Roots of unity and the lepton mixing patterns obtainable with finite flavor symmetries

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Outline

Math

I Roots of unity

ξ is a root of unity iff $\xi^n = 1$ for some positive integer n

Physics

II Residual symmetries of leptons

Math & Physics

III Classifying mixing matrices

IV Summary

Roots of unity

Some motivation: value of special angles

It is well known that the cosine function takes peculiar values for special angles θ

$(0 \leq \theta \leq \pi/2)$

$$\cos(0) = 1 \qquad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \qquad \cos\left(\frac{\pi}{2}\right) = 0$$

Are there more such angles?

To be precise, what are the solutions to the equation

$$\cos(q_1 \pi) = \sqrt{q_2} \text{ for } q_1, q_2 \in \mathbb{Q}$$

?

This is a Diophantine equation ...

Diophantine equations

These are equations where the variables are only allowed to take discrete values

Even simple Diophantine equations can be very hard to solve:

$$a^n + b^n = c^n$$
$$a, b, c, n \in \mathbb{N}$$

There are infinite solutions for n=1,2 but none for n>3 according to Fermat's Last Theorem

In Physics, we deal mostly with continuous parameters (derivatives, integrations, ...)

However, there are exceptions

As I will point out latter, discrete groups are related to Diophantine equations, and they have been used in particle physics (for example)

$\cos(q_1\pi) = \sqrt{q_2}$ for $q_1, q_2 \in \mathbb{Q}$

Coming back to our equation

$\cos(q_1\pi) = \frac{1}{2}(\xi + \xi^*)$ where $\xi = \exp(iq_1\pi)$ is a root of unity

$$\cos(q_1\pi) = \sqrt{q_2}$$

$$\Leftrightarrow$$

$$\xi + \xi^* = 2\sqrt{q_2}$$

$$\Leftrightarrow$$

$$\xi^2 + \xi^{*2} + 2 - 4q_2 = 0$$

The root of unity ξ^2 plus the root of unity ξ^{*2} plus $(2 - 4q_2)$ times the root of unity 1 equals zero.

Solutions

As a formal sum of roots of unity, this can only be

$\omega \equiv \exp\left(\frac{2\pi i}{3}\right)$

$0 = 0$ **or** $q\theta\left(1 + \omega + \omega^2\right) = 0$

$$|2 - 4q_2| = 1 \Rightarrow q_2 = \frac{1}{4}, \frac{3}{4}$$

$$\xi^2 = +\xi^{2*} : |2 - 4q_2| = 2 \Rightarrow q_2 = 0, 1$$

$$\xi^2 = -\xi^{2*} : 2 - 4q_2 = 0 \Rightarrow q_2 = \frac{1}{2}$$

This is it!

$$\cos(q_1\pi) = 0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1$$

Vanishing sums of roots of unity
(I)

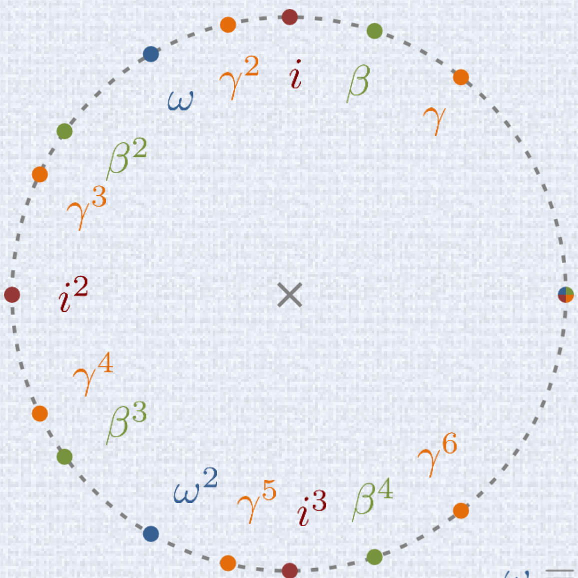
This is a problem that has received the attention of mathematicians

$$c_1\xi_1 + c_2\xi_2 + \cdots + c_n\xi_n = 0$$

(ξ_i =roots of unity; c_i =some coefficients)

For it to make sense, the coefficients must be restricted to some set (ring)

Consider henceforth $c_i \in \mathbb{Z}$ (or $\mathbb{Q}$)



We know of course of some vanishing sums of roots of unity with integer coefficients:

$$1 - 1 = 0 \qquad 1 + \omega + \omega^2 = 0$$

$$1 + i + i^2 + i^3 = 0 \qquad 1 + \beta + \beta^2 + \beta^3 + \beta^4 = 0$$

...

Are there more vanishing sums?

$\omega \equiv \exp\left(\frac{2\pi i}{3}\right) \quad \beta \equiv \exp\left(\frac{2\pi i}{5}\right) \quad \gamma \equiv \exp\left(\frac{2\pi i}{7}\right)$

Vanishing sums of roots of unity (II)

Consider **formal sums** of roots of unity → $S \equiv \sum_{i=1}^{n} c_i \xi_i$ **should be seen as a vector**, with the basis vectors given by the ξ_i

In this way, $S = 1 + \omega + \omega^2$ and $S' = 1 + i + i^2 + i^3$ are different, even thought they have the **same value**.

If S is a vanishing sum of roots of unity, then so is $c\theta S$ for any root of unity θ, and $c \in \mathbb{Q}$ → Factor out **rotations of S** and multiplications by $c \in \mathbb{Q}$

If S and S' are vanishing sums, then so is $S + S'$ → Consider only **minimal*** **sums** (those which do not have any vanishing [proper] subsum)

$1 + \omega + \omega^2 = 0$ ✓

$1 + i + i^2 + i^3 = 0$ ✗

$1 + \beta + \beta^2 + \beta^3 + \beta^4 = 0$ ✓

*Not to be confused with primitive sums

Vanishing sums of roots of unity
(III)

In Physics, see
Grimus (2013)

It turns out that there are not that many primitive vanishing sums ...

