Topology and Size of the Universe from CMB Temperature and Polarization

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Talk based on:


An overview of properties of the CMB.
Current status

- General Relativity is a local theory, and constrains the local properties of spacetime
- The Standard Model is also a local theory
- Nothing in the physical laws says anything about global properties
- We can only see a finite portion of space (horizon)
- Nothing known from the basic laws. Global structure has to be tested experimentally by observations on cosmological scales.
Many possibilities:

Take the usual flat, closed or open universes \((\mathbb{R}^3, S^3, H^3)\) and take the quotient by the action of a discrete group.

Result is a finite manifold of constant curvature which is locally indistinguishable from the original covering space.

Simple 1D example:

\[
\mathbb{R}/\mathbb{Z} = S^1, \quad \text{under action} \quad x \rightarrow x + n, \quad n \in \mathbb{Z}
\]
Universe Flat

Observations (Planck + BAO + highL + WP):

\[ \Omega_k = -0.0005^{+0.0065}_{-0.0066} \]

Theoretically:
The probability of quantum creation of positive curvature universes is exponentially suppressed.

Much easier to think of flat space.
3 Possible Flat manifolds

\[ M_0 = \mathbb{T}^3 \quad M_1 = \mathbb{T}^2 \times \mathbb{R} \quad M_2 = S^1 \times \mathbb{R}^2 \]

- Opposite edges are identified (periodic boundary conditions).
- All sides have equal length \( L \) (to reduce number of parameters)
- Break isotropy of space
Particle Horizon and LSS

The particle horizon (PH) is the portion of space from where light could have reached us.

The *last scattering surface* (LSS) is where the CMB comes from. Photons free stream after that.

\[ R_{\text{LSS}} \equiv L_0 = 14.4 \text{ Gpc} \]

If the global structure of space is smaller than the particle horizon, then we can (in principle) see it, and hence constrain it by observations.
Circles in the Sky

Main method — circles in the sky

LSS crosses with itself on circles if $2L_0 > L$. Looks for these patterns in the sky.

Cornish, Spergel, Starkman astro-ph/9801212
Limits

WMAP1 (Cornish, Spergel, Starkman, Komatsu, astro-ph/0310233)

\[ L > 24 \text{ Gpc} \]

WMAP7 (Bielewicz, Banday, arXiv:1012.3549)

\[ L > 27.9 \text{ Gpc} \]

Method does not constrain \( L > 2L_0 = 28.8 \text{ Gpc} \).

Can one can see beyond \( 2L_0 \)?

Turns out you can do better using the CMB.
COBE Temperature

COBE–DMR Map of CMB Anisotropy

North Galactic Hemisphere

South Galactic Hemisphere

$-100 \ \mu K$ $+100 \ \mu K$
WMAP9 Temperature
Planck Temperature
PLANCK'S POLARISATION OF THE COSMIC MICROWAVE BACKGROUND

Filtered at 5 degrees

Filtered at 20 arcminutes

Full sky map
Filtered at 5 degrees
Divide the sky into 12 pixels of equal solid angle, and then subdivide.

\[ N_{\text{pixel}} = 12 N_{\text{side}}^2 \]

with \( N_{\text{side}} = 1, 2, 4, \ldots \). Allows for a fast angular Fourier transform.
Temperature and Polarization

For $T$, subtract out the average temperature and the dipole, which gives our motion relative to the CMB.

