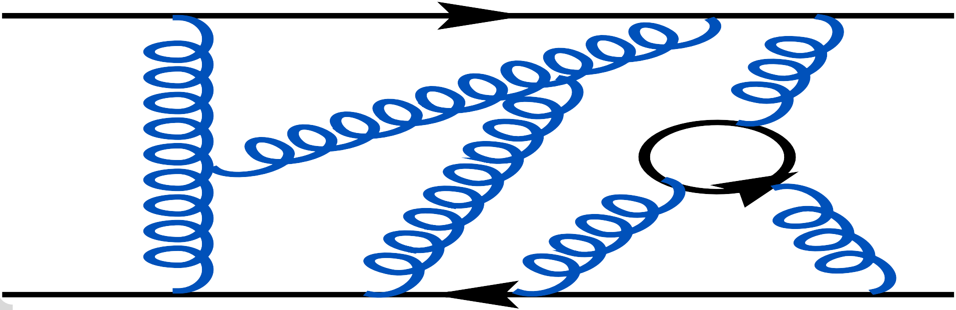
Top and bottom at threshold

Matthias Steinhauser | TTP Karlsruhe

Vienna, January 27, 2015



Outline

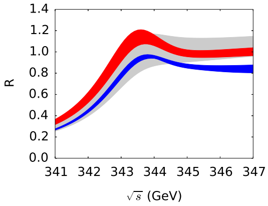
- Introduction
- a_3 to 3 loops
- c_v to 3 loops
- $\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$ to NNNLO
- positronium HFS
- Summary

Physical quantities

- energy levels and wave function of bound states
- $E_{q\bar{q}}$ [NNNLO: Penin,Steinhauser'02; Kiyo,Sumino'03; Beneke,Kiyo,Schuller'05]
  - $M_{\Upsilon(1S)} = 2m_b + E_{b\bar{b}}(n=1)$ ➪ m_b
  - $E_{\rm res} = 2m_t + E_{t\bar{t}}(n=1) + \delta^{\Gamma_t} E_{\rm res}$ ➪ m_t
- Υ sum rules ➪ m_b

  [Penin,Pivovarov'98; Hoang'98; Melnikov,Yelkhovsky'98, . . . , Penin,Zerf'14, Beneke,Maier,Piclum,Rauh'14]
- $\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$, NNNLO
- $\sigma(e^+e^- \to t\bar{t})$, NNNLO

  NNLL [Pineda,Signer'07; Hoang,Stahlhofen'13,. . .]



- Comparison of pQCD and LQCD

  [Necco,Sommer'01,Pineda'02,Sumino'05, . . . , Brambilla,Vairo,Garcia i Tormo,Soto'09, Donnellan,Knechtli,Leder,Sommer'10]

Framework: potential NRQCD

scales: mass, m: hard $\gg$ momentum, mv: soft $\gg$ energy, mv^2: ultrasoft $\gg$ $\Lambda_{\rm QCD}$

bottom:	5 GeV	2.5 GeV	0.5 GeV
top:	175 GeV	30 GeV	5 GeV

QCD ➪ NRQCD ➪ pNRQCD (potential non-relativistic QCD)

↑ $m \gg mv\,, mv^2$

[Caswell,Lepage'86;

Bodwin,Braaten,Lepage'95]

↑ integrate out all non-physical degrees of freedom

[Beneke,Smirnov'97; Pineda,Soto'98; Brambilla,Pineda,Soto,Vairo'00]

[alternative formulation: velocity NRQCD [Luke,Manohar,Rothstein'00; Hoang,Stewart'03]]

Effective Hamiltonian to N^{3}LO

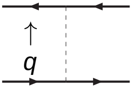
[Gupta,Radford'81,. . . ,Manohar'97,. . . ,Kniehl,Penin,Smirnov,Steinhauser'02,. . . ,Beneke,Kiyo,Schuller'13]

$$\mathcal{H} = (2\pi)^3\delta(\vec{q})\left(\frac{\vec{p}^{\,2}}{m} - \frac{\vec{p}^{\,4}}{4m^3}\right) + C_c(\alpha_s)V_C(|\vec{q}|) + C_{1/m}(\alpha_s)V_{1/m}(|\vec{q}|)$$
$$+\frac{\pi C_F\alpha_s(\mu)}{m^2}\left[C_\delta(\alpha_s) + C_p(\alpha_s)\frac{\vec{p}^{\,2}+\vec{p}^{\,\prime 2}}{2\vec{q}^{\,2}} + C_s(\alpha_s)\vec{S}^{\,2}\right]$$

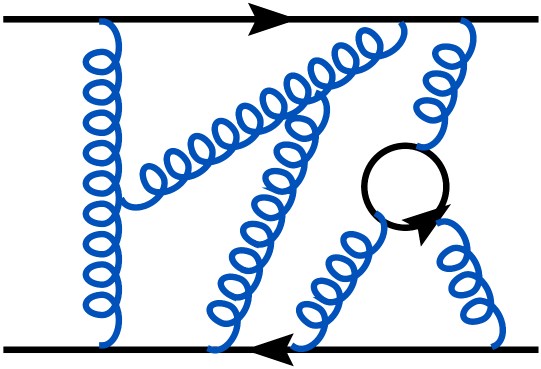
Static potential:	$V_C(\lvert\vec{q}\rvert) = -\frac{4\pi C_F\alpha_s(\lvert\vec{q}\rvert)}{\vec{q}^2}$	C_c	3 loops
$1/m$ potential:	$V_{1/m}(\lvert\vec{q}\rvert) = \frac{\pi^2 C_F\alpha_s^2(\lvert\vec{q}\rvert)}{m\lvert\vec{q}\rvert}$	$C_{1/m}$	2 loops
"Breit" potential:	$\propto 1/m^2$	$C_{\delta,p,s}$	1 loop

pNRQCD has dynamical degrees of freedom:
potential quarks and ultrasoft gluons

$\vec{q} = \vec{p}^{\,\prime} - \vec{p}$

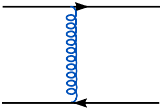


Static potential



Static potential

$$V_C(\mu = |\vec{q}\,|) \quad = \quad -\frac{4\pi C_F \alpha_s(|\vec{q}\,|)}{|\vec{q}\,|^2}\Bigg[1$$



[Appelquist,Politzer'75,Susskind'77]