Theorem. *Let S be a non-empty vanishing sum of length at most 9. Then either S involves θ, $\theta\omega$ and $\theta\omega^2$ for some root θ, or S is similar to one of*

R_5	$a)$ $1+\beta+\beta^2+\beta^3+\beta^4,$
$-R_3+R_5$	$b)$ $-\omega-\omega^2+\beta+\beta^2+\beta^3+\beta^4,$
R_7	$c)$ $1+\gamma+\gamma^2+\gamma^3+\gamma^4+\gamma^5+\gamma^6,$
$R_5-(\beta^2+\beta^3)\,R_3$	$d)$ $1+\beta+\beta^4-(\omega+\omega^2)(\beta^2+\beta^3),$
$-R_3+R_7$	$e)$ $-\omega-\omega^2+\gamma+\gamma^2+\gamma^3+\gamma^4+\gamma^5+\gamma^6,$
$R_5-(1+\beta^2+\beta^3)\,R_3$	$f)$ $\beta+\beta^4-(\omega+\omega^2)(1+\beta^2+\beta^3),$
$R_7-(\gamma+\gamma^6)\,R_3$	$g)$ $1+\gamma^2+\gamma^3+\gamma^4+\gamma^5-(\omega+\omega^2)(\gamma+\gamma^6),$
$R_3-(\omega+\omega^2)\,R_5$	$h)$ $1-(\omega+\omega^2)(\beta+\beta^2+\beta^3+\beta^4).$

Detail: **in this theorem, a root $-\alpha$ is considered to be the same as $-1\times(\alpha)$**

Curiosity: the coefficients of the roots in these sums are always ± 1

Conway, Jones 1976

Cyclotomic polynomial

It is known that there are primitive sums of roots of unity with coefficients different from ± 1: for example nth cyclotomic polynomial for big n's.

$$\Phi_n(x) = \prod_{\substack{n\text{th primitive} \\ \text{roots } \xi_i}} (x - \xi_i)$$

n **is the smallest positive integer such that** $(\xi_i)^n = 1$

For any primitive n-th root of unity ξ_i,
$\Phi_n(\xi_i)$ is a primitive vanishing sum of roots of unity

yet ...

$$\begin{aligned}\Phi_{105}(x) = {} & x^{48} + x^{47} + x^{46} - x^{43} - x^{42} - 2x^{41} - x^{40} - x^{39} + x^{36} + x^{35} + x^{34} \\ & + x^{33} + x^{32} + x^{31} - x^{28} - x^{26} - x^{24} - x^{22} - x^{20} + x^{17} + x^{16} + x^{15} \\ & + x^{14} + x^{13} + x^{12} - x^{9} - x^{8} - 2x^{7} - x^{6} - x^{5} + x^{2} + x + 1\end{aligned}$$

... so there are for sure primitive vanishing sums of roots of unity with length $>$32 and with coefficients of modulus different from one

Deciding if a number is a root of unity (I)

Consider the number $\varphi \equiv \frac{1-i\sqrt{7}}{\sqrt{8}}$

Clearly $|\varphi| = 1$

But **is this number a root of unity?**

For φ to be a root of unity, its **complex phase** must be of the form $q\pi,\ q \in \mathbb{Q}$

But is it? How can we check?

$$\frac{1}{\pi}\arg\varphi = -0.384973271918\cdots$$
$$-\frac{3}{8} = -0.375$$
$$-\frac{5}{13} = -0.384615\cdots$$
$$-\frac{1655}{4299} = -0.384973249\cdots$$
$$\cdots$$

Suggestion

Use the polynomial P for which $P(\varphi) = 0$ to extract information on φ

Deciding if a number is a root of unity (II)

Changing the sign of the real and/or imaginary part of ξ will always be allowed

In fact, instead of $\frac{1-i\sqrt{7}}{\sqrt{8}}$ consider the **more general case** $\xi \equiv \frac{\sqrt{n_1}+i\sqrt{n_2}}{\sqrt{n_1+n_2}}$

$$\left[n_i \in \mathbb{N}_0 \text{ with } (n_1,n_2) \neq (0,0)\right]$$

$$(n_1+n_2)\left(\xi^2+\xi^{*2}\right)+2\left(n_2-n_1\right)=0$$

This is a **very simple vanishing sum of roots of unity** with integer coefficients (very similar to the one in slide 6) !

Solutions: $(n_1,n_2)=(n,0)\,,(0,n)\,,(n,n)\,,\left(n,\frac{1}{3}n\right)\,,\left(\frac{1}{3}n,n\right)$

... or ...

$$\xi = 1, i, \frac{1+i}{\sqrt{2}}, \frac{\sqrt{3}+i}{2}, \frac{1+\sqrt{3}i}{2}$$

$-i\omega$ $\quad -\omega^2$

Considering − signs in the Re/Im parts, we get all 8th- and 12th-roots of unity:

$$\pm1, \pm i, \frac{\pm1\pm i}{\sqrt{2}}, \frac{\pm\sqrt{3}\pm i}{2}, \frac{\pm1\pm\sqrt{3}i}{2}$$

One can show this in other ways Bu (2014)

Residual symmetries of leptons

Lepton mixing

Charged current

$$\mathscr{L} = -\frac{g}{\sqrt{2}}\overline{\ell_{L,\alpha}}\gamma^{\mu}U_{\alpha i}\nu_i W_{\mu}^{-} + \text{h.c.}$$