Polarization:

$$E = \text{Re} \ E_0 e^{i \mathbf{k} \cdot \mathbf{r} - i \omega t}$$

For a wave travelling in the $z$ direction, let

$$|\psi\rangle = \begin{bmatrix} e_x e^{i \theta_x} \\ e_y e^{i \theta_y} \end{bmatrix}$$

so that

$$E_x = e_x \cos(\omega t - \theta_x)$$
$$E_y = e_y \cos(\omega t - \theta_y)$$
Polarization

The intensity matrix is
\[ \rho = |\psi\rangle \langle \psi| \]
\[ = \begin{bmatrix} \psi_x^2 & e_x e_y e^{i(\theta_x - \theta_y)} \\ e_x e_y e^{-i(\theta_x - \theta_y)} & \psi_y^2 \end{bmatrix} \]

Decompose the density matrix as
\[ \rho = \frac{1}{2} (a_0 + a \cdot \sigma) \]
\[ a^\mu = \text{Tr} \rho \sigma^\mu \]
\[ \sigma^0 = 1 \]

Stokes’ parameters defined by
\[ a^\mu = (I, U, V, Q) \]
\[ \rho = \frac{1}{2} \begin{bmatrix} I + Q & U - iV \\ U + iV & I - Q \end{bmatrix} \]
\[ I = e_x^2 + e_y^2 \]
\[ Q = e_x^2 - e_y^2 \]
\[ U = 2e_x e_y \cos(\theta_y - \theta_x) \]
\[ V = 2e_x e_y \sin(\theta_y - \theta_x) \]
Rotate the $x - y$ axes by $\theta$:

\[
I' = I \quad V' = V \quad \begin{bmatrix} Q' \\ U' \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} Q \\ U \end{bmatrix}
\]

\[
Q' \pm iU' = e^{\pm 2i\theta} (Q \pm iU)
\]

So can measure $Q$ and $U$ from difference in intensities for $x$ and $y$ polarization, and $x'$ and $y'$ polarization with $\theta = \pi/4$.

Acts like a helicity $\pm 2$ object because it is quadratic in the EM field.
Spherical harmonic decomposition:

\[ T(\hat{n}) = \sum_{lm} T_{lm} Y_{lm}(\hat{n}) \]

\[ Q(\hat{n}) \pm i U(\hat{n}) = \sum_{lm} (E_{lm} \pm i B_{lm}) \pm 2 Y_{lm}(\hat{n}) \]

where \( \pm 2 Y_{lm}(\hat{n}) \) are helicity spherical harmonics.

Under parity, \( E \rightarrow E \) and \( B \rightarrow -B \).

Figure 21: Examples of \( E \)-mode and \( B \)-mode patterns of polarization. Note that if reflected across a line going through the center the \( E \)-mode patterns are unchanged, while the positive and negative \( B \)-mode patterns get interchanged.

The angular power spectra are defined as before.

Figure 22 shows the latest measurement of the \( TE \) cross-correlation. The \( EE \) spectrum has now begun to be measured, but the errors are still large. So far there are only upper limits on the \( BB \) spectrum, but no detection.

The cosmological significance of the \( E/B \) decomposition of CMB polarization was realized by the authors of Refs. [31, 32], who proved the following remarkable facts:

i) scalar (density) perturbations create only \( E \)-modes and no \( B \)-modes.

ii) vector (vorticity) perturbations create mainly \( B \)-modes.

iii) tensor (gravitational wave) perturbations create both \( E \)-modes and \( B \)-modes.

The fact that scalars do not produce \( B \)-modes while tensors do is the basis for the statement that detection of \( B \)-modes is a smoking gun of tensor modes, and therefore of inflation.

However, vectors decay with the expansion of the universe and are therefore believed to be subdominant at recombination. We therefore do not consider them here.
Inflation

$T(\hat{n}, x)$: CMB temperature seen by an observer at $x$ in direction $\hat{n}$.

$$ T(\hat{n}, x) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} a_{lm}(x) Y_{lm}(\hat{n}) , $$

Fourier space temperature fluctuations $T(\hat{n}, k)$

$$ a_{lm}(x) = \int \frac{d^3k}{(2\pi)^3} e^{i k \cdot x} \int d\Omega \ Y_{lm}^*(\hat{n}) T(\hat{n}, k) . $$

Observed CMB fluctuations from $a_{lm}$ correlations:

$$ M_{l m' m'} \equiv \langle a_{lm}(x_0) a_{l' m'}^*(x_0) \rangle . $$

$$ T(k, k \cdot \hat{n}) = \frac{T(k, k \cdot \hat{n})}{\zeta(k)} \zeta(k) $$
Inflation