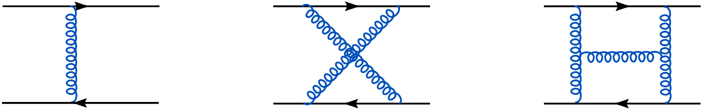
Static potential

$$V_C(\mu = |\vec{q}\,|) \quad = \quad -\frac{4\pi C_F \alpha_s(|\vec{q}\,|)}{|\vec{q}\,|^2}\left[1 + \frac{\alpha_s(|\vec{q}\,|)}{4\pi} a_1\right.$$

[Appelquist,Politzer'75,Susskind'77] [Fischler'77;Biloire'80]

Static potential

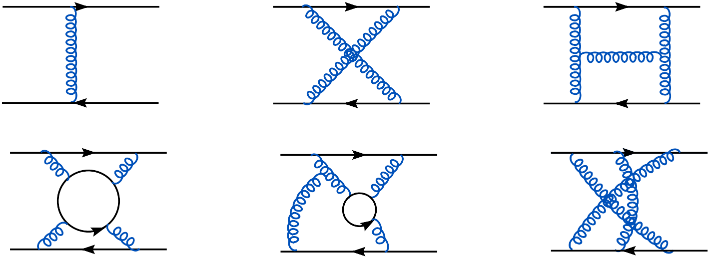
$$V_C(\mu = |\vec{q}\,|) \;=\; -\frac{4\pi C_F \alpha_s(|\vec{q}\,|)}{|\vec{q}\,|^2}\left[1 + \frac{\alpha_s(|\vec{q}\,|)}{4\pi}a_1 + \left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^2 a_2\right.$$



[Appelquist,Politzer'75,Susskind'77] [Fischler'77;Biloire'80] [Peter'96;Schröder'98]

Static potential

$$V_C(\mu = |\vec{q}\,|) = -\frac{4\pi C_F \alpha_s(|\vec{q}\,|)}{|\vec{q}\,|^2}\Bigg[1 + \frac{\alpha_s(|\vec{q}\,|)}{4\pi}a_1 + \left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^2 a_2 + \left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^3\left(a_3 + 8\pi^2 C_A^3 \ln\frac{\mu^2}{\vec{q}^{\,2}}\right) + \cdots\Bigg]$$



[Appelquist,Politzer'75,Susskind'77] [Fischler'77;Biloire'80] [Peter'96;Schröder'98]

[Smirnov,Smirnov,Steinhauser'08; Smirnov,Smirnov,Steinhauser'09; Anzai,Kiyo,Sumino'09]

Static potential

$$V_C(\mu = |\vec{q}\,|) \quad = \quad -\frac{4\pi C_F \alpha_s(|\vec{q}\,|)}{|\vec{q}\,|^2}\left[1 + \frac{\alpha_s(|\vec{q}\,|)}{4\pi}a_1 + \left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^2 a_2 \right.$$
$$\left. + \left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^3 \left(a_3 + 8\pi^2 C_A^3 \left(\frac{1}{3\epsilon} + \ln\frac{\mu^2}{\vec{q}^{\,2}}\right)\right) + \cdots\right]$$

- IR divergence [Appelquist,Dine,Muzinich'77]
  (in "naive perturbation theory")
- pNRQCD: ultra-soft contribution:
  (us)-gluons and $(Q\bar{Q})$ bound states as dynamical degrees of freedom
  ➪ IR finite result [Brambilla,Pineda,Soto,Vairo'99; Kniehl,Penin,Smirnov,Steinhauser'02]
- Finite physical quantities:
  - $V_{\rm QCD}(r) = V_{\rm QCD}^{\rm pert} + \delta V_{\rm QCD}^{\rm US}$ "static energy" ($\to$ lattice)
  - energy levels of $(q\bar{q})$ system: $E_{q\bar{q}} = \langle n|\mathcal{H}|n\rangle + \delta E_{q\bar{q}}^{\rm US}$
- H.O. log-contributions to V: [Pineda,Soto'00; Brambilla,Garcia i Tormo,Soto,Vairo'07]

Numerical results

$$
\begin{aligned}
V_C(|\vec{q}\,|) \quad = \quad & -\frac{4\pi C_F \alpha_s(|\vec{q}\,|)}{\vec{q}^{\,2}} \Bigg[1 + \frac{\alpha_s}{\pi} \left(2.5833 - 0.2778 n_l\right) \\
& + \left(\frac{\alpha_s}{\pi}\right)^2 \left(28.5468 - 4.1471 n_l + 0.0772 n_l^2\right) \\
& + \left(\frac{\alpha_s}{\pi}\right)^3 \left(209.884(1) - 51.4048 n_l + 2.9061 n_l^2 - 0.0214 n_l^3\right) + \cdots \Bigg]
\end{aligned}
$$

	n_l	$\alpha_s^{(n_l)}$	1 loop	2 loop	3 loop
charm	3	0.40	0.2228	0.2723	0.1677
bottom	4	0.25	0.1172	0.08354	0.02489
top	5	0.15	0.05703	0.02220	0.002485

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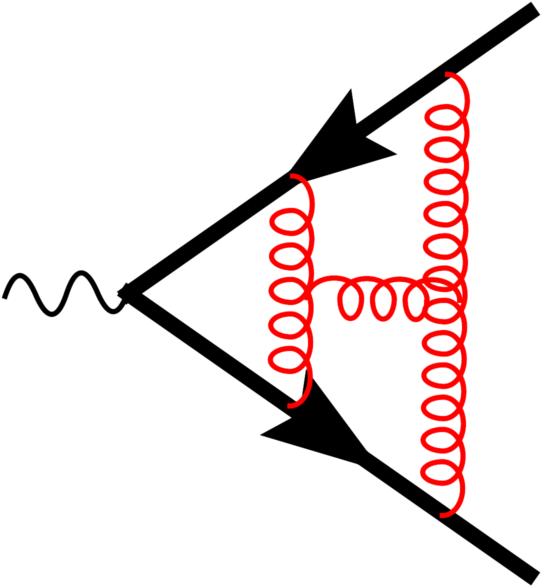
static potential for other colour configurations: [Kniehl,Penin,Schroder,Smirnov,Steinhauser’05; Anzai,Kiyo,Sumino’10; Collet,Steinhauser’11; Prausa,Steinhauser’13; Anzai,Prausa,Smirnov,Smirnov,Steinhauser’13]



c_v to 3 loops

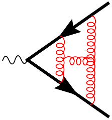
QCD $\longrightarrow$ NRQCD

$$j_v^\mu = \bar{Q}\gamma^\mu Q \quad \longrightarrow \quad \tilde{j}^i = \phi^\dagger \sigma^i \chi$$

$$j_v^i \quad = \quad c_v(\mu)\,\tilde{j}^i + \frac{d_v(\mu)}{6m_Q^2}\phi^\dagger\sigma^i \vec{D}^2\chi + \ldots$$

$$Z_2\Gamma_v = c_v \tilde{Z}_2 \tilde{Z}_v^{-1}\tilde{\Gamma}_v + \ldots$$

Γ_v:



$\tilde{\Gamma}_v \equiv 1 \quad \tilde{Z}_2 \equiv 1$

Z_2: [Melnikov,v.Ritbergen'00] [Marquard,Mihaila,Piclum,Steinhauser'07]

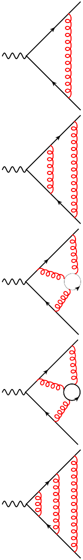
$\tilde{Z}_v = 1 + \mathcal{O}(\alpha_s^2)$

[Beneke,Signer,Smirnov'98; Kniehl,Penin,Steinhauser,Smirnov'03; Marquard,Piclum,Seidel,Steinhauser'06; Beneke,Kiyo,Penin'07]

($\overline{\mathrm{MS}}$ scheme)

c_V

$$c_V = 1 + \frac{\alpha_s}{\pi} c_V^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 c_V^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 c_V^{(3)} + \mathcal{O}(\alpha_s^4)$$



$c_V^{(1)}$ [Källen, Sarby'55]

$c_V^{(2)}$ [Czarnecki,Melnikov'97; Beneke,Signer,Smirnov'97]

$c_V^{(3),n_l}$ [Marquard,Piclum,Seidel,Steinhauser'06]

$c_V^{(3),n_h}$ [Marquard,Piclum,Seidel,Steinhauser'08]

$c_V^{(3)}$ [Marquard,Piclum,Seidel,Steinhauser'14]

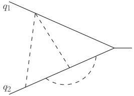
massive vertices

on-shell quarks: $q_1^2 = q_2^2 = M_Q^2$

$(q_1 + q_2)^2 = 4M_Q^2$

Some technical details ...

- automatic generation of Feynman diagrams, apply projector, reduce scalar products in numerator, ... ➪ (many) scalar integrals
- reduction to $\approx$ 100 master integrals with CRUSHER [Marquard,Seidel]
- compute MIs with FIESTA [Smirnov,Tentyukov'08; Smirnov,Smirnov,Tentyukov'11; Smirnov'13]



$$= \frac{e^{3\epsilon\gamma_E}}{m_Q^4}\left(\frac{\mu^2}{m_Q^2}\right)^{3\epsilon}$$

$$\left(\frac{0.411236(3)}{\epsilon^2} + \frac{3.4860(1)}{\epsilon} + 34.520(2) + 339.68(4)\epsilon + \mathcal{O}(\epsilon^2)\right)$$

- simpler integrals also known analytically
- add uncertainties of individual integrals quadratically
- multiply final uncertainty by factor 5 ("5σ")

Checks

- finiteness
- n_l contribution
- gauge parameter independence (ξ^1)
- change basis of MIs

	default basis	alternative basis
c_{FFF}	$36.55(0.11)$	$36.61(2.93)$
c_{FFA}	$-188.10(0.17)$	$-188.04(2.91)$
c_{FAA}	$-97.81(0.08)$	$-97.76(2.05)$
$c_v^{(3)}\ (n_l = 4)$	$-1621.7(0.4)$	$-1621(23)$
$c_v^{(3)}\ (n_l = 5)$	$-1508.4(0.4)$	$-1507(23)$

$\tilde{Z}_v$

$$
\begin{aligned}
\tilde{Z}_v \;=\;& 1 + \left(\frac{\alpha_s^{(n_l)}(\mu)}{\pi}\right)^2 \frac{C_F\pi^2}{\epsilon}\left(\frac{1}{12}C_F + \frac{1}{8}C_A\right) + \left(\frac{\alpha_s^{(n_l)}(\mu)}{\pi}\right)^3 C_F\pi^2 \\
&\times \left\{ C_F^2\left[\frac{5}{144\epsilon^2} + \left(\frac{43}{144} - \frac{1}{2}\ln 2 + \frac{5}{48}L_\mu\right)\frac{1}{\epsilon}\right]\right. \\
&+ C_F C_A\left[\frac{1}{864\epsilon^2} + \left(\frac{113}{324} + \frac{1}{4}\ln 2 + \frac{5}{32}L_\mu\right)\frac{1}{\epsilon}\right] \\
&+ C_A^2\left[-\frac{1}{16\epsilon^2} + \left(\frac{2}{27} + \frac{1}{4}\ln 2 + \frac{1}{24}L_\mu\right)\frac{1}{\epsilon}\right] \\
&+ Tn_l\left[C_F\left(\frac{1}{54\epsilon^2} - \frac{25}{324\epsilon}\right) + C_A\left(\frac{1}{36\epsilon^2} - \frac{37}{432\epsilon}\right)\right] \\
&\left. + C_F T n_h \frac{1}{60\epsilon}\right\} + \mathcal{O}(\alpha_s^4) \qquad\qquad L_\mu = \ln\frac{\mu^2}{m_Q^2}
\end{aligned}
$$

- analytic result from [Beneke,Signer,Smirnov'98; Kniehl,Penin,Smirnov,Steinhauser'02; Marquard,Piclum,Seidel,Steinhauser'06; Beneke,Kiyo,Penin'07]
- (numerical) agreement better than 1%

Results for c_v

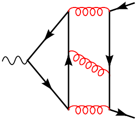
[Marquard,Piclum,Seidel,Steinhauser'14]

$$c_v(\mu = m_Q) = 1 - 2.67\frac{\alpha_s}{\pi} + [-44.55 + 0.41 n_l]\left(\frac{\alpha_s}{\pi}\right)^2 + \left[-2091(2) + 120.66 n_l - 0.82 n_l^2\right]\left(\frac{\alpha_s}{\pi}\right)^3 + \text{singlet terms}$$