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

Parameterization

if neutrinos are Majorana particles

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0 \\ 0 & 0 & e^{i\frac{\alpha_{31}}{2}} \end{pmatrix}$$

Current data

$$\theta_{12} = 34.6^\circ \pm 1.0^\circ \qquad \Delta m^2_{21} = 7.60^{+0.19}_{-0.18}\ (\times 10^{-5}\ \text{eV}^2)$$

$$\theta_{23} = 48.9^\circ{}^{+1.9^\circ}_{-7.4^\circ} \qquad |\Delta m^2_{31}| = 2.48^{+0.05}_{-0.07}\ (\times 10^{-3}\ \text{eV}^2)$$

$$\theta_{13} = 8.8^\circ \pm 0.4^\circ \qquad \delta/\pi = 1.34^{+0.64}_{-0.38}$$

Forero, Tórtola, Valle (2014)

Flavour symmetries

We do not understand family replication and the flavour structure of the Standard Model (both of quarks and leptons)

Given that symmetry has such an importance in fundamental physics, maybe it plays a role in this too ...

What kind of symmetry group?

Discrete

- Finite: A_n S_n D_n Z_n Q_n T_7 T' $\Delta(3n^2)$ $\Delta(6n^2)$ ...
- Infinite: ?

Continuous

- Global: $SU(2)$ $SU(3)$ $SO(3)$ ...
- Local: $SU(3)^n$ $SO(18)$ ...

Residual lepton flavour symmetry (I)

Lam 2008

Tell us assume that neutrinos are Majorana particles

$$\mathscr{L}_{\text{mass}} = -\bar{\ell}_L M_\ell \ell_R + \frac{1}{2}\nu_L^T C^{-1} M_\nu \nu_L + \text{h.c.}$$

Ignore ℓ_R: a flavour symmetry transforms ...

$$\ell_L \to T\ell_L \qquad \nu_L \to S\nu_L$$

There is always a $U(1)^3$ symmetry associated to $H_\ell \equiv M_\ell M_\ell^\dagger$:

$$T^\dagger H_\ell T = H_\ell \quad \Rightarrow \quad T = \begin{pmatrix} \lambda_1^{(0)} & 0 & 0 \\ 0 & \lambda_2^{(0)} & 0 \\ 0 & 0 & \lambda_3^{(0)} \end{pmatrix}$$

in the basis where H_ℓ is diagonal

There is always a Z_2^3 symmetry associated to M_ν:

$$S^T M_\nu S = M_\nu \quad \Rightarrow \quad S = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

in the basis where M_ν is diagonal

Residual lepton flavour symmetry
(II)

However, **lepton mixing** implies that H_ℓ and M_ν are not simultaneously diagonal, so either T or S must be rotated by U

Without loss of generality we can take ...

$$T = U^\dagger \cdot \begin{pmatrix} \lambda_1^{(0)} & 0 & 0 \\ 0 & \lambda_2^{(0)} & 0 \\ 0 & 0 & \lambda_3^{(0)} \end{pmatrix} \cdot U$$

$$S_1 = \begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad S_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad S_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix}$$

Importantly, with these matrices the **symmetry enforces*** U as the lepton mixing matrix

*It can be shown that only $|U|^2$ is enforced, and only up to row/column permutations

But for that to happen, the eigenvalues of T must be distinct, otherwise we can use two T matrices

When does $\{T, S_i\}$ generate a finite group?

?

Example: tri-bimaximal mixing (TBM)

Until the conclusive Daya Bay/RENO measurement of a non-zero reactor angle, the tri-bimaximal mixing ansatz was quite good:

An *et al.* 2012
Ahn *et al.* 2012

$$U_{TBM} = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Harrison, Perkins, Scott 2002

What is the symmetry group associated to TBM?

In the paper "*The unique horizontal symmetry of leptons*", C.S. Lam (2008), the answer given is S_4

Let us pause to discuss why this is a non-trivial result: to arrive at this answer one needs to make a scan over all possible eigenvalues of the T generator. These must be roots of unity (so there is a "discreteness" in the problem)

The finite symmetry group of TBM (I)

How was this result proven?

The author carefully **searched all the irreducible representations of all the finite subgroups** of SU(2) and SU(3) to find adequate representations

The **list of groups and representations** was taken from the following references:

Miller, Blichfeldt, Dickson (1916)
Blichfeldt (1917)
Fairbairn, Fulton, Klink (1964)
Bovier, Lüling, Wyler (1981)
Luhn, Nasri, Ramond (2007)

There are **subtleties** to this classification of groups

Ludl (2009, 2010, 2011)
Zwicky, Fischbacher (2009)
Grimus, Ludl (2013)

Various models had been proposed with TBM using the **group A_4** (which does not contain S_4)

So what is going on? S_4 should be seen as the **effective symmetry** associated to TBM

Grimus, Lavoura, Ludl (2009); Lam (2009)

The finite symmetry group of TBM
(II)

There is **no easy way** to present the **original proof** of the connections between TBM and S_4

However, there is a **quick alternative demonstration** !

Fonseca, Grimus (2014)

1

The **eigenvalues of all group matrices must be roots of unity**: the reason is that for any $g \in G$ finite, $g^n = e$.
Then consider the matrices TS_j with eigenvalues $\equiv \lambda_i^{(j)}$

2

From the **trace of** TS_2, we get the following vanishing sum of roots of unity:

$$\underbrace{\lambda_1^{(0)}+\lambda_2^{(0)}+\lambda_3^{(0)}}_{\text{eigenvalues of } T}+3\left(\underbrace{\lambda_1^{(2)}+\lambda_2^{(2)}+\lambda_3^{(2)}}_{\text{eigenvalues of } TS_2}\right)=0$$

3

The **solutions** are (according to what was discussed previously):

+ Physics...