The correlations between temperature anisotropies in $k$-space are related to the initial matter power spectrum:

$$\langle T\left(k, \hat{n}\right) T^*\left(k', \hat{n}'\right) \rangle = \left(2\pi\right)^3 \delta^3 (k - k') P_\zeta(k) \frac{T(k, k \cdot \hat{n})}{\zeta(k)} \frac{T^*(k, k \cdot \hat{n}')}{\zeta^*(k)} ,$$

The matter power spectrum is defined by

$$\langle \zeta(k) \zeta^*(k') \rangle \equiv \left(2\pi\right)^3 \delta^3 (k - k') P_\zeta(k) .$$

Transfer function computed by CAMB:

$$\frac{T(k, k \cdot \hat{n})}{\zeta(k)}$$

does not depend on initial conditions, since equations are linear.
Inflation

Expanding \( T(k, k \cdot \hat{n}) \) into Legendre polynomials

\[
T(k, k \cdot \hat{n}) = \sum_l (-i)^l (2l + 1) P_l(\hat{k} \cdot \hat{n}) T_l(k),
\]
gives

\[
M_{lml'm'} = (4\pi)^2 (-i)^{l''} \int \frac{d^3 k}{(2\pi)^3} P_\zeta(k) \frac{T_l(k)}{\zeta(k)} \frac{T_{l''}(k)}{\zeta'(k)} Y_{lm}^*(\hat{k}) Y_{l'm'}(\hat{k}).
\]

Usual infinite universe case:

\[
M_{lml'm'} = \delta_{ll'} \delta_{mm'} C_l,
\]

with

\[
C_l = \frac{2}{\pi} \int dk \ k^2 P(k) \left| \frac{T_l(k)}{\zeta(k)} \right|^2.
\]
Planck 2015

\[ D_\ell = \frac{\ell(\ell + 1)C_\ell}{2\pi} \]
Cosmic Variance

Measure $a_{lm}$ and get

$$C_{l}^{\text{obs}} \equiv \frac{1}{2l + 1} \sum_{m} |a_{lm}^{\text{obs}}|^2$$

Each $a_{lm}$ like a measurement of $C_l$. Large cosmic variance for small $l$.

$$\langle C_{l}^{\text{obs}} \rangle = C_l$$

$$\langle (\Delta C_{l}^{\text{obs}})^2 \rangle = \frac{2C_l}{2l + 1}$$

$\langle \cdot \rangle$ is w.r.t. universes.
Finite Case

\[ \int \frac{d^3k}{(2\pi)^3} \to \frac{1}{L_1 L_2 L_3} \sum_k \] 

over the discrete \( k \) values

\[ k_1 = \frac{2\pi}{L_1} n_1, \quad k_2 = \frac{2\pi}{L_2} n_2, \quad k_3 = \frac{2\pi}{L_3} n_3, \quad n_i \in \mathbb{Z} \]

The derivation remains the same in the finite case except that the \( k \) integral must be replaced by the sum

\[ M_{lml'm'} = (4\pi)^2 (-i)^{l+l''} \frac{1}{L_1 L_2 L_3} \sum_k P(k) \frac{T_l(k) T_{l'}^*(k)}{\zeta(k) \zeta^*(k)} Y^*_{lm}(\hat{k}) Y_{l'm'}(\hat{k}). \]

CAMB evolution is local, due to CMB scattering, etc. and is unaffected.
Finite Case

Find

\[ 2 | l' - l \quad \quad \quad 4 | m' - m \]

Plot of the ratio of \( C_l \) for \( M_0 = \mathbb{T}^3 \) with \( L/L_0 = 1.8 \) (blue), \( M_1 = \mathbb{T}^2 \times \mathbb{R}^1 \) with \( L/L_0 = 1.9 \) (red), and \( M_3 = S^1 \times \mathbb{R}^2 \) with \( L/L_0 = 1.9 \) (green), to that for infinite space \( \mathbb{R}^3 \).
Inflation