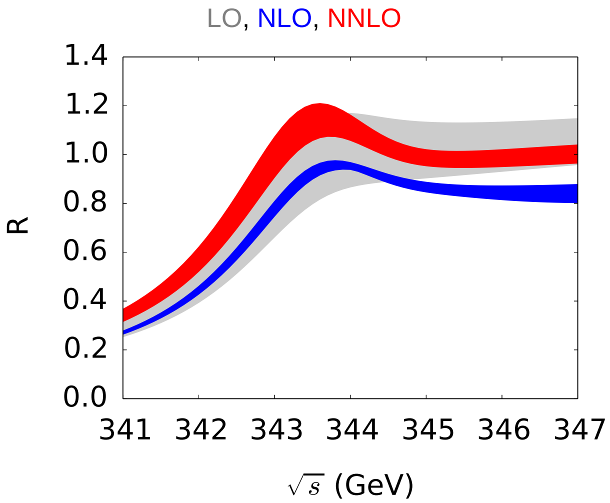
- $\overline{\mathrm{MS}}$ scheme; $\mu = m_Q$
- large corrections
- singlet terms: small ($\leq 3\%$ of $c^{(2)}$)

  at 2 loops (for axial-vector, scalar, pseudo-scalar current)

  [Kniehl,Onishchenko,Piclum,Steinhauser'06]



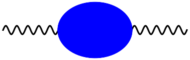
Application: height of $\sigma(e^+e^- \to t\bar{t})$



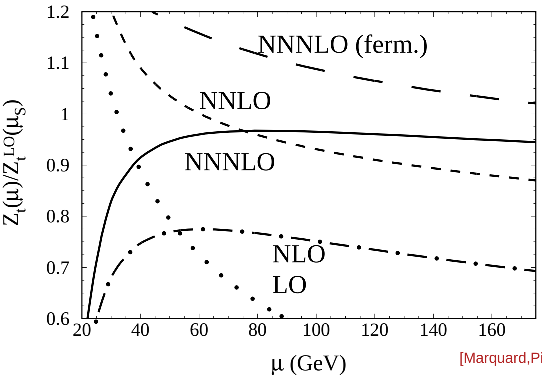
[Hoang,Beneke,Melnikov,Nagano,Ota,Penin,Pivovarov,Signer,Smirnov,Sumino,Teubner,Yakovlev,Yelkhovsky'00]

Application: Residue Z_t

$$\left(-q^2 g_{\mu\nu} + q_\mu q_\nu\right)\ \Pi(q^2) = i \int \mathrm{d}x\, e^{iqx} \langle 0|T j_\mu(x) j_\nu^\dagger(0)|0\rangle$$

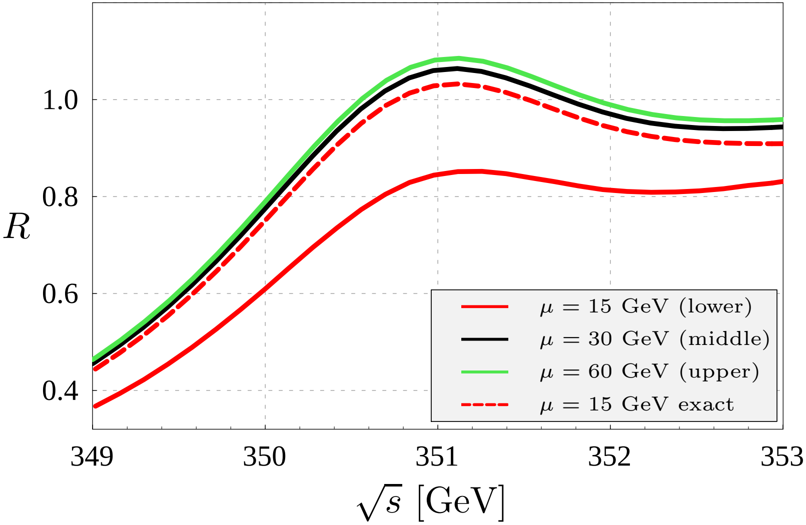


$$\Pi(q^2) \stackrel{E\to E_1}{=} \frac{N_c}{2m_t^2}\frac{Z_1}{E_1-(E+i0)} + \ldots \qquad Z_t = \left[c_v^2 - \frac{E_1}{m_t} c_v \left(c_v + \frac{d_v}{3}\right)\right] |\psi_1(0)|^2$$



[Marquard,Piclum,Seidel,Steinhauser'14]

Coulomb corrections to $\sigma(e^+e^- \to t\bar{t})$



[Beneke,Kiyo,Schuller'05]

$$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$$

$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$

- $\Upsilon(1S)$ meson: simplest heavy quark bound state
- $\Gamma(\Upsilon(1S) \to \ell^+\ell^-)_{\rm exp} = 1.340(18)$ keV
- $$\Gamma(\Upsilon(1S) \to \ell^+\ell^-) = \frac{4\pi\alpha^2}{9m_b^2} \left|\psi_1(0)\right|^2 c_v \left[c_v - \frac{E_1}{m_b}\left(c_v + \frac{d_v}{3}\right) + \ldots\right]$$

$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$

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- d_v: matching constant of sub-leading $b\bar{b}$ current 1 loop: [Luke,Savage'98]
- $\psi_1(0)$ wave function of the $(b\bar{b})$ system $\quad \left|\psi_1^{\rm LO}(0)\right|^2 = \frac{8m_b^3\alpha_s^3}{27\pi}$
- $M_{\Upsilon(1S)} = 2m_b + E_1 \quad E_1^{p,{\rm LO}} = -(4m_b\alpha_s^2)/9$

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- $M_{\Upsilon(1S)} = 2m_b + E_1 \quad E_1^{p,{\rm LO}} = -(4m_b\alpha_s^2)/9$
- many building blocks necessary; most recent ones:
  - a_3 [Smirnov,Smirnov,Steinhauser'08; Smirnov,Smirnov,Steinhauser'09; Anzai,Kiyo,Sumino'09]
  - c_v [Marquard,Piclum,Seidel,Steinhauser'14]
  - $\psi_1(0)$: ultrasoft contribution [Beneke,Kiyo,Penin'07]
  - $\psi_1(0)$: single- and double-potential insertions [Beneke,Kiyo,Schuller'08'13]
  - $\mathcal{O}(\epsilon)$ term of 2-loop $1/(m_b r^2)$ pNRQCD potential [Penin,Smirnov,Steinhauser'13]
- NLL: [Pineda'01]; NNLL-approx: [Pineda,Signer'07]; ...