(a) $\lambda_1^{(0)}=\lambda_2^{(0)}=\lambda_3^{(0)}=-\lambda_i^{(2)}$ and $\lambda_j^{(2)}=-\lambda_k^{(2)}$ $(i\neq j\neq k\neq i)$ ✗

(b) $\left(\lambda_1^{(0)},\lambda_2^{(0)},\lambda_3^{(0)}\right)=\underline{\xi\left(1,\omega,\omega^2\right)}$ and $\left(\lambda_1^{(2)},\lambda_2^{(2)},\lambda_3^{(2)}\right)=\xi'\left(1,\omega,\omega^2\right)$ ✓

Plugging in these eigenvalues in T yields the S_4 group

... or $S_4\times Z_n$

Classifying all mixing patterns

The challenge

$$T = U^\dagger \cdot \begin{pmatrix} \lambda_1^{(0)} & 0 & 0 \\ 0 & \lambda_2^{(0)} & 0 \\ 0 & 0 & \lambda_3^{(0)} \end{pmatrix} \cdot U$$

Just now, we have fixed the mixing matrix U and scanned over the eigenvalues of T

We wanted to scan over everything: both U and the eigenvalues of T

How to do it? One could try again to search through all SU(3) finite subgroups.

However, we realized that the complete list of possibilities could be found relying just of some mathematical results related to roots of unity.

Fonseca, Grimus (2014)

Advantages: there is no need to rely on (I) previous listings of groups and representations nor (II) group/representation theory in general (e.g., that irreducible an 3-dimensional representation is needed, ...).
The result can be understood in an analytical, self-contained way.

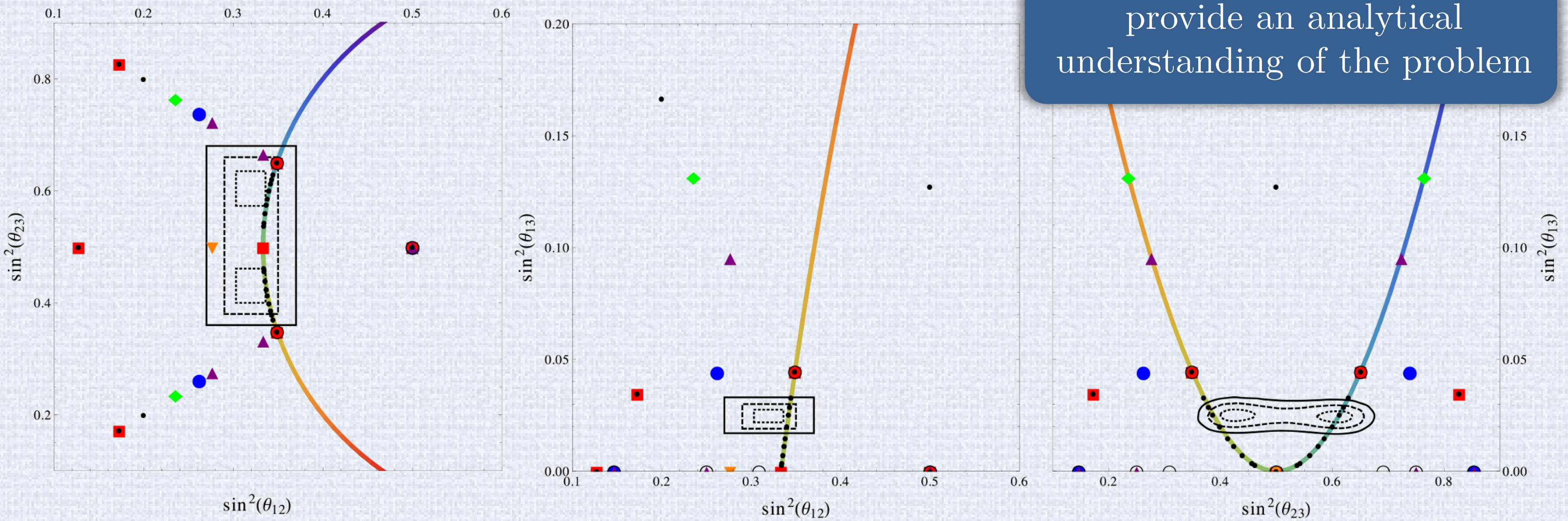
Computer scans

Computer scans have been made, using the GAP program to try and catalogue the possible lepton mixing patterns

From a scan of all groups with order smaller than 511 (1536)
More than a million groups in total

Parattu, Wingerter (2011)
Holthausen, Lim, Lindner (2013)

Still, these scan are not complete and they do not provide an analytical understanding of the problem



Classifying mixing matrices

Obtaining the $|T_{ij}|$

Consider the 9 matrices $Y^{(ij)} = T^\dagger S_i T S_j$

The eigenvalues of $Y^{(ij)}$ are $1,\, \lambda^{(ij)},\, (\lambda^{(ij)})^*$ because ...

$$\begin{cases} \det Y^{(ij)} = 1 \\ S_j^{-1} Y^{(ij)} S_j = \left(Y^{(ij)}\right)^\dagger \end{cases}$$

It is easily seen that $|T_{ij}|^2 = \frac{1}{2}\left(1 + \mathrm{Re}\,\lambda^{(ij)}\right)$

In other words, from the eigenvalues of $Y^{(ij)}$ we get the **absolute value of the entries of** T

