1)} = \gamma_{uu}^{(1)} - \gamma_{dd}^{(1)} \quad \leftarrow \text{cancellation of } \hat{\gamma} \text{ terms} \]

\[ \frac{\beta_{\tan \beta}}{\tan \beta}^{(2)} = \gamma_{uu}^{(2)} - \gamma_{dd}^{(2)} + \frac{\xi}{(4\pi)^2} \left( \frac{3}{10} g_1^2 + \frac{3}{2} g_2^2 \right) \frac{\beta_{\tan \beta}}{\tan \beta}^{(1)} \]

[Yamada 02]

**Understanding:**

- $\frac{\delta \bar{v}_u}{v_u} - \frac{\delta \bar{v}_d}{v_d} = \text{finite} \iff \hat{\gamma}_u - \hat{\gamma}_d = 0$
- Only true at one-loop because squared gauge couplings are equal
- Similar in 2HDM, NMSSM
Results – $\beta_{\tan \beta}$ in the E$_6$SSM

**E$_6$SSM:** extra $U(1)_N$ gauge symmetry

\[
\frac{\beta^{(1)}_{\tan \beta}}{\tan \beta} = \gamma^{(1)}_{uu} - \gamma^{(1)}_{dd} + \frac{\xi}{(4\pi)^2} 2g_N^2 \left[ \left( \frac{N_{Hu}}{2} \right)^2 - \left( \frac{N_{Hd}}{2} \right)^2 \right]
\]

- $\frac{\delta \bar{v}_u}{v_u} - \frac{\delta \bar{v}_d}{v_d} = \text{divergent} \iff \hat{\gamma}_u = \hat{\gamma}_d \neq 0$
- Already one-loop difference due to different $U(1)_N$-charges
General method of extended BRS/STI

Here: used $\hat{\phi}$, $s\hat{\phi} = \hat{q}$ to obtain information on $\delta \bar{V}$

Other possibilities

- $s^\xi \neq 0 \Rightarrow \partial^\xi (\ldots)$ calculable [Kraus, Häussling '95][Grassi, Gambino '00][Freitas, DS '02]
- $se \neq 0 \Rightarrow \delta e^{\text{div}}$ calculable [Flume, Kraus; Kraus; Kraus, DS '01]
Outline

1. Regularization: DREG, FDH, DRED: UV, IR
2. Renormalization of VEVs
3. Conclusions
Conclusions

DREG, HV, FDH, DRED: all consistent schemes
- old puzzles resolved
- DRED, FDH = DREG with additional $\epsilon$-scalar parton
- IR structure and renormalization in FDH, DRED understood
- practical transition rules
- DRED supersymmetric in many (not all?) important cases

Divergence structure of $\delta v$ (example of extended BRS/STI)
- if tadpoles=0: VEV is gauge- and $\mu$-dependent quantity
- $\delta \bar{v}$ from gauge fixing, $\propto$ squared gauge couplings and $\propto \xi$
- $\delta \hat{Z}_{Hu} - \delta \hat{Z}_{Hd} \equiv$finite in MSSM at 1-loop accidentally
- 2-loop $\beta$ functions, $\gamma^{SUSY}$ complete
1-loop CDR | 1-loop FDH, DRED
2-loop CDR | 2-loop FDH, DRED

Dominik Stöckinger: Renormalization of VEVs, IR structure in FDH
Key insight: All IR divs $\rightarrow$ factor $Z$, $Z^{-1}|\mathcal{A}\rangle = \text{fin.}$

\[ \frac{d}{d\ln \mu} Z = -\Gamma Z \]