Result

$$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)|_{\rm pole} =$$

$$\frac{2^5\alpha^2\alpha_s^3 m_b}{3^5}\Big[1 + \alpha_s\,(-2.003 + 3.979\,L) + \alpha_s^2\,(9.05 - 7.44\,\ln\alpha_s - 13.95\,L + 10.55\,L^2)$$

$$+\alpha_s^3\,(-0.91 + 4.78_{a_3} + 22.07_{b_2\epsilon} + 30.22_{c_f} - 134.8(1)_{c_g} - 14.33\,\ln\alpha_s - 17.36\,\ln^2\alpha_s$$

$$+(62.08 - 49.32\,\ln\alpha_s)\,L - 55.08\,L^2 + 23.33\,L^3) + \mathcal{O}(\alpha_s^4)\Big]$$

$$\stackrel{\mu=3.5\text{ GeV}}{=} \frac{2^5\alpha^2\alpha_s^3 m_b}{3^5}\,[1 + 1.166\alpha_s + 15.2\alpha_s^2 + (66.5 + 4.8_{a_3}$$

$$+22.1_{b_2\epsilon} + 30.2_{c_f} - 134.8(1)_{c_g})\,\alpha_s^3 + \mathcal{O}(\alpha_s^4)]$$

$$= \frac{2^5\alpha^2\alpha_s^3 m_b^{\rm pole}}{3^5}\,[1 + 0.28 + 0.88 - 0.16]$$

$$= [1.04 \pm 0.04(\alpha_s)^{+0.02}_{-0.15}(\mu)]\text{ keV}$$

$$= \frac{2^5\alpha^2\alpha_s^3 m_b^{\rm PS}}{3^5}\,[1 + 0.37 + 0.95 - 0.04]$$

$$= [1.08 \pm 0.05(\alpha_s)^{+0.01}_{-0.20}(\mu)]\text{ keV}$$

$$\alpha_s = 0.1184 \pm 0.001$$

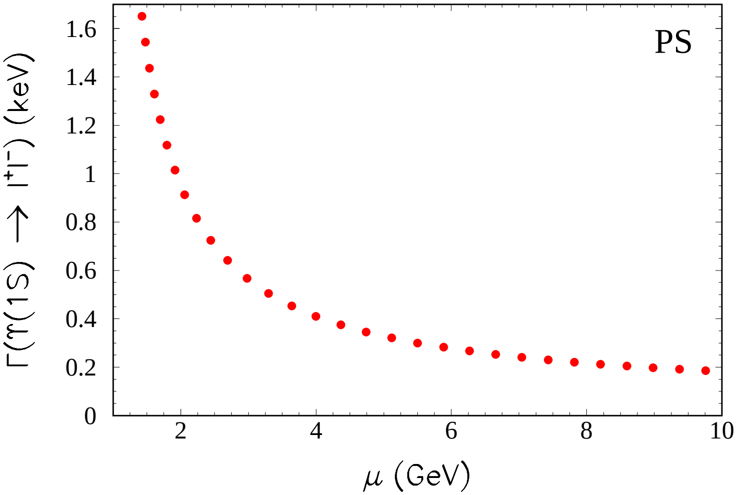
$$3 \le \mu \le 10\text{ GeV}$$

$$L = \ln[\mu/(4m_b\alpha_s/3)]$$

[Beneke,Kiyo,Marquard,Penin,Piclum,Seidel,Steinhauser'14]

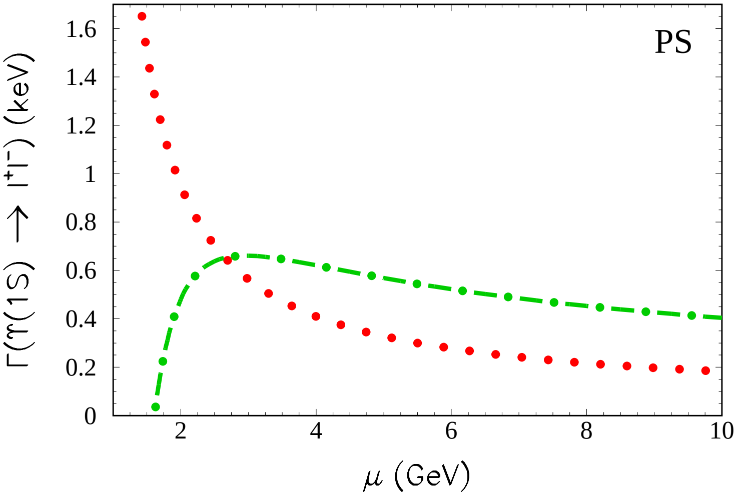
$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$: μ dependence

[Beneke,Kiyo,Marquard,Penin,Piclum,Seidel,Steinhauser'14]



$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$: μ dependence

[Beneke,Kiyo,Marquard,Penin,Piclum,Seidel,Steinhauser'14]



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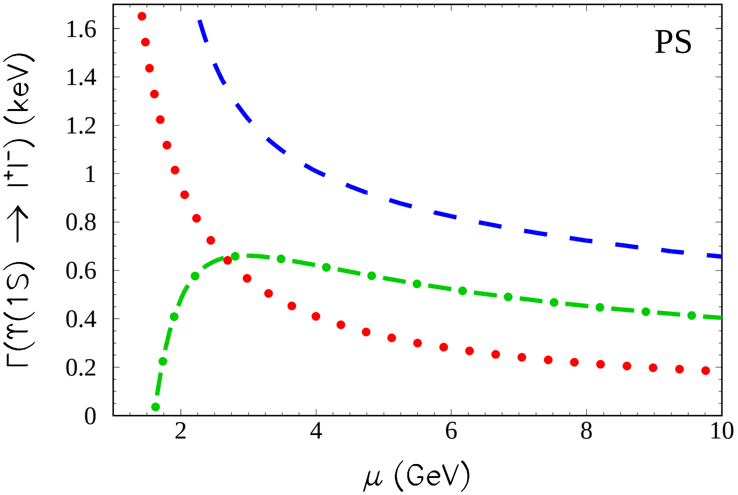

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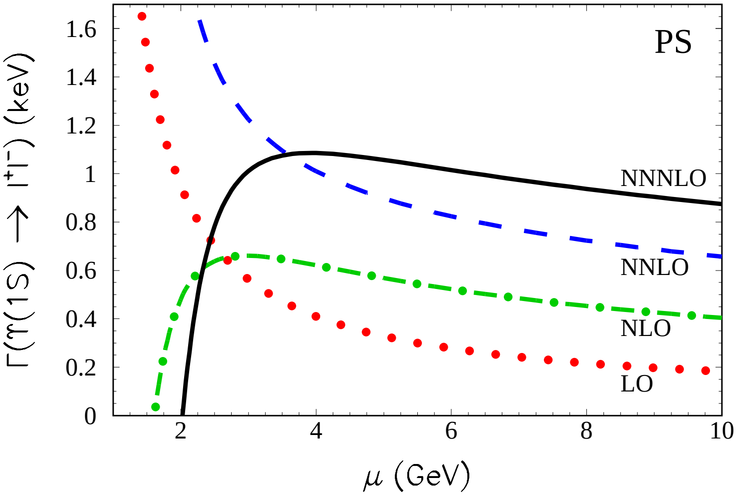