But **there is a way to get the** $\lambda^{(ij)}$...

$$\sum_{k=1}^{3} \mathrm{Tr}\, Y^{(kj)} = 1 \qquad \sum_{k=1}^{3} \mathrm{Tr}\, Y^{(ik)} = 1 \quad \Rightarrow \quad \sum_{k=1}^{3}\left(\lambda^{(kj)} + \lambda^{(kj)*}\right) + 2 = 0 \qquad \sum_{k=1}^{3}\left(\lambda^{(ik)} + \lambda^{(ik)*}\right) + 2 = 0$$

These are **sums of roots of unity** of the form

$$\sum_{k=1}^{3}(\lambda_k + \lambda_k^*) + 2 = 0 \xrightarrow{\text{Solutions}} (\lambda_1, \lambda_2, \lambda_3) = \begin{cases} (i,\, \omega,\, \omega) \\ (\omega,\, \beta,\, \beta^2) \\ (-1,\, \lambda,\, -\lambda) \end{cases}$$

Obtaining the $|T_{ij}|$

Taking into account that T is a unitary, we get **just 5 possible basic forms** for $|T|$

Column and row permutations are allowed

Form 1

$$|T| = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Form 2

$$|T| = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Form 3

$$|T| = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} & \frac{1}{2} \end{pmatrix}$$

Form 4

$$|T| = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{1}{2} & \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} \end{pmatrix}$$

Form 5

$$|T| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \qquad \theta = q\pi,\ q \in \mathbb{Q}$$

However, $|T|$ **is not enough**

To get the mixing matrix U **we need the missing phases of** T

Reminder

$$T = U^\dagger \cdot \begin{pmatrix} \lambda_1^{(0)} & 0 & 0 \\ 0 & \lambda_2^{(0)} & 0 \\ 0 & 0 & \lambda_3^{(0)} \end{pmatrix} \cdot U$$

Obtaining the $|T_{ij}|$

Taking into account that T is a unitary, we get **just 5 possible basic forms** for $|T|$

Column and row permutations are allowed

Form 1

$$|T| = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Form 2

$$|T| = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Form 3

$$|T| = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} & \frac{1}{2} \end{pmatrix}$$

<u>All</u> (representation of the) elements of the group must have one of these forms !

Form 4

$$|T| = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{1}{2} & \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} \end{pmatrix}$$

Form 5

$$|T| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$\theta = q\pi,\ q \in \mathbb{Q}$

Reminder

$$T = U^\dagger \cdot \begin{pmatrix} \lambda_1^{(0)} & 0 & 0 \\ 0 & \lambda_2^{(0)} & 0 \\ 0 & 0 & \lambda_3^{(0)} \end{pmatrix} \cdot U$$

However, $|T|$ is **not enough**

To get the mixing matrix U we need the **missing phases of T**

Different row/column permutations

Form 1

$$|T|_A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad |T|_B = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \quad |T|_C = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Form 2

$$|T|_A = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$|T|_B = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

Form 3

$$|T|_A = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} & \frac{1}{2} \end{pmatrix} \quad |T|_B = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} & \frac{1}{2} \\ \frac{\sqrt{5}-1}{4} & \frac{1}{2} & \frac{\sqrt{5}+1}{4} \end{pmatrix}$$

$$|T|_C = \begin{pmatrix} \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \end{pmatrix} \quad |T|_D = \begin{pmatrix} \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \end{pmatrix}$$

Form 4

$$|T|_A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{1}{2} & \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} \end{pmatrix} \quad |T|_B = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \end{pmatrix}$$

$$|T|_C = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{\sqrt{5}-1}{4} & \frac{1}{2} & \frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \end{pmatrix} \quad |T|_D = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & \frac{1}{2} & \frac{\sqrt{5}+1}{4} \end{pmatrix}$$

Form 5

$$|T|_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$|T|_B = \begin{pmatrix} 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \end{pmatrix}$$

It is enough to consider these cases

Adding internal phases

To make T unitary

It is easy to figure out the internal phases which will **make T unitary**

Form 1

$$\widetilde{T}_A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2}\varphi \\ \frac{1}{2}\varphi^2 & \frac{1}{\sqrt{2}} & -\frac{1}{2}\varphi^2 \\ \frac{1}{2}\varphi & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Form 2

$$\widetilde{T}_A = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Form 3

$$\widetilde{T}_A = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{5}-1}{4} & -\frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & -\frac{\sqrt{5}+1}{4} & \frac{1}{2} \end{pmatrix}$$

Form 4

$$\widetilde{T}_A = \begin{pmatrix} \frac{1}{\sqrt{2}}\rho_0 & \frac{1}{2}\omega & \frac{1}{2}\omega^2 \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{1}{2}\omega^2 & \frac{\sqrt{5}+1}{4}\omega & \frac{\sqrt{5}-1}{4} \end{pmatrix}$$

Form 5

$$\widetilde{T}_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

... and the **same for the other permutations**

$$\varphi = \frac{1 - i\sqrt{7}}{\sqrt{8}} \qquad \rho_0 = \frac{\sqrt{5} + i\sqrt{3}}{\sqrt{8}}$$

Recall that these are not roots of unity

But ... not all forms are allowed

With extra considerations, we will see that form 1 and 4 (all row/column permutations) do not generate finite groups

Form 1

$$\widetilde{T}_A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2}\text{[illegible]} & \frac{1}{2}\varphi \\ \frac{1}{2}\varphi^2 & \frac{\text{[illegible]}}{\sqrt{2}} & -\frac{1}{2}\varphi^2 \\ \frac{1}{2}\text{[illegible]} & -\frac{1}{2} & \frac{\text{[illegible]}}{\sqrt{2}} \end{pmatrix}$$