2nd insight [valid at least up to 2-loop order]

\[ \Gamma = \sum_{(ij)} \frac{T_i \cdot T_j}{2} \gamma_{\text{cusp}} \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i \]
Key insight: All IR divs → factor $Z, Z^{-1}|A\rangle = \text{fin.}$

$$\frac{d}{d \ln \mu} Z = -\Gamma Z$$

2nd insight [valid at least up to 2-loop order] $\mu$-dependence of $\Gamma$ in $\alpha_s$ and $\ln \mu$:

$$\Gamma = \left(\frac{\alpha_s}{4\pi}\right)^n \Gamma_n \uparrow_{cusp, \gamma^n} = \left(\frac{\alpha_s}{4\pi}\right)^n \left[ \Gamma'_n \ln \mu + \Gamma''_n \right] \uparrow_{cusp, \gamma^n}$$
<table>
<thead>
<tr>
<th>1-loop CDR</th>
<th>1-loop FDH, DRED</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-loop CDR</td>
<td>2-loop FDH, DRED</td>
</tr>
</tbody>
</table>

Integrate $\frac{d}{d \ln \mu} Z = -\Gamma Z$, use $\frac{d}{d \ln \mu^2} \alpha_s(\mu) = -\epsilon \alpha_s(\mu) + \beta_0^s \frac{\alpha_s^2}{4\pi} + \ldots$
Integrate \( \frac{d}{d \ln \mu} Z = -\Gamma Z \), use \( \frac{d}{d \ln \mu^2} \alpha_s(\mu) = -\epsilon \alpha_s(\mu) + \beta_{20} \frac{\alpha_s^2}{4\pi} + \ldots \)

Result:

\[
\ln Z = \left( \frac{\alpha_s}{4\pi} \right) \left( \frac{\Gamma'_1}{4\epsilon^2} + \frac{\Gamma_1}{2\epsilon} \right) + \left( \frac{\alpha_s}{4\pi} \right)^2 \left( -\frac{3\beta_{20}\Gamma'_1}{16\epsilon^3} + \frac{\Gamma'_2 - 4\beta_{20}\Gamma_1}{16\epsilon^2} + \frac{\Gamma_2}{4\epsilon} \right) + \ldots
\]

Same three constants \( \gamma^\text{cusp}, \gamma^q, \gamma^g \) describe all IR divergences

1-loop agrees with previous result, at 2-loop also \( \beta \) needed
Changes in FDH and DRED: Same structure . . .

\[ \frac{d}{d \ln \mu} Z = -\Gamma Z \]

. . . but the \( \gamma \)'s change and everything depends on \( \alpha_s, \alpha_e, \alpha_4 \epsilon \)

\[ \Gamma = \sum_{(ij)} \frac{T_i \cdot T_j}{2} \gamma^{\text{cusp}} \ln \frac{\mu^2}{-S_{ij}} + \sum_i \gamma^i \]
Integrate \( \frac{d}{d \ln \mu} Z = -\Gamma Z \), use \( \beta^s, \beta^e \), etc

\[
\ln Z^{\text{FDH}}_{2\text{-loop}} = \left( \frac{\alpha_s}{4\pi} \right)^2 \left( -\frac{3\beta_{20}\Gamma'_{10}}{16\epsilon^3} + \frac{\Gamma'_{20} - 4\beta_{20}\Gamma_{10}}{16\epsilon^2} + \frac{\Gamma_{20}}{4\epsilon} \right) \\
+ \left( \frac{\alpha_s}{4\pi} \right) \left( \frac{\alpha_e}{4\pi} \right) \left( -\frac{3\beta^e_{11}\Gamma'_{01}}{16\epsilon^3} + \frac{\Gamma'_{11} - 4\beta^e_{11}\Gamma_{01}}{16\epsilon^2} + \frac{\Gamma_{11}}{4\epsilon} \right) \\
+ \left( \frac{\alpha_e}{4\pi} \right)^2 \left( -\frac{3\beta^e_{02}\Gamma'_{01}}{16\epsilon^3} + \frac{\Gamma'_{02} - 4\beta^e_{02}\Gamma_{01}}{16\epsilon^2} + \frac{\Gamma_{02}}{4\epsilon} \right) + O(\alpha_4\epsilon, \alpha^3).
\]
Integrate $\frac{d}{d\ln \mu} Z = -\Gamma Z$, use $\beta^s, \beta^e$, etc

