Effective Field Theories

Lectures 1, 2

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04 Nov 2014 / Vienna

References

- AM, hep-ph/9606222
- I. Stewart, online MITx course 8.EFTx
- A. Pich, hep-ph/9806303

Outline

- Introduction
- Dimensional Analysis and Power Counting
- Examples
- Loops
- Decoupling
- Standard Model as an EFT

Basic Idea

Effective field theory ideas are "obvious," but non-trivial to actually use them correctly in quantum field theory.

- You can make quantitative predictions of observable phenomena without knowing everything.
- The computations have some small (non-zero) error.
- Can improve on the accuracy by adding a finite number of additional parameters, in a systematic way.
- Key concept is locality — as a result one can factorize quantities into some short distance parameters (coefficients in the Lagrangian), and long distance operator matrix elements.

Examples: H atom

Chemistry and atomic physics depend on the interactions of atoms.

The interaction Hamiltonian contains non-relativistic electrons and nuclei interacting via a Coulomb potential, plus electromagnetic radiation.

The only property of the nucleus we need is the electric charge Z.

The quark structure of the proton, weak interactions, GUTs, etc. are irrelevant.

Quantum Mehanics: H atom

A more accurate calculation includes recoil corrections and needs the reduced mass.

The fine structure needs m_e, m_p.

The Hyperfine interaction needs the proton magnetic moment μ_p

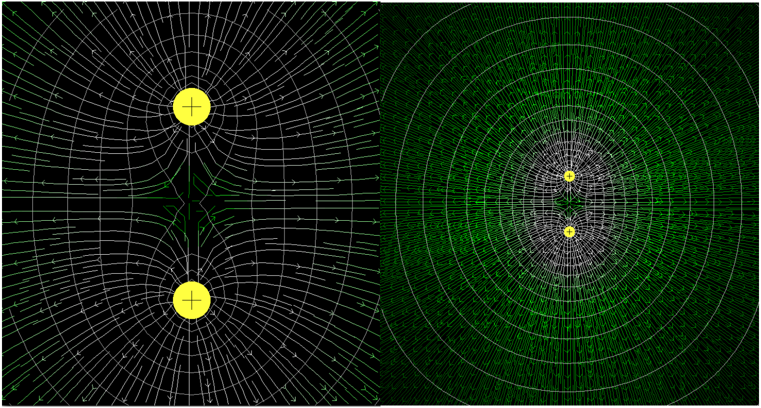
A more accurate calculation needs $g-2$ for the electron and QED radiative corrections.

Charge radius, ...

Weak interactions, ...

If one is interested in atomic parity violation, weak interactions are the leading contribution, and cannot be treated as a small correction.

Multipole Expansion



The field far away looks just like a point charge.

$$V(r) = \frac{1}{r} \sum c_{lm} Y_{lm}(\Omega) \left(\frac{a}{r}\right)^l$$

At the classical level: expand in a/r.

c_{lm} expected to be of order unity, once a^l has been factored out.

Need more multipoles for a better description of the field.

Effective theory is a local quantum field theory with a finite number of low energy parameters.

There is a systematic expansion in a small parameter like a/r for the multipole expansion. [called power counting]

All the non-trivial effects are due to quantum corrections, i.e. loops
At the classical level, just series expand.

Examples of EFT in High Energy Physics

In some cases, one can compute the EFT from a more fundamental theory (typically, if it is weakly coupled).

- The Fermi theory of weak interactions is an expansion in p/M_W, and can be computed from the $SU(2) \times U(1)$ electroweak theory in powers of $1/M_W$, $\alpha_s(M_W)$, $\alpha(M_W)$ and $\sin^2\theta$.
- The heavy quark Lagrangian (HQET) can be computed in powers of $\alpha_s(m_Q)$ and $1/m_Q$ from QCD.
- NRQCD/NRQED: Non-relativistic QCD/QED

Examples

Chiral perturbation theory: Describes the low energy interactions of mesons and baryons.

The full theory is QCD, but the relation between the two theories (and the degrees of freedom) is non-perturbative.

χPT has parameters that are fit to experiment. Has been enormously useful.

Standard Model — don't know the more fundamental theory, and we all hope there is one.

Can use EFT ideas to parameterize new physics in terms of a few operators in studying, for example, precision electroweak measurements.

Reasons for using EFT

- Every theory is an effective theory: Can compute in the standard model, even if there are new interactions at (not much) higher energies.
- Greatly simplifies the calculation by only including the relevant interactions: Gives an explicit power counting estimate for the interactions.
- Deal with only one scale at a time: For example the *B* meson decay rate depends on M_W, m_b and Λ_{QCD}, and one can get horribly complicated functions of the ratios of these scales. In an EFT, deal with only one scale at a time, so there are no functions, only constants.

- Makes symmetries manifest: QCD has spontaneously broken chiral symmetry, which is manifest in the chiral Lagrangian, and heavy quark spin-flavor symmetry which is manifest in HQET. These symmetries are only true for certain limits of QCD, and so are hidden in the QCD Lagrangian.

$$b\uparrow,\ b\downarrow, c\uparrow, c\downarrow$$

- Sum logs: Use renormalization group improved perturbation theory. The running of constants is not small, e.g.

$$\alpha_s(M_Z) \sim 0.118, \qquad \alpha_s(m_b) \sim 0.22.$$

Fixed order perturbation theory breaks down. Sum logs of the ratios of scales (such as M_Z/m_b).

- Efficient way to characterize new physics: Can include the effects of new physics in terms of higher dimension operators. All the information about the dynamics is encoded in the coefficients. [This also shows it is difficult to discover new physics using low-energy measurements.]

- Include non-perturbative effects: Can include Λ_{QCD}/m corrections in a systematic way through matrix elements of higher dimension operators. The perturbative corrections and power corrections are tied together. [Renormalons]

Dimensional Analysis

Effective Lagrangian (neglect topological terms)

$$L = \sum c_i O_i = \sum L_D$$

is a sum of local, gauge and Lorentz invariant operators.

The functional integral is

$$\int \mathcal{D}\phi \; e^{iS}$$

so S is dimensionless.