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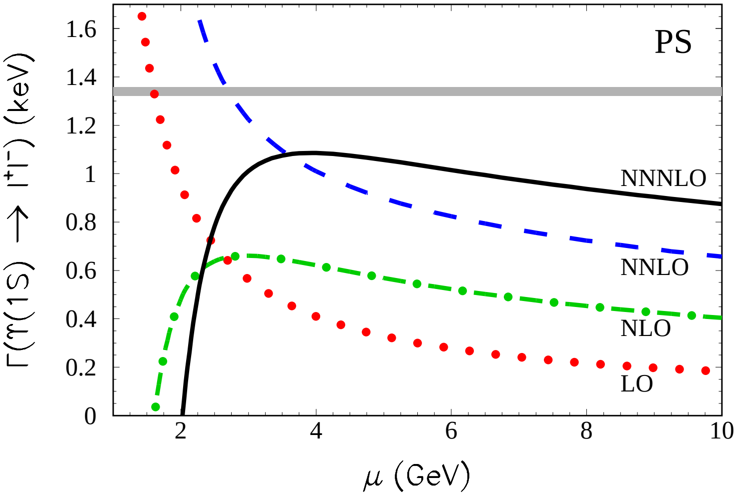
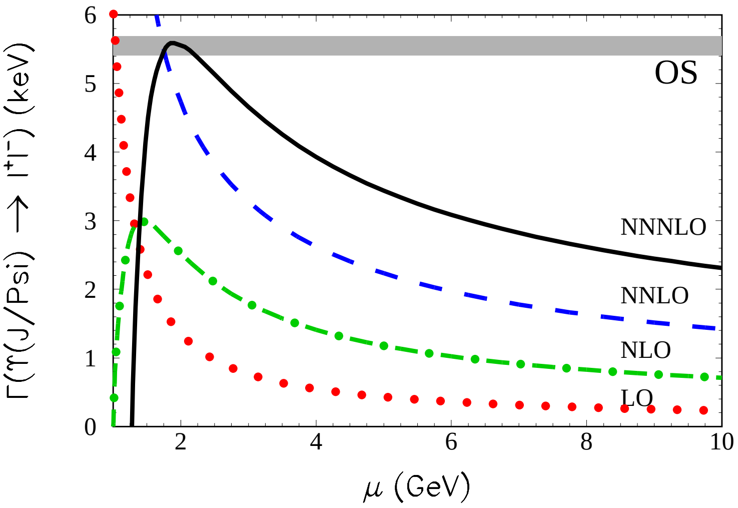
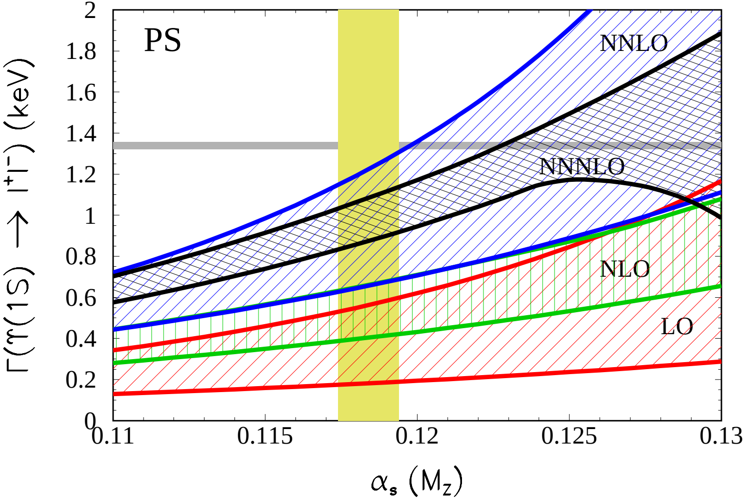


$\Gamma(J/\Psi \to \ell^+\ell^-)$: μ dependence



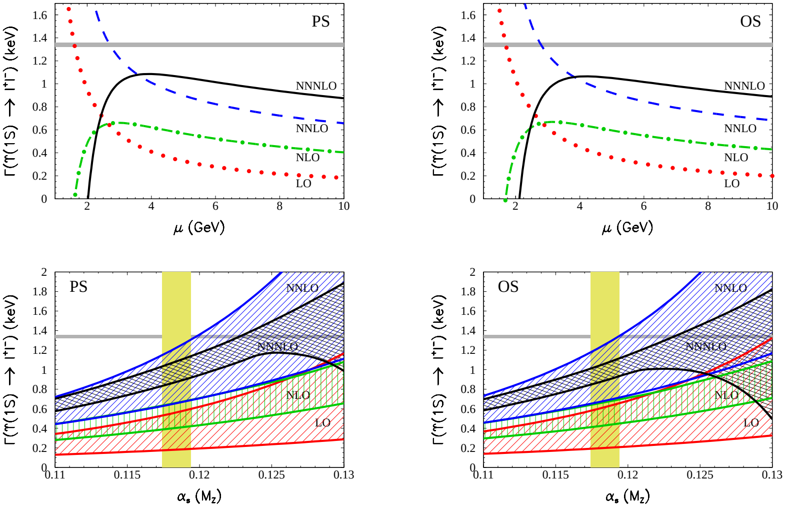
$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$: α_s dependence

[Beneke,Kiyo,Marquard,Penin,Piclum,Seidel,Steinhauser'14]



$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$: PS vs. OS

[Beneke,Kiyo,Marquard,Penin,Piclum,Seidel,Steinhauser'14]



Result

- $\Gamma(\Upsilon(1S) \to \ell^+\ell^-)|_{\rm pole} = [1.04 \pm 0.04(\alpha_s)^{+0.02}_{-0.15}(\mu)]$ keV
- $\Gamma(\Upsilon(1S) \to \ell^+\ell^-)|_{\rm PS} = [1.08 \pm 0.05(\alpha_s)^{+0.01}_{-0.20}(\mu)]$ keV
- $\Gamma(\Upsilon(1S) \to \ell^+\ell^-)|_{\rm exp} = [1.340 \pm 0.018]$ keV

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  - add NLO and NNLO m_c terms to $\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$

    [Hoang'00; Beneke,Maier,Piclum,Rauh'14]
  - decouple charm: $\alpha_s^{(4)} \to \alpha_s^{(3)}$ [Pineda]