$$\ln Z_{\text{FDH}}^{\text{2-loop}} = \left( \frac{\alpha_s}{4\pi} \right)^2 \left( - \frac{3\beta_20\Gamma'_{10}}{16\epsilon^3} + \frac{\Gamma'_{20} - 4\beta_{20}\Gamma_{10}}{16\epsilon^2} + \frac{\Gamma_{20}}{4\epsilon} \right)$$

$$+ \left( \frac{\alpha_s}{4\pi} \right) \left( \frac{\alpha_e}{4\pi} \right) \left( - \frac{3\beta_{11}\Gamma'_{01}}{16\epsilon^3} + \frac{\Gamma'_{11} - 4\beta_{11}\Gamma_{01}}{16\epsilon^2} + \frac{\Gamma_{11}}{4\epsilon} \right)$$

$$+ \left( \frac{\alpha_e}{4\pi} \right)^2 \left( - \frac{3\beta_{02}\Gamma'_{01}}{16\epsilon^3} + \frac{\Gamma'_{02} - 4\beta_{02}\Gamma_{01}}{16\epsilon^2} + \frac{\Gamma_{02}}{4\epsilon} \right) + O(\alpha_4, \alpha^3).$$

**FDH: three constants $\gamma^{\text{cusp}}, \gamma^q, \gamma^g = CDR$-values + $O(N_\epsilon)$**

- need $\beta^e, \alpha_e$-dependence separately
- Leads to translation rules between CDR, HV, FDH
Integrate $\frac{d}{d\ln \mu} Z = -\Gamma Z$, use $\beta^s, \beta^e$, etc

$$\ln Z_{2\text{-loop}}^{FDH} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left( -\frac{3\beta_20 \Gamma_{10}'}{16\epsilon^3} + \frac{\Gamma_{20}'}{16\epsilon^2} + \frac{\Gamma_{20}}{4\epsilon} \right)$$

$$+ \left(\frac{\alpha_s}{4\pi}\right) \left(\frac{\alpha_e}{4\pi}\right) \left( -\frac{3\beta_{11} \Gamma_{01}'}{16\epsilon^3} + \frac{\Gamma_{11}'}{16\epsilon^2} + \frac{\Gamma_{11}}{4\epsilon} \right)$$

$$+ \left(\frac{\alpha_e}{4\pi}\right)^2 \left( -\frac{3\beta_{02} \Gamma_{01}'}{16\epsilon^3} + \frac{\Gamma_{02}'}{16\epsilon^2} + \frac{\Gamma_{02}}{4\epsilon} \right) + \mathcal{O}(\alpha_4\epsilon, \alpha_3^3).$$

**FDH:** three constants $\gamma^{\text{cusp}}, \gamma^q, \gamma^g = \text{CDR-values } + \mathcal{O}(N_\epsilon)$

- Logic and result fully compatible with [Kilgore ’12]
- Differences to [Kilgore ’12]: slightly different $\gamma^i$, other sample processes
Integrate \( \frac{d}{d \ln \mu} Z = -\Gamma Z \), use \( \beta^s \), \( \beta^e \), etc.

\[
\ln Z_{\text{FDH}}^{\text{2-loop}} = \left( \frac{\alpha_s}{4\pi} \right)^2 \left( -\frac{3\beta_{20} \Gamma'_{10}}{16\epsilon^3} + \frac{\Gamma'_{20} - 4\beta_{20} \Gamma_{10}}{16\epsilon^2} + \frac{\Gamma_{20}}{4\epsilon} \right) \\
+ \left( \frac{\alpha_s}{4\pi} \right) \left( \frac{\alpha_e}{4\pi} \right) \left( -\frac{3\beta_{11}^e \Gamma'_{01}}{16\epsilon^3} + \frac{\Gamma'_{11} - 4\beta_{11}^e \Gamma_{01}}{16\epsilon^2} + \frac{\Gamma_{11}}{4\epsilon} \right) \\
+ \left( \frac{\alpha_e}{4\pi} \right)^2 \left( -\frac{3\beta_{02}^e \Gamma'_{01}}{16\epsilon^3} + \frac{\Gamma'_{02} - 4\beta_{02}^e \Gamma_{01}}{16\epsilon^2} + \frac{\Gamma_{02}}{4\epsilon} \right) + \mathcal{O}(\alpha_4\epsilon, \alpha^3).
\]