Kinetic terms:

$$S = \int \mathrm{d}^d x \, \bar{\psi} \, i \not{D} \, \psi, \qquad S = \int \mathrm{d}^d x \, \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi$$

so

$$0 = -d + 2 \, [\psi] + 1, \qquad 0 = -d + 2 \, [\phi] + 2$$

Dimensions given by

$$[\phi] = (d-2)/2, \qquad [\psi] = (d-1)/2, \qquad [D] = 1, \qquad [g A_\mu] = 1$$

Field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \ldots$ so A_μ has the same dimension as a scalar field.

$$[g] = 1 - (d-2)/2 = (4-d)/2$$

In $d = 4$,

$$[\phi] = 1, \qquad [\psi] = 3/2, \qquad [A_\mu] = 1, \qquad [D] = 1, \qquad [g] = 0$$

Only Lorentz invariant renormalizable interactions (with $D \le 4$) are

$$\begin{aligned}
D = 0: &\qquad 1 \\
D = 1: &\qquad \phi \\
D = 2: &\qquad \phi^2 \\
D = 3: &\qquad \phi^3, \bar\psi\psi \\
D = 4: &\qquad \phi\bar\psi\psi, \phi^4
\end{aligned}$$

and kinetic terms which include gauge interactions.

Renormalizable interactions have coefficients with mass dim ≥ 0.

In $d = 2$,

$$[\phi] = 0, \qquad [\psi] = 1/2, \qquad [A_\mu] = 0, \qquad [D] = 1, \qquad [g] = 1$$

so an arbitrary potential $V(\phi)$ is renormalizable. Also $\left(\bar\psi\psi\right)^2$ is renormalizable.

In $d = 6$,

$$[\phi] = 2, \qquad [\psi] = 5/2, \qquad [A_\mu] = 2, \qquad [D] = 1, \qquad [g] = -1$$

Only allowed interaction is ϕ^3.

What Fields to use for EFT?

Not always obvious: Low energy QCD described in terms of meson fields.

NRQCD/NRQED and SCET: Naive guess does not work. Need multiple gluon fields.

The Sine-Gordon model is the massive Thirring model.
Two theories in 1+1 dimensions

$$L = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{\alpha}{\beta^2}\cos\beta\phi, \qquad L = \bar\psi\left(i\partial\!\!\!/ - m\right)\psi - \frac{1}{2}g\left(\bar\psi\gamma^\mu\psi\right)^2,$$

$$\frac{\beta^2}{4\pi} = \frac{1}{1+g/\pi}.$$

Effective Lagrangian:

$$L_D = \sum_D \frac{O_D}{M^{D-d}}$$

so in $d = 4$,

$$L_{eft} = L_{D\leq 4} + \frac{O_5}{M} + \frac{O_6}{M^2} + \ldots$$

An infinite number of terms (and parameters)

Power Counting

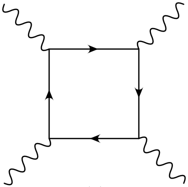
If one works at some typical momentum scale p, and neglects terms of dimension D and higher, then the error in the amplitudes is of order

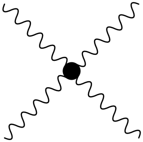
$$\left(\frac{p}{M}\right)^{D-4}$$

A non-renormalizable theory is just as good as a renormalizable theory for computations, provided one is satisfied with a finite accuracy.

Usual renormalizable case given by taking $M \to \infty$.

Photon-Photon Scattering


(a)


(b)

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\alpha^2}{m_e^4} \left[c_1 \left(F_{\mu\nu} F^{\mu\nu} \right)^2 + c_2 \left(F_{\mu\nu} \tilde{F}^{\mu\nu} \right)^2 \right].$$

(Terms with only three field strengths are forbidden by charge conjugation symmetry.)

e^4 from vertices, and $1/16\pi^2$ from the loop.

An explicit computation gives

$$c_1 = \frac{1}{90}, \qquad c_2 = \frac{7}{90}.$$

Scattering amplitude

$$A \sim \frac{\alpha^2 \omega^4}{m_e^4}$$

and

$$\sigma \sim \left(\frac{\alpha^2 \omega^4}{m_e^4} \right)^2 \frac{1}{\omega^2} \frac{1}{16\pi} \sim \frac{\alpha^4 \omega^6}{16\pi m_e^8} \times \frac{15568}{22275}$$

$$A \propto \frac{1}{m_e^4}$$

determined by the operator dimension.

Proton Decay

The lowest dimension operator in the standard model which violates baryon number is dimension 6. Natural explanation of baryon number conservation.

$$L \sim \frac{qqql}{M_G^2}$$

This gives the proton decay rate $p \to e^+\pi^0$ as

$$\Gamma \sim \frac{m_p^5}{16\pi M_G^4}$$

or

$$\tau \sim \left(\frac{M_G}{10^{15}\,\text{GeV}}\right)^4 \times 10^{30}\ \text{years}$$

Neutrino Masses

The lowest dimension operator in the standard model which gives a neutrino mass is dimension five,

$$\mathcal{L} \sim \frac{(H^\dagger L)(H^\dagger L)}{M_S}$$

and violates lepton number.

This gives a Majorana neutrino mass of ($v \sim 246$ GeV)

$$m_\nu \sim \frac{v^2}{M_S}$$

or a seesaw scale of 6×10^{15} GeV for $m_\nu \sim 10^{-2}$ eV.

Absolute scale of masses not known. Only Δm^2 measured.

Rayleigh Scattering

Scattering of light from atoms

$$L = \psi^\dagger \left(i\partial_t - \frac{p^2}{2M} \right) \psi + a_0^3 \, \psi^\dagger \psi \left(c_1 E^2 + c_2 B^2 \right)$$

$$A \sim c_i \, a_0^3 \, \omega^2$$

$$\sigma \propto a_0^6 \, \omega^4 .$$

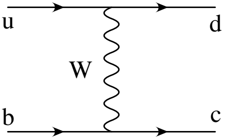
Scattering goes as the fourth power of the frequency, so blue light is scattered about 16 times mores strongly than red.

a_0^3 dimensional analysis.