Define the transfer functions

$$g_i^X(k) = \frac{X_i(k)}{\zeta(k)}, \quad X = T, E, B$$

Power spectrum:

$$M_{lm l' m'}^{XY} = (4\pi)^2 (-i)^l l' i'' \frac{1}{L_1 L_2 L_3} \sum_k P_\zeta(k) g_i^X(k) g_i^Y*(k) Y_{lm}(\hat{k}) Y_{l'm'}(\hat{k}).$$

and $TT$, $TE$, $EE$ and $BB$ are non-zero. $BB$ requires gravitational waves (tensor perturbations) in the initial conditions.

$$M^{TB} = 0 \quad M^{EB} = 0 \quad M^{BB} = 0$$
Likelihood

The WMAP and Planck collaborations make public the data in the form of maps in pixel space.

\[ m = (T, Q, U) \]

are the temperature and polarization maps treated as a column vector with index \( \hat{n}_i \) in pixel space.

Define \( S \) and \( N \) as the signal and noise correlation matrices, with indices in pixel space.

\( N \) is provided by the experimental collaborations

\( S \) is the theoretical prediction based on the cosmological model.
The likelihood is given by

\[ \mathcal{L}(m|S)dm = \exp \left[ -\frac{1}{2} m^t (S + N)^{-1} m \right] \frac{(2\pi)^{3n_p/2}}{\det(S + N)^{1/2}} dm \]

Spherical harmonic decomposition:

\[ T(\hat{n}) = \sum_{lm} T_{lm} Y_{lm}(\hat{n}) \]

\[ Q(\hat{n}) \pm i U(\hat{n}) = \sum_{lm} (E_{lm} \pm i B_{lm}) Y_{lm}(\hat{n}) \]

so we can compute pixel correlations in terms of \( M_{lm,l'm'}^{XY} \)

\[ \langle X(\hat{n}) Y(\hat{n}')^* \rangle \]

\( lm \leftrightarrow \hat{n} \) is like a change of basis with transformation \( Y_{lm}(\hat{n}) \)
Noise in low resolution temperature maps is negligible. Can separate out the temperature from the polarization maps.

\[
\tilde{E}_{lm} = E_{lm} - \left[ M^{ET} \left( M^{TT} \right)^{-1} T \right]_{lm}
\]

so that

\[
\langle \tilde{E}_{lm} T^*_{lm} \rangle = 0
\]

Define

\[
\tilde{Q}(\hat{n}) = \frac{1}{2} \sum_{lm} \tilde{E}_{lm} \left[ 2 Y_{lm}(\hat{n}) + -2 Y_{lm}(\hat{n}) \right] + i \tilde{B}_{lm} \left[ 2 Y_{lm}(\hat{n}) - -2 Y_{lm}(\hat{n}) \right]
\]

\[
i \tilde{U}(\hat{n}) = \frac{1}{2} \sum_{lm} \tilde{E}_{lm} \left[ 2 Y_{lm}(\hat{n}) - -2 Y_{lm}(\hat{n}) \right] + i \tilde{B}_{lm} \left[ 2 Y_{lm}(\hat{n}) + -2 Y_{lm}(\hat{n}) \right]
\]

\[
\langle \tilde{Q}(\hat{n}) T(\hat{n}') \rangle = 0 \quad \langle \tilde{U}(\hat{n}) T(\hat{n}') \rangle = 0
\]
Define $P$ to be the $Q, U$ part, and $T$ to be the temperature part. Then

$$
\mathcal{L}(m|S)dm = \frac{\exp \left[ -\frac{1}{2} \tilde{m}_P^T (\tilde{S}_P + N_P)^{-1} m_P \right]}{(2\pi)^{n_p} \det(\tilde{S}_P + N_P)^{1/2}} d\tilde{m}_P \frac{\exp \left[ -\frac{1}{2} \tilde{T}^T (S_T)^{-1} T \right]}{(2\pi)^{n_p/2} \det(S_T)^{1/2}} dT
$$