⇨ $\Gamma(\Upsilon(1S) \to \ell^+\ell^-)|_{\rm pole} = [1.20 \pm 0.06(\alpha_s)^{+0.01}_{-0.24}(\mu)]$ keV

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- $\Gamma(\Upsilon(1S) \to \ell^+\ell^-) = [1.361^{+0.154}_{-0.214}]$ keV [Shen,Wu,Ma,Bi,Wang'15]

  apply "principle of maximum conformality" (PMC) to N^{3}LO result

  ➪ $\Gamma(\Upsilon(1S) \to \ell^+\ell^-) \sim r_1\alpha_s^3(Q_1) + r_2\alpha_s^4(Q_2) + r_3\alpha_s^5(Q_3) + r_4\alpha_s^6(Q_4)$

  with $Q_1 \approx 1.3$ GeV, $Q_2 \approx 2.0$ GeV, $Q_3 = Q_4 \approx 5.1$ GeV

Non-perturbative contribution

KIT Karlsruhe Institute of Technology

1. $\delta_{\rm np}|\psi_1(0)|^2 = |\psi_1^{\rm LO}(0)|^2 \times 17.54\pi^2 K$ [Leutwyler'81,Voloshin'82]

- $K = \frac{\langle \frac{\alpha_s}{\pi} G^2 \rangle}{m_b^4 (\alpha_s C_F)^6}$
- $\langle \frac{\alpha_s}{\pi} G^2 \rangle = 0.012\ \text{GeV}^4$ ⇨ $\boxed{\delta_{\rm np}\Gamma_{\ell\ell}(\Upsilon(1S)) = 1.67_{\rm pole}/2.20_{\rm PS}\ \text{keV}}$
  $[\alpha_s(3.5\ \text{GeV}) \approx 0.24]$

? value of $\langle \frac{\alpha_s}{\pi} G^2 \rangle$
? scale of α_s
? dimension-6 condensate contribution [Pineda'97]

$$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)\Big|_{\rm exp} - \Gamma(\Upsilon(1S) \to \ell^+\ell^-)\Big|_{\rm pert. N^3LO} \approx 0.3\ \text{keV}$$

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   - $K = \frac{\langle \frac{\alpha_s}{\pi} G^2 \rangle}{m_b^4 (\alpha_s C_F)^6}$
2. $M_{\Upsilon(1S)} = 2m_b + E_1^{\rm p} + \frac{624\pi^2}{425} m_b (\alpha_s C_F)^2 K$
   - use PS scheme ➪ perturbation theory converges [Beneke,Kiyo,Schuller'05]

$$➪ \; \delta_{\rm np}\Gamma_{\ell\ell}(\Upsilon(1S)) = \frac{4\alpha^2\alpha_s}{9} \frac{17.54\times 425}{3744} \delta M_{\Upsilon(1S)}^{\rm np}$$

$$\approx \left[1.28^{+0.17}_{-0.18}(\alpha_s) \pm 0.42(m_b)^{+0.20}_{-0.57}(\mu) \pm 0.12(m_c)\right] \text{keV}$$

(charm effects [Hoang'00])

? $m_b^{\overline{\rm MS}} = 4.163$ GeV $\rightarrow 4.203$ GeV ➪ $\delta_{\rm np}\Gamma_{\ell\ell}(\Upsilon(1S)) \approx 0.3$ keV

$\alpha_s = 0.1184 \pm 0.001$, $\quad m_b = 4.163 \pm 0.016$ GeV, $\quad 3 \leq \mu \leq 10$ GeV

[PDG] [Chetyrkin et al.'09]

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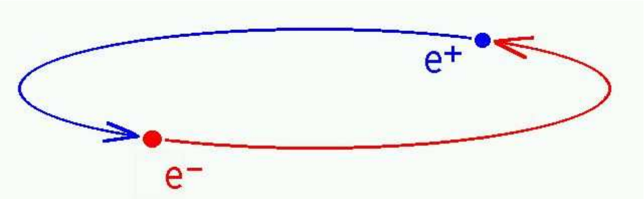
Summary: perturbation theory: solid prediction
non-perturbative contribution: unclear


e+
e−

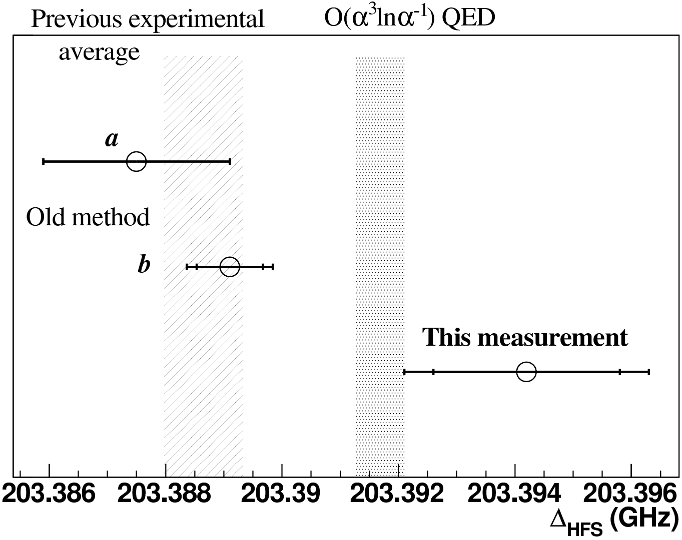
Positronium hyperfine splitting

- $\Delta\nu = E_{\uparrow\uparrow}(e^+e^-) - E_{\uparrow\downarrow}(e^+e^-)$
- precise test of QED
- Experiment:
  - $203.387\,5(1\,6)$ GHz [Mills et al.'75'83]
  - $203.389\,10(74)$ GHz [Ritter et al.'84]

    $\Updownarrow$ 2.7σ

  - $203.394\,2(16)_{\rm stat.}(13)_{\rm syst.}$ GHz [Ishida et al.'13]
- $\Delta_{\rm Zeeman} \sim \Delta_{\rm HFS}\left(\sqrt{1+4q^2}-1\right) \qquad q \sim B$
- material effects? non-uniform B field? ...