Low energy weak interactions

W boson interacts with a current:

$$-\frac{ig}{\sqrt{2}}\; V_{ij}\, \bar{q}_i\, \gamma^\mu\, P_L\, q_j,$$



The tree-level amplitude is

$$A = \left(\frac{ig}{\sqrt{2}}\right)^2 V_{cb} V_{ud}^* \;\left(\bar{c}\,\gamma^\mu\, P_L\, b\right)\left(\bar{d}\,\gamma^\nu\, P_L\, u\right)\left(\frac{-i g_{\mu\nu}}{p^2 - M_W^2}\right),$$

For low momentum transfers, $p \ll M_W$:

$$\frac{1}{p^2 - M_W^2} = -\frac{1}{M_W^2}\left(1 + \frac{p^2}{M_W^2} + \frac{p^4}{M_W^4} + \ldots\right),$$

and retaining only a finite number of terms.

$$A = \frac{i}{M_W^2}\left(\frac{ig}{\sqrt{2}}\right)^2 V_{cb} V_{ud}^* \; (\bar{c}\,\gamma^\mu\, P_L\, b)\,(\bar{d}\,\gamma_\mu\, P_L\, u) + \mathcal{O}\left(\frac{1}{M_W^4}\right).$$

$$L = -\frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^* \; (\bar{c}\,\gamma^\mu\, P_L\, b)\,(\bar{d}\,\gamma_\mu\, P_L\, u) + \mathcal{O}\left(\frac{1}{M_W^4}\right),$$

$$\frac{G_F}{\sqrt{2}} \equiv \frac{g^2}{8M_W^2}.$$

Effective Lagrangian for μ decay

$$L = -\frac{4G_F}{\sqrt{2}}\ \left(\bar{e}\,\gamma^\mu\,P_L\,\nu_e\right)\left(\bar{\nu}_\mu\,\gamma^\mu\,P_L\,\mu\right) + \mathcal{O}\left(\frac{1}{M_W^4}\right),$$

Gives the standard result for the muon lifetime at lowest order,

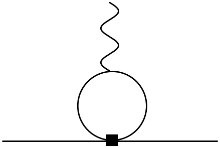
$$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3}\ f\left(\frac{m_e^2}{m_\mu^2}\right)$$

EFT gives the full dependence on low energy parameters.

$$f(\rho) = 1 - 8\rho + 8\rho^3 - \rho^4 - 12\rho^2 \ln\rho, \qquad \rho = \frac{m_e^2}{m_\mu^2}$$

The advantages of EFT show up in higher order calculations

Loops



Gives a contribution

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - M_W^2} \frac{1}{k^2 - m^2} \sim \frac{1}{M_W^2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2} \sim \frac{\Lambda^2}{M_W^2} \sim \mathcal{O}(1)$$

Similarly, a dimension eight operator has vertex k^2/M_W^4, and gives a contribution

$$I' \sim \frac{1}{M_W^4} \int d^4k \, \frac{k^2}{k^2 - m^2} \sim \frac{\Lambda^4}{M_W^4} \sim \mathcal{O}(1)$$

Would need to know the entire effective Lagrangian, since all terms are equally important. The reason for this breakdown is using a cutoff procedure with a dimensionful parameter Λ.

More generally, need to make sure that dimensionful parameters at the high scale do not occur in the numerator after evaluating Feynman diagrams.

In doing weak interactions, one should not have M_G or M_P appear in the numerator.

Need a renormalization scheme which maintains the power counting.

Dimensional Regularization

$d = 4 - 2\epsilon$:

$$\mu^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\left(k^2\right)^a}{\left(k^2 - M^2\right)^b}$$

$$= \frac{\mu^{2\epsilon}}{(4\pi)^{d/2}} \frac{(-1)^{a-b}\Gamma(d/2 + a)\Gamma(b - a - d/2)}{\Gamma(d/2)\Gamma(b)} \left(M^2\right)^{d/2+a+b}$$

Integral defined by analytic continuation.

Convert all integrals to this form using Feynman parameters for the denominator.

$\overline{\text{MS}}$

Need to use a mass independent subtraction scheme such as $\overline{\text{MS}}$:

μ can only occur in logarithms, so

$$\frac{1}{M_W^2}\mu^{2\epsilon}\int d^dk\,\frac{1}{k^2-m^2}\sim\frac{m^2}{M_W^2}\log\frac{\mu^2}{m^2},$$

$$\frac{1}{M_W^4}\mu^{2\epsilon}\int d^dk\,\frac{k^2}{k^2-m^2}\sim\frac{m^4}{M_W^4}\log\frac{\mu^2}{m^2},$$

Expanding $1/(k^2-M_W^2)$ in a power series ensures that there is no pole for $k\sim M_W$, and so M_W cannot appear in the numerator.

Dimensional regularization is like doing integrals using residues. Relevant scales given by poles of the denominator.

Power Counting Formula

Manifest power counting in p/M.

Loop graphs consistent with the power counting, since one can never get any M's in the numerator.

If the vertices have $1/M^a$, $1/M^b$, etc. then any amplitude (including loops) will have

$$\frac{1}{M^a}\frac{1}{M^b}\ldots = \frac{1}{M^{a+b+\ldots}}$$

Correct dimensions due to factors of the low scale in the numerator, represented generically by p. (Could be a mass)

Power Counting Formula

Only a finite number of terms to any given order in $1/M$.

Order $1/M$: L_5 at tree level

Order $1/M^2$: L_6 at tree level,
or loop graphs with two insertions of L_5.

General power counting result:

- you can count the powers of M.
- you can count powers of p

Power counting formula for χPT:

$$A \sim p^r, \qquad r = 2L + 2 + \sum_k n_k (k-2)$$

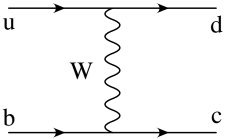
where n_k is the number of vertices of order p^k.

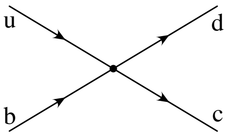
Toy Model (Integral)

Rather than do an explicit EFT example, look at a simple integral which illustrates what happens.

Tree-level:

$$-\frac{1}{k^2 - M^2} = c_1 \frac{1}{M^2} + c_2 \frac{k^2}{M^4} + \ldots \qquad\qquad c_i = 1$$



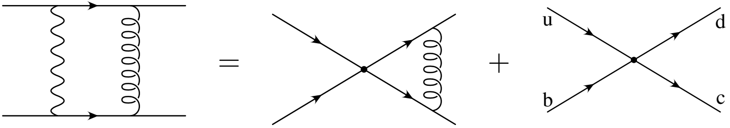


Toy Model (Integral)

One Loop:

$$I_F = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)(k^2 - M^2)}$$

Integral arises as a one-loop graph in a field theory, has some couplings in front.