$\tilde{S}_P$ can be derived from

$$
\langle \tilde{E}_{lm} \tilde{E}_{lm}' \rangle = M_{lml'm'}^{E_E} = \left[ M^{EE} - M^{ET} \left( M^{TT} \right)^{-1} M^{TE} \right]_{lml'm'}
$$

Main difficulty is numerical: the maps have $N_p \sim 3072$ and so the covariance matrices are $3072 \times 3072$. Takes a lot of computing power.
Previous Results based on Temperature

COBE: de Oliviera Costa, Smoot, astro-ph/9412003:

\[ L > 4.32h^{-1}\text{Gpc for } M_0 \text{ (95\%)} \]

de Oliviera Costa, Smoot, Starobinsky, astro-ph/9510109:

\[ L > 3.0h^{-1}\text{Gpc for } M_{1,2} \text{ (95\%)} \]

WMAP1: Phillips, Kogut, astro-ph/0404400:

\[ M_0 : L > 1.2L_0(95\%), \quad L > 2.1L_0(68\%) \quad \text{best fit } L = 2.1L_0 \]

Kunz et al. astro-ph/0510164:

\[ M_0 : L > 19.3\text{Gpc} \quad \quad M_2 : L > 14.4\text{Gpc} \]
Data Used

WMAP temperature maps: Low resolution ILC, V, W, Q bands at $N_{\text{side}} = 16$ smoothed to 9.183° degrees and masked with Kp2. In pixel space, just drop the pixels corresponding to the mask. 1 $\mu$K white noise added to regularize the numerical inversion of the covariance matrix.

Polarization maps: Combination of Ka, Q, V bands at $N_{\text{side}} = 8$ masked with P06.

Analyze WMAP7 and WMAP9 data
CAMB used to calculate the transfer functions
WMAP likelihood code used
CAMB modified to use a sum on $k$, and WMAP modified to use the new covariance matrices.
Isotropy of space broken, so have to look at orientations of the compactification direction, in terms of Euler angles $\phi, \theta, \psi$.

Huge increase in computer time to map out these orientations. Some simplification using symmetries of a cube,

$$R(\phi, \theta, \psi) = R(g)R(\phi', \theta', \psi')$$

where $R(g)$ is a symmetry of $M$, e.g. a $\pi/2$ rotation about an axis of the cube.

Only studied all sides equal.
Variation with Orientation

\[-2 \ln \mathcal{L} = \chi^2 + \ln \det C / C_f + \ln \det (2\pi C_f)\]

Look at $M_0$:

**Det varies weakly**

$L/L_0 = 1.8$ with $\theta, \phi$ fixed.

**Likelihood varies strongly**

-  red is best fit direction, blue is random direction for $L/L_0 = 1.8$ with $\theta, \phi$ fixed.
- green is best fit direction for $L/L_0 = 2.2$. 

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Analysis

Make a scan over angle steps $0.05\pi$ and then minimize using MINUIT.

Use the Feldman-Cousins method (physics/9711021) to construct confidence intervals

1. For a point $P$ in parameter space, construct a set of simulations $S_i$ and find the best fit for each one.
2. Calculate likelihood ratios $\Delta \mathcal{L} = -2 \ln \mathcal{L}(S_i|P) + 2 \ln \mathcal{L}(S_i|P_{i,\text{best}})$
3. Find $\Delta \mathcal{L}_c$ such that a fraction $\alpha$ of simulations have $\Delta \mathcal{L} < \Delta \mathcal{L}_c$
4. For real data $D$ calculate $\Delta \mathcal{L}_D = -2 \ln \mathcal{L}(D|P) + 2 \ln \mathcal{L}(D|P_{D,\text{best}})$
5. Accept $P$ at confidence level $\alpha$ if $\Delta \mathcal{L}_D < \Delta \mathcal{L}_c$