Form 2

$$\widetilde{T}_A = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Form 3

$$\widetilde{T}_A = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{5}-1}{4} & -\frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & -\frac{\sqrt{5}+1}{4} & \frac{1}{2} \end{pmatrix}$$

Form 4

$$\widetilde{T}_A = \begin{pmatrix} \frac{1}{\sqrt{2}}\rho_0 & \text{[illegible]} & \frac{1}{2}\omega^2 \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{1}{2}\text{[illegible]} & \frac{\sqrt{5}+1}{4}\omega & \frac{\sqrt{5}-1}{4} \end{pmatrix}$$

Form 5

$$\widetilde{T}_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

Interestingly, we are then left with T's with no internal complex phases

Forms 1 and 4: not valid

Form 1A

$$T=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2}\varphi\\ \frac{1}{2}\varphi^2 & \frac{1}{\sqrt{2}} & -\frac{1}{2}\varphi^2\\ \frac{1}{2}\varphi & -\frac{1}{2} & \frac{1}{\sqrt{2}}\end{pmatrix}\mathrm{diag}\left(\kappa_1,\kappa_2,\kappa_3\right)$$

Some **external phases** (no need to consider an extra phase matrix on the left)

A priori these κ_i **complex phases are completely general**

Theorem. *Let G be a finite group with $T\in G_\ell$ and let c be one of the numbers $1/2$, $(\sqrt{5}+1)/4$ or $(\sqrt{5}-1)/4$. Moreover, T_{jj} is a diagonal element and T_{kl}, T_{lk} are off-diagonal elements of T. Then the following holds:*

$$|T_{jj}|=c\ \Rightarrow\ T_{jj}=c\xi,\quad |T_{kl}T_{lk}|=\frac{1}{4}\ \Rightarrow\ T_{kl}T_{lk}=\frac{\xi'}{4}$$

with roots of unity ξ, ξ'.

(we shall see a **sketch of the proof** shortly)

What does this imply for form 1A?

$$\kappa_1\kappa_2\varphi^2\equiv\xi'_{12}$$
$$\kappa_1\kappa_3\varphi^2\equiv\xi'_{13}$$
$$\kappa_2\kappa_3\varphi^2\equiv\xi'_{23}$$

Some roots of unity

$$\Longrightarrow\ (\det T)^2=\frac{1+3\sqrt{7}i}{8}\times(\text{root of unity})$$

Not a root of unity

Contradiction !

Forms 1 and 4: not valid

Reminder

Theorem. *Let G be a finite group with $T \in G_\ell$ and let c be one of the numbers $1/2$, $(\sqrt{5}+1)/4$ or $(\sqrt{5}-1)/4$. Moreover, T_{jj} is a diagonal element and T_{kl}, T_{lk} are off-diagonal elements of T. Then the following holds:*

$$|T_{jj}| = c \;\Rightarrow\; T_{jj} = c\xi, \quad |T_{kl}T_{lk}| = \frac{1}{4} \;\Rightarrow\; T_{kl}T_{lk} = \frac{\xi'}{4}$$

with roots of unity ξ, ξ'.

Theorem. *Let ζ be an n-th root of unity, i.e. $\zeta = e^{2\pi i/n}$, and let $\mathcal{S} = \sum_{k=0}^{n-1} a_k \zeta^k$ be a sum with integer coefficients a_k. If $|\mathcal{S}| = 1$, then $\mathcal{S}$ is itself a root of unity.*

Sum of roots of unity

$$\mathrm{Tr}\,(TS_j) + \mathrm{Tr}\,T = 2T_{jj}$$

$$\left(\frac{\sqrt{5}\pm 1}{2}\right)^{-1}[\mathrm{Tr}\,(TS_j) + \mathrm{Tr}\,T] = 2T_{jj}\left(\frac{\sqrt{5}\pm 1}{2}\right)^{-1}$$

Sum of two 5th-roots of unity

$$\frac{1}{\det T}(T_{kk}T_{ll} - T_{kl}T_{lk}) = (T_{jj})^*$$

(because T is unitary)

Forms 1A, 1B, 1C, 4A, 4B, 4C, 4D can all be shown to be invalid

The valid cases

For the remaining forms, we still need the **external phases**

We can obtain these phases because **we know what forms both $|T|$ and $|T|^2$ can take**

Corresponding to **three flavour mixing**, we get **17 sporadic mixing patterns**, and **one infinite family**

Associated groups:

Sporadic cases

$A_4 \quad S_4 \quad A_5$

$PSL(2,7) \quad \Sigma(360\times 3)$

Infinite family

$\left(\mathbb{Z}_m \times \mathbb{Z}_{m/3}\right) \rtimes S_3$, $m = \text{lcm}(2,n)$ when 9 divides n

$\Delta(6m^2)$, $m = \text{lcm}(6,n)/3$ otherwise

About the **sporadic cases:**

$$|U|^2 = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \\ \frac{1}{12}\left(5-\sqrt{21}\right) & \frac{1}{12}\left(5+\sqrt{21}\right) & \frac{1}{6} \\ \frac{1}{12}\left(5+\sqrt{21}\right) & \frac{1}{12}\left(5-\sqrt{21}\right) & \frac{1}{6} \end{pmatrix}$$

≈ 0.035

Smallest non-zero value in all sporadic cases...