Expanding does not commute with loop integration

Do the integral exactly in $d = 4 - 2\epsilon$ dimensions:

$$
\begin{aligned}
I_F &= \mu^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)(k^2 - M^2)} \\
&= \frac{i}{16\pi^2} \left[\frac{1}{\epsilon} + \frac{m^2 \log(m^2/\mu^2) - M^2 \log(M^2/\mu^2)}{M^2 - m^2} + 1 \right]
\end{aligned}
$$

Relatively simple because only 2 denominators. Three denominators gives Spence functions (dilogs).

Expand, do the integral term by term, and then sum up the result:

$$
\begin{aligned}
I_{\text{eft}} &= \mu^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)} \left[-\frac{1}{M^2} - \frac{k^2}{M^4} - \ldots \right] \\
&= \frac{i}{16\pi^2} \left[-\frac{1}{\epsilon} \frac{m^2}{M^2 - m^2} + \frac{m^2}{M^2 - m^2} \log \frac{m^2}{\mu^2} - \frac{m^2}{M^2 - m^2} \right]
\end{aligned}
$$

Points to Note

- Missing the non-analytic terms in M.
- The $1/\epsilon$ terms do not agree, they are cancelled by counterterms which differ in the full and EFT.
- The two theories have different anomalous dimensions.
- The term non-analytic in the IR scale, $\log(m^2)$ agrees in the two theories. This is the part which must be reproduced in the EFT.
- The analytic parts are local, and can be included as matching contributions to the Lagrangian.
- Sum $\log M^2/m^2$ terms using RG evolution.

No Non-Analytic Terms in M

$$\log \frac{m^2}{M^2} = \log \frac{m^2}{\mu^2} - \log \frac{M^2}{\mu^2}$$

$$I_F = \frac{i}{16\pi^2}\left[\frac{1}{\epsilon} + \frac{m^2}{M^2-m^2}\log\frac{m^2}{\mu^2} - \frac{M^2}{M^2-m^2}\log\frac{M^2}{\mu^2} + 1\right]$$

$$I_{eft} = \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon}\,\frac{m^2}{M^2-m^2} + \frac{m^2}{M^2-m^2}\log\frac{m^2}{\mu^2} - \frac{m^2}{M^2-m^2}\right]$$

$1/\epsilon$ terms are different

$$I_F = \frac{i}{16\pi^2}\left[\frac{1}{\epsilon} + \frac{m^2}{M^2-m^2}\log\frac{m^2}{\mu^2} - \frac{M^2}{M^2-m^2}\log\frac{M^2}{\mu^2} + 1\right]$$

$$I_{eft} = \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon}\,\frac{m^2}{M^2-m^2} + \frac{m^2}{M^2-m^2}\log\frac{m^2}{\mu^2} - \frac{m^2}{M^2-m^2}\right]$$

Each theory has its own counterterms (renormalization).

Different anomalous dimensions

Full theory:

$$\frac{1}{\epsilon}$$

The amplitude has an anomalous dimensions

EFT:

$$-\frac{1}{\epsilon}\,\frac{m^2}{M^2-m^2} = -\frac{1}{\epsilon}\,\frac{m^2}{M^2} - \frac{1}{\epsilon}\,\frac{m^4}{M^4} + \ldots$$

Each EFT order in $1/M$ has its own anomalous dimension.

Non-analytic Terms in m Agree

$$I_F = \frac{i}{16\pi^2}\left[\frac{1}{\epsilon} + \frac{m^2}{M^2-m^2}\log\frac{m^2}{\mu^2} - \frac{M^2}{M^2-m^2}\log\frac{M^2}{\mu^2} + 1\right]$$

$$I_{eft} = \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon}\,\frac{m^2}{M^2-m^2} + \frac{m^2}{M^2-m^2}\log\frac{m^2}{\mu^2} - \frac{m^2}{M^2-m^2}\right]$$

The EFT reproduces the complete low-energy limit of the full theory, including all the dependence on low energy (IR) scales.

If there are multiple IR scales $m_1, m_2, \ldots$, reproduces the complete m_i/m_j dependence.

Matching

Infinite parts cancelled by counterterms.

The difference between the finite parts of the two results is

$$
\begin{aligned}
I_F - I_{eft} &= \frac{i}{16\pi^2}\left[\log\frac{\mu^2}{M^2} + \frac{m^2\log(\mu^2/M^2)}{M^2-m^2} + \frac{M^2}{M^2-m^2}\right] \\
&= \frac{i}{16\pi^2}\left[\left(\log\frac{\mu^2}{M^2}+1\right) + \frac{m^2}{M^2}\left(\log\frac{\mu^2}{M^2}+1\right) + \ldots\right]
\end{aligned}
$$

The terms in parentheses are matching coefficients to a coefficient of order 1, order $1/M^2$, etc. They are analytic in m.

Note:

$$
\log\frac{m}{M} \to -\log\frac{M}{\mu} + \log\frac{m}{\mu}
$$

with the first part in the matching, and the second part in the EFT.

Summing Large Logs

The full theory has $\log M^2/m^2$ terms. At higher orders, get

$$\alpha_s^n \log^n M^2/m^2$$

- If $M \gg m$, perturbation theory breaks down as $\alpha_s \log M/m \sim 1$.
- Full theory involves two widely separated scales.
- Calculations become very difficult at higher orders.

Divide one calculation into two calculations, each involving one scale.

- Each calculation much easer since it involves a single scale
- For the matching to be accurate, want $\mu = M$.
- For the EFT to be accurate, want $\mu = m$.

- For the matching use $\mu = M$. $\log M/\mu$ small
- For the EFT calculation, pick $\mu = m$. $\log m/\mu$ small
- Use the EFT renormalization group to convert the Lagrangian from $\mu = M$ to $\mu = m$.
- RG perturbation theory valid as long as α_s small. Do not need $\alpha_s \log$ to be small.