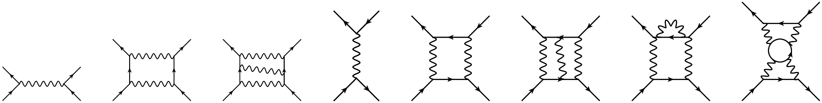
Comparison experiment theory



Theory

- $\Delta\nu^{\rm LO} = \left[\frac{1}{3}_{sct} + \frac{1}{4}_{ann}\right] \alpha^4 m_e^2$ $\qquad H_{\rm HFS} \sim \frac{\vec{S}^2}{m_e^2}$
- Higher orders

$$\begin{aligned}\Delta\nu &= \Delta\nu^{LO}\Bigg\{1 - \frac{\alpha}{\pi}\left(\frac{32}{21} + \frac{6}{7}\ln 2\right) - \frac{5}{14}\alpha^2 \ln\alpha \\ &+ \left(\frac{\alpha}{\pi}\right)^2 \left[\frac{1367}{378} - \frac{5197}{2016}\pi^2 + \left(\frac{6}{7} + \frac{221}{84}\pi^2\right)\ln 2 \right. \\ &\left. - \frac{159}{56}\zeta(3)\right] - \frac{3}{2}\frac{\alpha^3}{\pi}\ln^2\alpha + \left(-\frac{62}{15} + \frac{68}{7}\ln 2\right)\frac{\alpha^3}{\pi}\ln\alpha \\ &+ D\left(\frac{\alpha}{\pi}\right)^3\Bigg\}\end{aligned}$$



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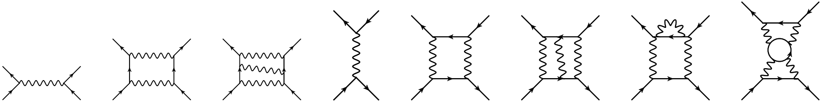
[Pirenne'47]



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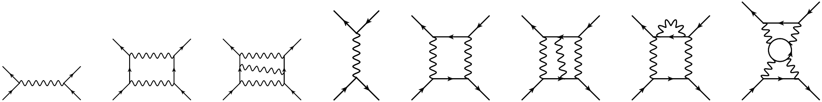
[Brodsky,Erickson'66; . . . ; Hoang,Labelle,Zebarjad'97; Czanecki,Melnikov,Yelkhovsky'99]



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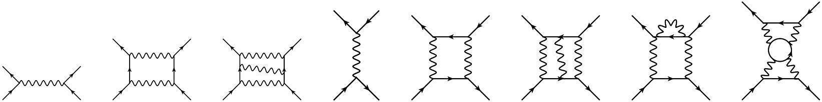
[Karshenboim'93]



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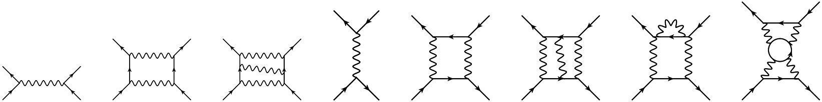
&+ D\left(\frac{\alpha}{\pi}\right)^3 \Big\}
\end{aligned}$$

[Kniehl,Penin'00; Hill'01; Melnikov,Yelkhovsky'01]



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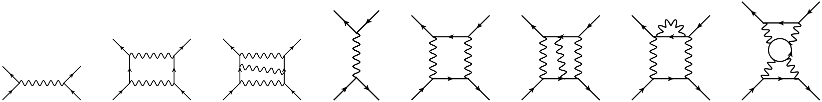
error estimate: "*D*" from muonium atom [Nio,Kinoshita'97]



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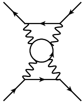
[Adkins,Fell'14] light-by-light 2γ exchange contribution to D: tiny



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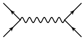
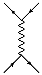
$$\left. + D\left(\frac{\alpha}{\pi}\right)^3\right\}$$

- aim: 1-γ annihilation contribution to D
- at $\mathcal{O}(\alpha^6 m_e)$:

  1-γ annihilation contribution $\approx 32\%$



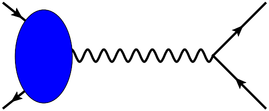
  non-annihilation $\approx 47\%$

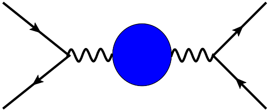


$D_{\mathrm{ann}}^{1-\gamma}$: 1-$\gamma$ annihilation contribution

[Baker,Marquard,Penin,Piclum,Steinhauser'14]

- pNRQED
- dimensional regularization

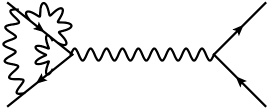


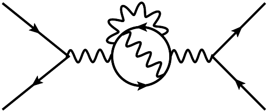


➪ QED part of c_v

➪ $\Pi_{\gamma\gamma}(q^2 = 4m_e^2)$

computed up to 3 loops





$D_{\mathrm{ann}}^{1-\gamma}$: result

- $D_{ann}^{1-\gamma} = 84.8 \pm 0.5$ [Baker,Marquard,Penin,Piclum,Steinhauser'14]
  ➪
  $\Delta\nu^{\mathrm{th}} = 203.391\,69 + 0.000\,217$ GHz $= 203.391\,91(22)$ GHz
  error estimate: size of the evaluated 1-γ annihilation contribution
- moves closer to [Ishida et al.'13]
  $\Delta\nu^{\mathrm{exp}} = 203.394\,2(16)_{\mathrm{stat.}}(13)_{\mathrm{syst.}}$ GHz

$D_{\rm ann}^{1-\gamma}$: result

Karlsruhe Institute of Technology

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  ➪
  $\Delta\nu^{\rm th} = 203.391\,69 + 0.000\,217$ GHz $=$ 203.391 91(22) GHz
  error estimate: size of the evaluated 1-γ annihilation contribution
- moves closer to [Ishida et al.'13]
  $\Delta\nu^{\rm exp} = 203.394\,2(16)_{\rm stat.}(13)_{\rm syst.}$ GHz
- 95% from ultra-soft contribution: $D_{ann}^{1-\gamma,us} \approx \frac{3\pi^2}{7}\delta_o^{us}$
- Assumption: also nonannihilation correction is dominated by ultra-soft contribution: $D_{sct}^{1-\gamma,us} \approx \frac{4\pi^2}{7}\delta_o^{us} \approx 106$
  ➪ $D_{\rm tot} \approx 191$
  ➪ $\Delta\nu^{\rm th} =$ 203.392 11 GHz
  ➪ agreement within 1σ with [Ishida et al.'13]

Conclusions

- a_3, c_v to 3 loops
- $\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$ to NNNLO

  first NNNLO calculation to physical quantity in quarkonium physics
- positronium HFS:

  1-γ annihilation to $\mathcal{O}(\alpha^7 m_e)$
- Coming soon: $e^+e^- \to t\bar{t}$ at NNNLO