**DRED:** again same structure but four constants \( \gamma_{\text{cusp}}, \gamma^q, \gamma^g, \gamma^{\tilde{g}} \)

- turns out to depend even on \( \alpha_4\epsilon \)
IR structure in all four schemes

\[
\uparrow \\
\text{common structure, depends on anomalous dimensions } \gamma \text{ and } \beta \\
\text{functions of all couplings}
\]

Allows transition rules between schemes, once all \( \gamma \)'s are known
Two simple processes/form factors

\[ \gamma \rightarrow q\bar{q} \]

\[ H \rightarrow gg \]
Two simple processes/form factors

\[ \gamma \rightarrow q\bar{q} \]

\[ H \rightarrow gg \]

- IR structure, transition rules
- determine \( \gamma \)'s
- study concrete calculation/renormalization
Two simple processes/form factors

Simple kinematics

\[ \gamma \rightarrow q\bar{q} = F^q(s) \]

\[ H \rightarrow gg = F^g(s) \]
Two simple processes/form factors

Simple kinematics

\[ \gamma \rightarrow q\bar{q} = F^q(s) \]

\[ H \rightarrow gg = F^\hat{g}(s) + F^\tilde{g}(s) \]
Two simple processes/form factors

\[
\gamma \rightarrow q\bar{q}
\]

\[
H \rightarrow gg
\]

Simple colour/IR structure:

\[
\Gamma' = -2C_{F,A}\gamma^{\text{cusp}}
\]

\[
\Gamma = 2\gamma^i (i = q, g, \hat{g}, \tilde{g})
\]
Two simple processes/form factors

\[ \gamma \rightarrow q\bar{q} \]

\[ H \rightarrow gg \]

Simple colour/IR structure:

\[ \Gamma' = -2C_F A \gamma^{\text{cusp}} \]

\[ \Gamma = 2\gamma^i (i = q, g, \hat{g}, \tilde{g}) \]

\[ Z^{-1} F = \text{fin.} \]

\[ \ln F - \ln Z = \text{fin.} \]

\[ \left[ F^{2L} - \frac{1}{2} (F^{1L})^2 \right] - \ln Z|^{2L} = \text{fin.} \]
Example: Hgg in FDH/DRED. Starting point: gauge invariant $\mathcal{L}_{\text{eff}}$

$$\mathcal{L}_{\text{eff}} = \lambda H O_1 + \lambda_\epsilon H \tilde{O}_1 + \ldots$$

with

$$O_1 = -\frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu}, \quad \tilde{O}_1 = -\frac{1}{2} (\hat{D}_\mu \tilde{A}_\nu) (\hat{D}_\mu \tilde{A}_\nu).$$
Need two-loop diagrams and counterterm diagrams such as

- all couplings $\lambda$, $\lambda_\epsilon$, $\alpha_s$, $\alpha_e$, $\alpha_{4\epsilon}$ appear
- Need also: all one-loop renormalization constants for these couplings
- Need also: two-loop $\delta\lambda$, $\delta\lambda_\epsilon$
Need two-loop diagrams and counterterm diagrams such as

- all couplings $\lambda, \lambda_\epsilon, \alpha_s, \alpha_e, \alpha_{4\epsilon}$ appear
- Need also: all one-loop renormalization constants for these couplings
- Need also: two-loop $\delta \lambda, \delta \lambda_\epsilon$
Problem: renormalization of $\mathcal{L}_{\text{eff}}, \lambda, \lambda_\epsilon$

Situation in CDR, HV:

- $\frac{\lambda + \delta \lambda}{\lambda} = 1 + \alpha_s \frac{\partial}{\partial \alpha_s} \ln Z_{\alpha_s}$ [Spiridonov '84, Chetyrkin, Kniehl, Steinhauser '97]
- known from literature, no dedicated computation required

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- **Reason:** full basis of local operators $O_{1...5}$ can all be written as $\frac{\partial}{\partial \text{fields, parameters}} \mathcal{L}_{\text{QCD}}$

Situation in FDH, DRED:

Dominik Stöckinger
Renormalization of VEVs, IR structure in FDH
Explicit computation and renormalization at the two-loop level
Problem: renormalization of $L_{\text{eff}}$, $\lambda$, $\lambda_\varepsilon$

Situation in CDR, HV:
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Situation in FDH, DRED:
- e.g. $\Box \tilde{A}_\mu \tilde{A}_\mu$ exists in operator basis
- has no counterpart in $L_{\text{QCD}} \Rightarrow$ Spiridonov's method fails
Problem: renormalization of $\mathcal{L}_{\text{eff}}, \lambda, \lambda_\varepsilon$

Situation in CDR, HV:
- $\frac{\lambda + \delta \lambda}{\lambda} = 1 + \alpha_s \frac{\partial}{\partial \alpha_s} \ln Z_{\alpha_s}$ [Spiridonov '84, Chetyrkin, Kniehl, Steinhauser '97]
- known from literature, no dedicated computation required
- **Reason:** full basis of local operators $O_1\ldots 5$ can all be written as $\frac{\partial}{\partial \text{fields, parameters}} \mathcal{L}_{\text{QCD}}$

Situation in FDH, DRED:
- e.g. $\Box \tilde{A}_\mu \tilde{A}_\mu$ exists in operator basis
- has **no counterpart in** $\mathcal{L}_{\text{QCD}} \Rightarrow$ Spiridonov’s method fails
- way out: determine operator mixing renormalization constants from explicit off-shell calculations (one- and two-loop)
- off-shell means: non-gauge invariant operators also needed
**Hgg in FDH: Sample results**

\[
\delta \lambda^{(2)} = \lambda \left( \frac{\alpha_s}{4\pi} \right)^2 \left( \frac{\beta_{20}^s}{\epsilon^2} - \frac{\beta_{30}^s}{\epsilon} \right) + \left( \lambda + \lambda \epsilon \right) \left( \frac{\alpha_s}{4\pi} \right) \left( \frac{\alpha_e}{4\pi} \right) \left( - \frac{\beta_{21}^s}{2\epsilon} \right)
\]

[Spiridonov '84] applicable, but mixing occurs

\[
\delta \lambda^{(1)}_\epsilon = \lambda \epsilon \left[ \left( \frac{\alpha_s}{4\pi} \right) \left( - \frac{3C_A}{\epsilon} \right) + \left( \frac{\alpha_e}{4\pi} \right) \frac{N_F}{\epsilon} + \left( \frac{\alpha_{4\epsilon}}{4\pi} \right) C_A \left( - \frac{1 + N_\epsilon}{\epsilon} \right) \right]
\]

explicit calculation
\[ Hgg \text{ in FDH: Result for } \bar{G}^{(2)} = Fg,^{2L} - \frac{1}{2}(Fg,^{1L})^2 \]

\[
\bar{G}^{(2)}(\alpha_s, \alpha_e) = G^{(2),CDR}(\alpha_s) + \left( \frac{\alpha_s}{4\pi} \right)^2 N_\epsilon \left\{ C_A^2 \left[ -\frac{1}{4\epsilon^3} + \frac{-7 + \frac{N_\epsilon}{72}}{\epsilon^2} + \frac{49 - \pi^2}{72 \epsilon} \right] + C_A N_F \frac{1}{9\epsilon^2} \right\} \\
+ \left( \frac{\alpha_s}{4\pi} \right) \left( \frac{\alpha_e}{4\pi} \right) N_\epsilon \left\{ -\frac{C_F N_F}{2\epsilon} \right\} + O(N_\epsilon \epsilon^0).
\]