RG Improved Perturbation Theory

$$
\begin{aligned}
\mu \frac{d}{d\mu} c &= \left[\frac{g^2}{16\pi^2} \gamma_0 + \mathcal{O}\left(\frac{g^4}{(16\pi^2)^2} \right) \right] c \\
\mu \frac{d}{d\mu} g &= -b_0 \frac{g^3}{16\pi^2} + \mathcal{O}\left(\frac{g^5}{(16\pi^2)^2} \right)
\end{aligned}
$$

with solution

$$
\frac{c(\mu_1)}{c(\mu_2)} = \left[\frac{\alpha(\mu_1)}{\alpha(\mu_2)} \right]^{-\gamma_0/(2b_0)}, \qquad \alpha = \frac{g^2}{4\pi}
$$

Correction can be big (factors of two or more), but perturbation theory is valid as long as $\alpha/(4\pi)$ is small.

Radiative Corrections: Operator Mixing

$$L = -\frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^* \; (c_1 O_1 + c_2 O_2)$$

$$\begin{aligned} O_1 &= (\bar{c}^\alpha \gamma^\mu P_L b_\alpha) \left(\bar{d}^\beta \gamma_\mu P_L u_\beta\right) & \qquad c_1 &= 1 + \mathcal{O}(\alpha_s) \\ O_2 &= (\bar{c}^\alpha \gamma^\mu P_L b_\beta) \left(\bar{d}^\beta \gamma_\mu P_L u_\alpha\right) & \qquad c_2 &= 0 + \mathcal{O}(\alpha_s) \end{aligned}$$

$$\mu \frac{d}{d\mu} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{\alpha_s}{4\pi} \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Can integrate by finding the eigenvalues and eigenvectors of γ.

Matching

$$I_M = \frac{i}{16\pi^2}\left[\left(\log\frac{\mu^2}{M^2}+1\right)+\frac{m^2}{M^2}\left(\log\frac{\mu^2}{M^2}+1\right)+\ldots\right]$$

We computed the matching from $I_F - I_{eft}$.

But there is an easier way which does not involve computing the two scale integral I_F.

I_M is analytic in m. Therefore, we can compute

$$I_F(m=0) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{1}{(k^2)(k^2-M^2)}$$

$$\frac{\partial I_F}{\partial m^2}(m=0) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{1}{(k^2)^2(k^2-M^2)}$$

Keep only the finite terms. More and more IR divergent.

Summary of EFT procedure

1. Write down the most general EFT Lagrangian with coefficients c_i.
2. Compute the EFT Lagrangian $c_i(\mu = M)$ by expanding in $1/M$ around $M = \infty$.
3. Compute the EFT coefficients δc_i by matching:
   Expand in powers of m around $m = 0$
4. RG improve the EFT result by running c_i from $\mu = M$ to $\mu = m$.
5. Compute EFT graphs in terms of $c_i(\mu = m)$ using L_{eft}.

We have added the two contributions from expanding in $1/M$ and expanding in m.

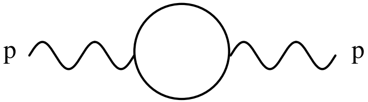
$$\frac{1}{k^2 - m^2} \frac{1}{k^2 - M^2}$$

This gives I_F, not $2I_F$.

Decoupling of Heavy Particles

Heavy particles decouple from low energy physics.
Obvious?

Not explicit in a mass independent scheme such as $\overline{MS}$.



$$i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\frac{1}{6\epsilon} - \int_0^1 dx\, x(1-x)\, \log\frac{m^2 - p^2 x(1-x)}{\mu^2}\right]$$

and we want to look at $p^2 \ll m^2$.

The graph is UV divergent.

Momentum Subtraction Scheme

Note that renormalization involves doing the integrals, and then performing a subtraction using some scheme to render the amplitudes finite.

Subtract the value of the graph at the Euclidean momentum point $p^2 = -M^2$ (the $1/\epsilon$ drops out)

$$-i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\int_0^1 dx\, x(1-x)\,\log\frac{m^2 - p^2 x(1-x)}{m^2 + M^2 x(1-x)}\right].$$

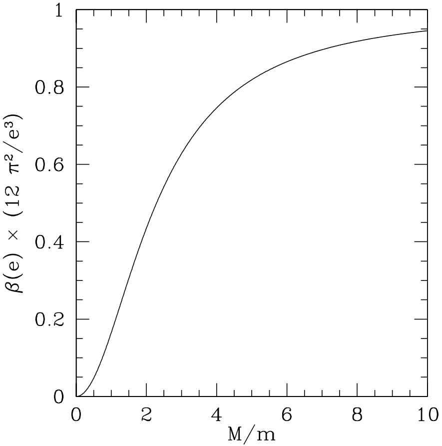
$$\begin{aligned}\beta(e) &= -\frac{e}{2} M \frac{\mathrm{d}}{\mathrm{d}M}\frac{e^2}{2\pi^2}\left[\int_0^1 dx\, x(1-x)\,\log\frac{m^2 - p^2 x(1-x)}{m^2 + M^2 x(1-x)}\right] \\ &= \frac{e^3}{2\pi^2}\int_0^1 dx\, x(1-x)\,\frac{M^2 x(1-x)}{m^2 + M^2 x(1-x)}.\end{aligned}$$

$m \ll M$ (light fermion):

$$\beta(e) \approx \frac{e^3}{2\pi^2} \int_0^1 dx\, x(1-x) = \frac{e^3}{12\pi^2}.$$

$M \ll m$ (heavy fermion):

$$\beta(e) \approx \frac{e^3}{2\pi^2} \int_0^1 dx\, x(1-x) \frac{M^2 x(1-x)}{m^2} = \frac{e^3}{60\pi^2} \frac{M^2}{m^2}.$$



cross-over

In the $\overline{\text{MS}}$ scheme:

$$-i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\int_0^1 dx\ x(1-x)\log\frac{m^2 - p^2 x(1-x)}{\mu^2}\right].$$

$$\begin{aligned}\beta(e) &= -\frac{e}{2}\mu\frac{d}{d\mu}\frac{e^2}{2\pi^2}\left[\int_0^1 dx\ x(1-x)\log\frac{m^2 - p^2 x(1-x)}{\mu^2}\right] \\ &= \frac{e^3}{2\pi^2}\int_0^1 dx\ x(1-x) = \frac{e^3}{12\pi^2},\end{aligned}$$

Is the first term in the β-function scheme independent?