Do 500 simulations for each size and topology, varying size in steps of $0.1L_0$. 
Results (Temp only)

\[ M_0 = T^3 \quad M_1 = T^2 \times R^1 \quad M_2 = S^1 \times R^2 \]

upper: WMAP7, lower: WMAP9. green triangles: ILC map, the blue squares: combined data from V, W, and Q maps.
Lower bounds on $L/L_0$:

<table>
<thead>
<tr>
<th>Map</th>
<th>$M_0 (68%)$</th>
<th>$M_0 (95%)$</th>
<th>$M_1 (68%)$</th>
<th>$M_1 (95%)$</th>
<th>$M_2 (68%)$</th>
<th>$M_2 (95%)$</th>
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</thead>
<tbody>
<tr>
<td>ILC (7)</td>
<td>1.71</td>
<td>1.50</td>
<td>1.49</td>
<td>1.40</td>
<td>1.49</td>
<td>1.11</td>
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<tr>
<td>VWQ (7)</td>
<td>1.71</td>
<td>1.50</td>
<td>1.48</td>
<td>1.38</td>
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<td>1.10</td>
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<tr>
<td>ILC (9)</td>
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<td>1.49</td>
<td>1.41</td>
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<tr>
<td>VWQ (9)</td>
<td>1.76</td>
<td>1.66</td>
<td>1.47</td>
<td>1.30</td>
<td>1.51</td>
<td>1.10</td>
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<tr>
<td>Planck</td>
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<td>1.42</td>
<td></td>
<td></td>
<td>1.00</td>
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Planck 2013: XXVI 1303.5086

<table>
<thead>
<tr>
<th>Topology</th>
<th>Map</th>
<th>$\Delta\mathcal{L}$</th>
<th>$L/L_0$</th>
<th>$\phi$</th>
<th>$\theta$</th>
<th>$\psi$</th>
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<tbody>
<tr>
<td>$M_0$</td>
<td>ILC (7)</td>
<td>18.89</td>
<td>2.0 ± 0.05</td>
<td>2.328 ± 0.036</td>
<td>2.512 ± 0.012</td>
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<td>ILC (9)</td>
<td>19.45</td>
<td>2.0 ± 0.05</td>
<td>2.330 ± 0.035</td>
<td>2.512 ± 0.012</td>
<td>0.380 ± 0.033</td>
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<tr>
<td>$M_1$</td>
<td>ILC (7)</td>
<td>19.30</td>
<td>1.9 ± 0.05</td>
<td>0.356 ± 0.023</td>
<td>0.932 ± 0.024</td>
<td>1.061 ± 0.020</td>
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<td>$M_2$</td>
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<td>ILC (9)</td>
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<td>1.9 ± 0.05</td>
<td>1.704 ± 0.016</td>
<td>2.166 ± 0.016</td>
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Best Fit
Temperature + Polarization

WMAP polarization maps are very noisy, and not much improvement. Did a forecast for Planck using 100 simulations

![Graph showing polarization maps and CL values](image)

Bounds go up $1.73 \rightarrow 1.92$ (68%) and $1.68 \rightarrow 1.89$ (95.5%). Detect $L/L_0 = 1.7$ at $3\sigma$ and $L/L_0 = 2.0$ at $2\sigma$.

Recent paper: Planck 1502.01593 consistent with these estimates.
Checks

Not related to the lowest multipoles. We took $M_{lml'm'}$ and replaced pieces by the infinite universe case. Seems that the signal depends on off-diagonal terms for $5 \leq \ell \leq 25$.

Not due to low $\ell$, i.e. not from the quadrupole or octupole.
Conclusions

- Tested for global topology using CMB data
- Rule out $L/L_0 \sim 1.1 - 1.7$
- An indication of a dip at around $L/L_0 \sim 2$
- Unlikely to get much better bounds
- Also tested for deviations from inflation, can put limits in a similar way.