- IR prediction was: \([G^{(2)} - \ln Z|^{2L}] = \text{fin. for both FDH and CDR}\)
**Hgg in FDH: Result for** \( \bar{G}^{(2)} = Fg,^{2L} - \frac{1}{2} (Fg,^{1L})^2 \)

\[
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\]

- IR prediction was: \( [G^{(2)} - \ln Z]^{2L} \) = fin. for both FDH and CDR
- This is valid!
- can read off e.g. \( \tilde{\gamma}_{g,20}^{g} = \gamma_{g,20}^{g} + N_\epsilon \left( \frac{98}{27} - \frac{\pi^2}{36} \right) C_A^2 \)
**Hgg in FDH: Result for** \( \bar{G}^{(2)} = F g,^{2L} - \frac{1}{2}(F g,^{1L})^2 \)

\[
\bar{G}^{(2)}(\alpha_s, \alpha_e) = G^{(2),\text{CDR}}(\alpha_s) + \left(\frac{\alpha_s}{4\pi}\right)^2 N_\epsilon \left\{ C_A^2 \left[ -\frac{1}{4\epsilon^3} + \frac{7}{18} + \frac{N_\epsilon}{72} + \frac{49}{27} - \frac{\pi^2}{72} \right] + C_A N_F \frac{1}{9\epsilon^2} \right\}
+
\left(\frac{\alpha_s}{4\pi}\right) \left(\frac{\alpha_e}{4\pi}\right) N_\epsilon \left\{ -\frac{C_F N_F}{2\epsilon} \right\} + O(N_\epsilon \epsilon^0).
\]

- IR prediction was: \([G^{(2)} - \ln Z|^{2L}] = \text{fin. for both FDH and CDR}
- This is valid!
- can read off e.g. \[\bar{\gamma}_{20}^{g} = \gamma_{20}^{g} + N_\epsilon \left(\frac{98}{27} - \frac{\pi^2}{36}\right) C_A^2\]
- after subtracting IR divs, the difference is \(O(N_\epsilon \epsilon^0)\)
- \(\Rightarrow\) translation rules
**Hgg in FDH:** Result for $\tilde{G}^{(2)} = Fg_{2L} - \frac{1}{2}(Fg_{1L})^2$

$$
\tilde{G}^{(2)}(\alpha_s, \alpha_e) = G^{(2),\text{CDR}}(\alpha_s) + \left(\frac{\alpha_s}{4\pi}\right)^2 N_{\epsilon} \left\{ C_A^2 \left[ -\frac{1}{4e^3} + \frac{7}{18} + \frac{N_{\epsilon}}{72} + \frac{49 - \pi^2}{18} \right] + C_A N_F \frac{1}{9e^2} \right\} \\
+ \left(\frac{\alpha_s}{4\pi}\right) \left(\frac{\alpha_e}{4\pi}\right) N_{\epsilon} \left\{ -\frac{C_F N_F}{2e} \right\} + O\left(N_{\epsilon}\epsilon^0\right).
$$

- IR prediction was: $[G^{(2)} - \ln Z|^{2L}] = \text{fin. for both FDH and CDR}$
- This is valid!
- can read off e.g. $\tilde{\gamma}_{20} = \gamma_{20} + N_{\epsilon} \left(\frac{98}{27} - \frac{\pi^2}{36}\right) C_A^2$
- after subtracting IR divs, the difference is $O\left(N_{\epsilon}\epsilon^0\right)$
- $\Rightarrow$ translation rules
- Similarly, obtain all two-loop $F^q$, $F^g$, $\gamma$'s in all schemes

$$
\gamma_{\text{FDH,DRED}}^{\text{cusp}}, \gamma_{\text{FDH,DRED}}^{q}, \gamma_{\text{FDH,DRED}}^{g}, \gamma_{\text{DRED}}^{\hat{g}}, \gamma_{\text{DRED}}^{\tilde{g}}
$$