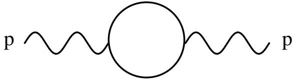
$$-i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\int_0^1 dx\ x(1-x)\log\frac{m^2}{\mu^2}\right],$$

Large logs cancel the wrong β-function contributions.

Explicitly integrate out heavy particles and go to an EFT.

Full theory: Includes fermion with mass m.

EFT: drop the heavy fermion (it no longer contributes to β)



Present in theory above m, but not in theory below m. Assume that $p \ll m$, so

$$
\begin{aligned}
&\int_0^1 dx\ x(1-x)\ \log \frac{m^2 - p^2 x(1-x)}{\mu^2} \\
=\ &\int_0^1 dx\ x(1-x)\ \left[\log \frac{m^2}{\mu^2} + \frac{p^2 x(1-x)}{m^2} + \ldots\right] \\
=\ &\frac{1}{6}\log \frac{m^2}{\mu^2} + \frac{p^2}{30 m^2} + \ldots
\end{aligned}
$$

So in theory above m:

$$i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\frac{1}{6\epsilon} - \frac{1}{6}\log\frac{m^2}{\mu^2} - \frac{p^2}{30m^2} + \ldots\right] + c.t.$$

Counterterm cancels $1/\epsilon$ term (and also contributes to the β function).

$$i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[-\frac{1}{6}\log\frac{m^2}{\mu^2} - \frac{p^2}{30m^2} + \ldots\right]$$

The log term gives

$$Z = 1 - \frac{e^2}{12\pi^2} \log \frac{m^2}{\mu^2}$$

so that in the effective theory,

$$\frac{1}{e_L^2(\mu)} = \frac{1}{e_H^2(\mu)} \left[1 - \frac{e_H^2(\mu)}{12\pi^2} \log \frac{m^2}{\mu^2} \right]$$

One usually integrates out heavy fermions at $\mu = m$, so that (at one loop), the coupling constant has no matching correction.

The p^2 term gives the dimension six operator

$$-\frac{1}{4}\frac{e^2}{2\pi^2}\frac{1}{30m^2}F_{\mu\nu}\partial^2 F^{\mu\nu}$$

and so on.

Even if the structure of the graphs is the same in the full and effective theories, one still needs to compute the difference to compute possible matching corrections, because the integrals need not have the same value. (next example)

This difference is independent of IR physics, since both theories have the same IR behavior, so the matching corrections are IR finite.

Note that nothing discontinuous is happening to any physical quantity at m.

We have changed our description of the theory from the full theory including m to an effective theory without m. By construction, the EFT gives the same amplitude as the full theory, so the amplitudes are continuous through m.

All m dependence in the effective theory is manifest through the explicit $1/m$ factors and through logarithmic dependence in the matching coefficients (in e_L).

Have to treat the p^2/m^2 term as a perturbation

Otherwise

$$\frac{1}{p^2 - e^2 p^4/(60\pi^2 m^2)}$$

has a pole at

$$p^2 = \frac{60\pi^2 m^2}{e^2} = \frac{15\pi m^2}{\alpha}$$

This new pole will violate the power counting. Also can get ghosts from quantizing a higher derivative theory.

SMEFT

A model independent way to include the effects of new physics. The assumption is that there are no new particles at the electroweak scale. This is what the data indicates.

Fields are the SM fields. $\langle H \rangle$ breaks $SU(2) \times U(1)$.

$$L = L_{SM} + \frac{L_5}{\Lambda} + \frac{L_6}{\Lambda^2} + \ldots$$

Need to include the $+\ldots$. Cannot just stop at a given dimension.

SMEFT: dim 5

$$L_5 = c_{rs} \left(H^{\dagger\, i}\, l_{i\alpha r} \right) \left(H^{\dagger\, j}\, l_{j\beta s} \right) \epsilon^{\alpha\beta}$$

i : $SU(2)$ index

α : Lorentz index

r : flavor index

Violates lepton number $\Delta L = 2$. The scale is the seesaw scale and is high. Not relevant for Higgs physics at LHC.

SMEFT: dim 6

Lots of operators at dimension six. 59 operators that preserve baryon number, and 4/5 that violate baryon number.

Buchmuller and Wyler, Nucl.Phys. B268 (1986) 621

Grzadkowski et al. JHEP 1010 (2010) 085

Flavor indices run over $n_g = 3$ values.
2499 baryon number conserving operators, including flavor indices.
1350 CP-even and 1149 CP-odd operators.

Notation:

$$\mathrm{L} : q_r,\ l_r, \qquad \mathrm{R} : u_r,\ d_r,\ e_r \qquad r = 1, \ldots, n_g = 3$$

$$H_j,\ \widetilde{H}_j = \epsilon_{ij} H^{\dagger\, j}$$

$$X_{\mu\nu} : G^A_{\mu\nu},\ W^I_{\mu\nu}, B_{\mu\nu}$$

Fierz Identities

Write down all possible gauge invariant operators of dimension 6.

Use Fierz identities:

$$\overline{\psi}_1 \gamma^\mu P_L \psi_2 \, \overline{\psi}_3 \gamma^\mu P_L \psi_4 = \overline{\psi}_3 \gamma^\mu P_L \psi_2 \, \overline{\psi}_1 \gamma^\mu P_L \psi_4$$

So in the four-quark operators:

$$\begin{aligned}
&\overline{q}_p^{\alpha i} \gamma^\mu q_{r\alpha i} \, \overline{q}_s^{\beta j} \gamma_\mu q_{t\beta j} \\
&\overline{q}_p^{\alpha i} \gamma^\mu q_{r\beta i} \, \overline{q}_s^{\beta j} \gamma_\mu q_{t\alpha j} \to \overline{q}_s^{\beta j} \gamma^\mu q_{r\beta i} \, \overline{q}_p^{\alpha i} \gamma_\mu q_{t\alpha j} \\
&\overline{q}_p^{\alpha i} \gamma^\mu q_{r\beta j} \, \overline{q}_s^{\beta j} \gamma_\mu q_{t\alpha i} \to \overline{q}_s^{\beta j} \gamma^\mu q_{r\beta j} \, \overline{q}_p^{\alpha i} \gamma_\mu q_{t\alpha i}
\end{aligned}$$