... too large to be $\sin^2\theta_{13}$

The infinite family

$$|U|^2 = \frac{1}{3}\begin{pmatrix} 1 & 1+\mathrm{Re}\,\sigma & 1-\mathrm{Re}\,\sigma \\ 1 & 1+\mathrm{Re}\,(\omega\sigma) & 1-\mathrm{Re}\,(\omega\sigma) \\ 1 & 1+\mathrm{Re}\,(\omega^2\sigma) & 1-\mathrm{Re}\,(\omega^2\sigma) \end{pmatrix}$$

$\delta = 0$

σ : any root of unity

This is the **only infinite family** of mixing patterns involving three flavours

Mentioned before in the literature:

Toorop, Feruglio, Hagedorn (2012)
Holthausen, Lim, Lindner (2013)
Grimus, Lavoura (2014)

Taking into account that we can make 36 row/column permutations of $|U|^2$, **not all distinct σ's will lead to distinct $|U|^2$**

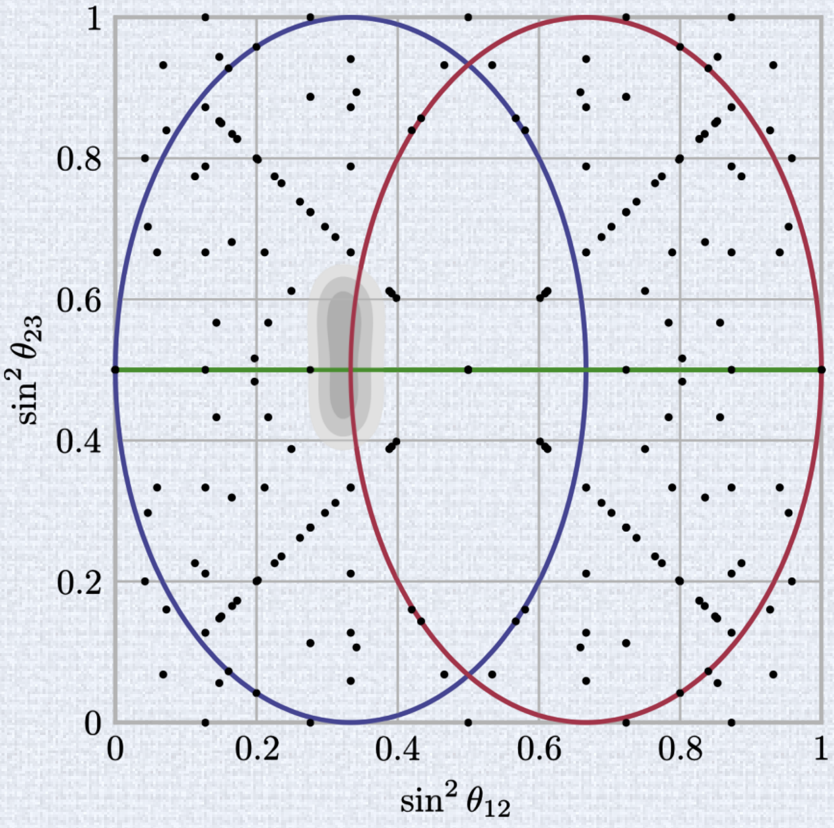
Indeed, σ and σ' will only lead to distinct mixing patterns **if and only if**

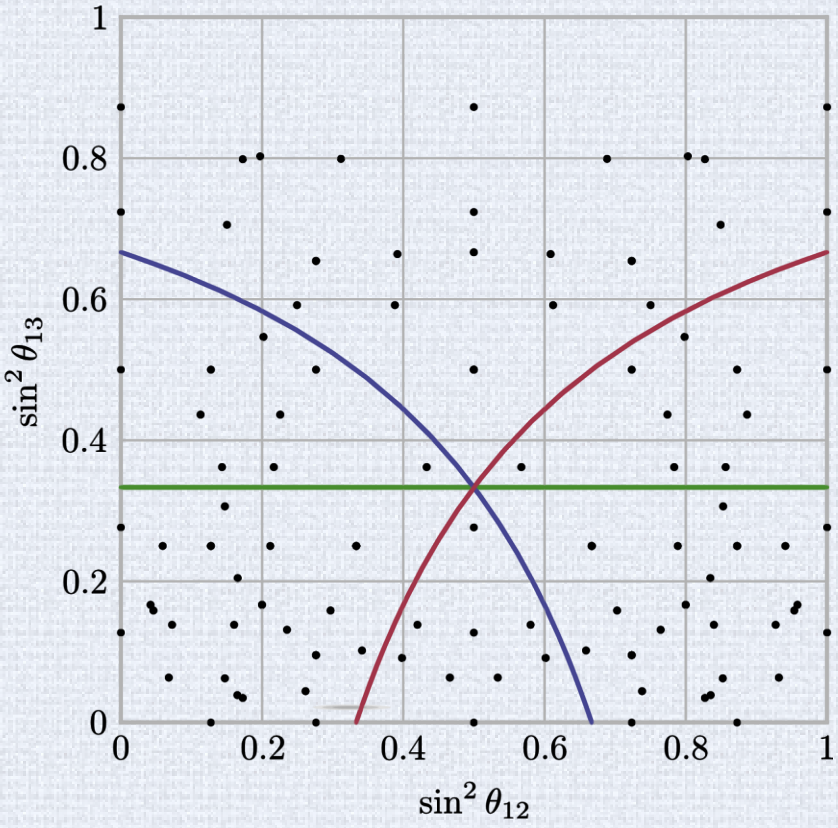
$$\mathrm{Re}\,(\sigma^6) \neq \mathrm{Re}\,(\sigma'^6)$$

Tri-bimaximal mixing is a special case of this family of patterns

$$\mathrm{Re}\,(\sigma^6) = 1 \quad \Longrightarrow \quad |U|^2 = \frac{1}{3}\begin{pmatrix} 1 & 2 & 0 \\ 1 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