Two independent contractions:

$$\begin{aligned}
Q_{qq}^{(1)} &= \overline{q}_p \gamma^\mu q_r \, \overline{q}_s \gamma_\mu q_t \\
Q_{qq}^{(3)} &= \overline{q}_p \gamma^\mu \tau^I q_r \, \overline{q}_s \gamma_\mu \tau^I q_t
\end{aligned}$$

Equations of Motion

$$0 = D^2 H_a + m^2 H_a + 2\lambda(H^\dagger H)H_a + \overline{q}_b Y_u^\dagger u \epsilon_{ba} + \overline{d} Y_d q_a + \overline{e} Y_e l_a$$

$$i\not{D} q_a = Y_u^\dagger u \widetilde{H}_a + Y_d^\dagger d H_a \qquad i\not{D} u = Y_u q_a \widetilde{H}_a^\dagger$$

$$i\not{D} d = Y_d q_a H_a^\dagger \qquad i\not{D} l_a = Y_e^\dagger e H_a$$

$$i\not{D} e = Y_e l_a H_a^\dagger \qquad D^\alpha F_{\alpha\beta} = g j_\beta$$

$$\begin{aligned}
E_{H\Box} &= (H^\dagger H)(H^\dagger D^2 H + D^2 H^\dagger H) \\
&= -(H^\dagger H)(m^2 H^\dagger H + 2\lambda(H^\dagger H)^2 + \overline{q}_b Y_u^\dagger u \epsilon_{ba} H_a^\dagger + \overline{d} Y_d q_a H_a^\dagger \\
&+ \overline{e} Y_e l_a H_a^\dagger + h.c.) \\
&= -2m^2 (H^\dagger H)^2 - 4\lambda Q_H \\
&- \left([Y_u^\dagger]_{rs} [Q_{uH}]_{rs} + [Y_d^\dagger]_{rs} [Q_{dH}]_{rs} + [Y_e^\dagger]_{rs} [Q_{eH}]_{rs} + h.c. \right)
\end{aligned}$$

Equations of Motion

Can use classical equations of motion in a quantum field theory.

Green's functions can change, but the S-matrix does not.

There are no quantum corrections needed.

The two versions of the operator do not agree diagram by diagram. Only the total contribution to the S-matrix is unchanged.

Dimension Six Operators

Buchmüller and Wyler, Nucl.Phys. B268 (1986) 621

Grzadkowski et al. JHEP 1010 (2010) 085

X^3		H^6 and $H^4 D^2$		$\psi^2 H^3$	
Q_G	$f^{ABC} G_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu}$	Q_H	$(H^\dagger H)^3$	Q_{eH}	$(H^\dagger H)(\bar l_p e_r H)$
$Q_{\widetilde G}$	$f^{ABC} \widetilde G_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu}$	$Q_{H\Box}$	$(H^\dagger H)\Box(H^\dagger H)$	Q_{uH}	$(H^\dagger H)(\bar q_p u_r \widetilde H)$
Q_W	$\varepsilon^{IJK} W_\mu^{I\nu} W_\nu^{J\rho} W_\rho^{K\mu}$	Q_{HD}	$\left(H^\dagger D^\mu H\right)^\star \left(H^\dagger D_\mu H\right)$	Q_{dH}	$(H^\dagger H)(\bar q_p d_r H)$
$Q_{\widetilde W}$	$\varepsilon^{IJK} \widetilde W_\mu^{I\nu} W_\nu^{J\rho} W_\rho^{K\mu}$				
$X^2 H^2$		$\psi^2 X H$		$\psi^2 H^2 D$	
Q_{HG}	$H^\dagger H\, G^A_{\mu\nu} G^{A\mu\nu}$	Q_{eW}	$(\bar l_p \sigma^{\mu\nu} e_r)\tau^I H W^I_{\mu\nu}$	$Q_{Hl}^{(1)}$	$(H^\dagger i\overleftrightarrow{D}_\mu H)(\bar l_p \gamma^\mu l_r)$
$Q_{H\widetilde G}$	$H^\dagger H\, \widetilde G^A_{\mu\nu} G^{A\mu\nu}$	Q_{eB}	$(\bar l_p \sigma^{\mu\nu} e_r) H B_{\mu\nu}$	$Q_{Hl}^{(3)}$	$(H^\dagger i\overleftrightarrow{D}^I_\mu H)(\bar l_p \tau^I \gamma^\mu l_r)$
Q_{HW}	$H^\dagger H\, W^I_{\mu\nu} W^{I\mu\nu}$	Q_{uG}	$(\bar q_p \sigma^{\mu\nu} T^A u_r)\widetilde H\, G^A_{\mu\nu}$	Q_{He}	$(H^\dagger i\overleftrightarrow{D}_\mu H)(\bar e_p \gamma^\mu e_r)$
$Q_{H\widetilde W}$	$H^\dagger H\, \widetilde W^I_{\mu\nu} W^{I\mu\nu}$	Q_{uW}	$(\bar q_p \sigma^{\mu\nu} u_r)\tau^I \widetilde H\, W^I_{\mu\nu}$	$Q_{Hq}^{(1)}$	$(H^\dagger i\overleftrightarrow{D}_\mu H)(\bar q_p \gamma^\mu q_r)$
Q_{HB}	$H^\dagger H\, B_{\mu\nu} B^{\mu\nu}$	Q_{uB}	$(\bar q_p \sigma^{\mu\nu} u_r)\widetilde H\, B_{\mu\nu}$	$Q_{Hq}^{(3)}$	$(H^\dagger i\overleftrightarrow{D}^I_\mu H)(\bar q_p \tau^I \gamma^\mu q_r)$
$Q_{H\widetilde B}$	$H^\dagger H\, \widetilde B_{\mu\nu} B^{\mu\nu}$	Q_{dG}	$(\bar q_p \sigma^{\mu\nu} T^A d_r) H\, G^A_{\mu\nu}$	Q_{Hu}	$(H^\dagger i\overleftrightarrow{D}_\mu H)(\bar u_p \gamma^\mu u_r)$
Q_{HWB}	$H^\dagger \tau^I H\, W^I_{\mu\nu} B^{\mu\nu}$	Q_{dW}	$(\bar q_p \sigma^{\mu\nu} d_r)\tau^I H\, W^I_{\mu\nu}$	Q_{Hd}	$(H^\dagger i\overleftrightarrow{D}_\mu H)(\bar d_p \gamma^\mu d_r)$
$Q_{H\widetilde W B}$	$H^\dagger \tau^I H\, \widetilde W^I_{\mu\nu} B^{\mu\nu}$	Q_{dB}	$(\bar q_p \sigma^{\mu\nu} d_r) H\, B_{\mu\nu}$	Q_{Hud}	$i(\widetilde H^\dagger D_\mu H)(\bar u_p \gamma^\mu d_r)$

$(\bar{L}L)(\bar{L}L)$		$(\bar{R}R)(\bar{R}R)$		$(\bar{L}L)(\bar{R}R)$	
Q_{ll}	$(\bar{l}_p\gamma_\mu l_r)(\bar{l}_s\gamma^\mu l_t)$	Q_{ee}	$(\bar{e}_p\gamma_\mu e_r)(\bar{e}_s\gamma^\mu e_t)$	Q_{le}	$(\bar{l}_p\gamma_\mu l_r)(\bar{e}_s\gamma^\mu e_t)$
$Q_{qq}^{(1)}$	$(\bar{q}_p\gamma_\mu q_r)(\bar{q}_s\gamma^\mu q_t)$	Q_{uu}	$(\bar{u}_p\gamma_\mu u_r)(\bar{u}_s\gamma^\mu u_t)$	Q_{lu}	$(\bar{l}_p\gamma_\mu l_r)(\bar{u}_s\gamma^\mu u_t)$
$Q_{qq}^{(3)}$	$(\bar{q}_p\gamma_\mu \tau^I q_r)(\bar{q}_s\gamma^\mu \tau^I q_t)$	Q_{dd}	$(\bar{d}_p\gamma_\mu d_r)(\bar{d}_s\gamma^\mu d_t)$	Q_{ld}	$(\bar{l}_p\gamma_\mu l_r)(\bar{d}_s\gamma^\mu d_t)$
$Q_{lq}^{(1)}$	$(\bar{l}_p\gamma_\mu l_r)(\bar{q}_s\gamma^\mu q_t)$	Q_{eu}	$(\bar{e}_p\gamma_\mu e_r)(\bar{u}_s\gamma^\mu u_t)$	Q_{qe}	$(\bar{q}_p\gamma_\mu q_r)(\bar{e}_s\gamma^\mu e_t)$
$Q_{lq}^{(3)}$	$(\bar{l}_p\gamma_\mu \tau^I l_r)(\bar{q}_s\gamma^\mu \tau^I q_t)$	Q_{ed}	$(\bar{e}_p\gamma_\mu e_r)(\bar{d}_s\gamma^\mu d_t)$	$Q_{qu}^{(1)}$	$(\bar{q}_p\gamma_\mu q_r)(\bar{u}_s\gamma^\mu u_t)$
		$Q_{ud}^{(1)}$	$(\bar{u}_p\gamma_\mu u_r)(\bar{d}_s\gamma^\mu d_t)$	$Q_{qu}^{(8)}$	$(\bar{q}_p\gamma_\mu T^A q_r)(\bar{u}_s\gamma^\mu T^A u_t)$
		$Q_{ud}^{(8)}$	$(\bar{u}_p\gamma_\mu T^A u_r)(\bar{d}_s\gamma^\mu T^A d_t)$	$Q_{qd}^{(1)}$	$(\bar{q}_p\gamma_\mu q_r)(\bar{d}_s\gamma^\mu d_t)$
				$Q_{qd}^{(8)}$	$(\bar{q}_p\gamma_\mu T^A q_r)(\bar{d}_s\gamma^\mu T^A d_t)$

$(\bar{L}R)(\bar{R}L)$ and $(\bar{L}R)(\bar{L}R)$		*B*-violating	
Q_{ledq}	$(\bar{l}_p^j e_r)(\bar{d}_s q_t^j)$	Q_{duq}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[(d_p^\alpha)^T C u_r^\beta\right]\left[(q_s^{\gamma j})^T C l_t^k\right]$
$Q_{quqd}^{(1)}$	$(\bar{q}_p^j u_r)\varepsilon_{jk}(\bar{q}_s^k d_t)$	Q_{qqu}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(u_s^\gamma)^T C e_t\right]$
$Q_{quqd}^{(8)}$	$(\bar{q}_p^j T^A u_r)\varepsilon_{jk}(\bar{q}_s^k T^A d_t)$	$Q_{qqq}^{(1)}$	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\varepsilon_{mn}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(q_s^{\gamma m})^T C l_t^n\right]$
$Q_{lequ}^{(1)}$	$(\bar{l}_p^j e_r)\varepsilon_{jk}(\bar{q}_s^k u_t)$	$Q_{qqq}^{(3)}$	$\varepsilon^{\alpha\beta\gamma}(\tau^I\varepsilon)_{jk}(\tau^I\varepsilon)_{mn}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(q_s^{\gamma m})^T C l_t^n\right]$
$Q_{lequ}^{(3)}$	$(\bar{l}_p^j \sigma_{\mu\nu} e_r)\varepsilon_{jk}(\bar{q}_s^k \sigma^{\mu\nu} u_t)$	Q_{duu}	$\varepsilon^{\alpha\beta\gamma}\left[(d_p^\alpha)^T C u_r^\beta\right]\left[(u_s^\gamma)^T C e_t\right]$

2499 independent coefficients: 1350 *CP*-even and 1149 *CP*-odd. Over 150 operators if you break them up into flavor representations.

Eight classes: X^3, H^6, $H^4 D^2$, $X^2 H^2$, $\psi^2 H^3$, $\psi^2 X H$, $\psi^2 H^2 D$, ψ^4.

In the broken phase:

$$H = \begin{bmatrix} \varphi^+ \\ \frac{1}{\sqrt{2}}\left(v + h + \varphi^0\right) \end{bmatrix}$$

$$X^2 H^2 : \quad h \to \gamma\gamma, h \to gg$$

$$\psi^2 XH : \quad \bar{l}_r \, \sigma_{\mu\nu} \, e_s \, F^{\mu\nu}$$

$$\psi^2 H^3 : \frac{\text{higgs coupling}}{\text{mass}} = \frac{3}{1}$$