All mixing patterns in plots



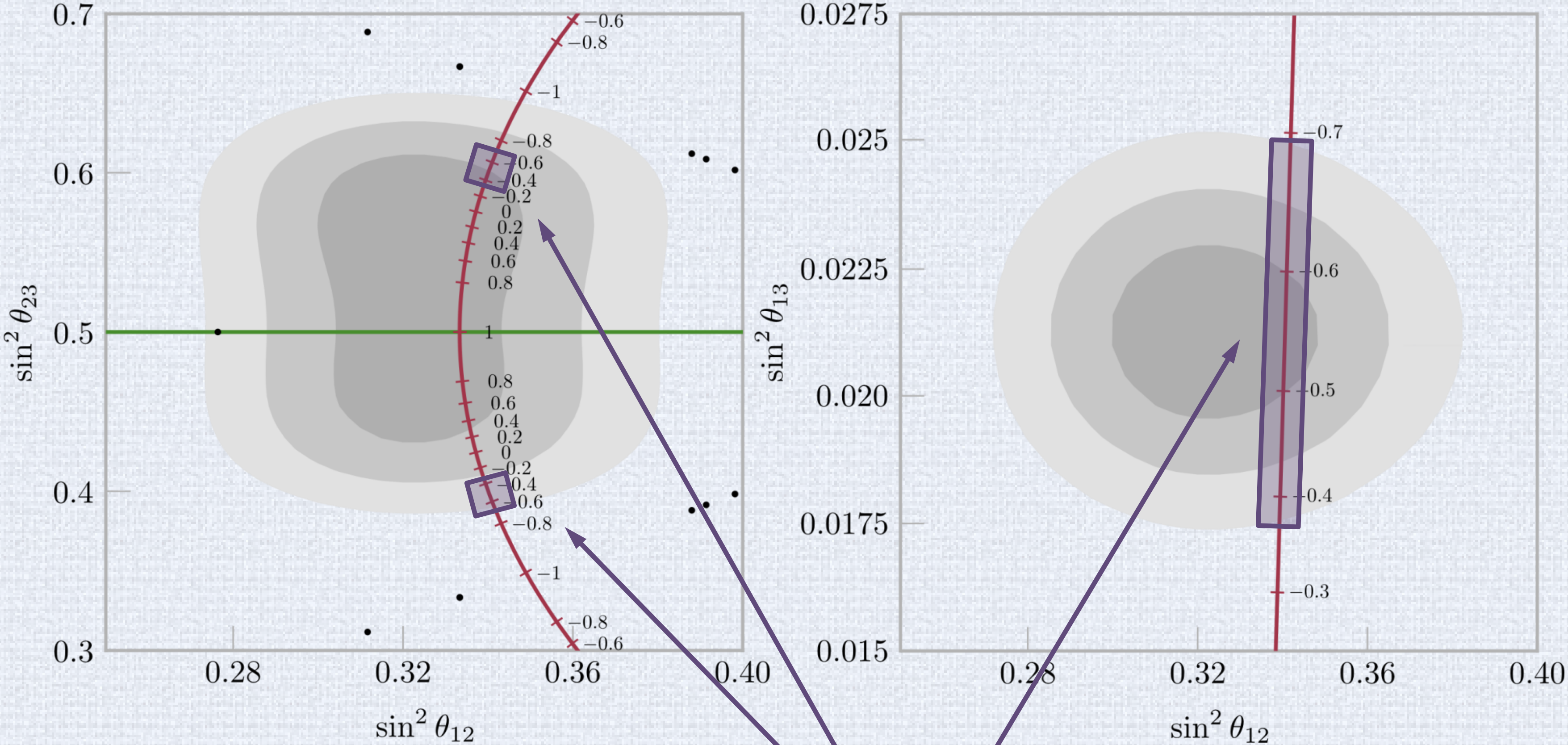


Dots: sporadic cases

Red/Green/Blue lines: different column permutations of the infinite series →

$$|U|^2 = \frac{1}{3}\begin{pmatrix} 1 & 1+\mathrm{Re}\,\sigma & 1-\mathrm{Re}\,\sigma \\ 1 & 1+\mathrm{Re}\,(\omega\sigma) & 1-\mathrm{Re}\,(\omega\sigma) \\ 1 & 1+\mathrm{Re}\,(\omega^2\sigma) & 1-\mathrm{Re}\,(\omega^2\sigma) \end{pmatrix}$$

All mixing patterns in plots



Numbers on red line: $\mathrm{Re}\left(\sigma^6\right)$

$$|U|^2 = \frac{1}{3}\begin{pmatrix} 1 & 1+\mathrm{Re}\,\sigma & 1-\mathrm{Re}\,\sigma \\ 1 & 1+\mathrm{Re}\,(\omega\sigma) & 1-\mathrm{Re}\,(\omega\sigma) \\ 1 & 1+\mathrm{Re}\,(\omega^2\sigma) & 1-\mathrm{Re}\,(\omega^2\sigma) \end{pmatrix}$$

3σ allowed

Concrete prediction:
θ_{23} **is not maximal**

Summary

Summary

1

The flavour structure of the SM remains a mystery

2

Finite symmetries have been used to constrain the lepton mixing matrix to a mass independent form

3

There was no systematic study of all mixing matrices achievable in this way

4

In a recent work, we did such an analysis, relying on simple mathematical results related to roots of unity

The only phenomenologically viable case is the infinite series of mixing patterns with ...

... $\delta = 0, \sin^2\theta_{23} \sim 0.4, 0.6$

Thank you