LL in low-energy EFTs

Johan Bijnens

LEADING LOGARITHMS IN LOW-ENERGY EFFECTIVE FIELD THEORIES

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Overview

Main question: can we get information on high orders?

1. Leading logarithms (LL)
2. Effective field theory (EFT)
3. Weinberg’s argument
4. $O(N + 1)/O(N)$
   - Masses, decay
   - large $N$
   - Other expansions/Numerics
   - Other work
   - $\pi\pi$-scattering
5. Anomaly $O(4)/O(3)$
6. $SU(N) \times SU(N)/SU(N)$
7. Nucleon
8. Conclusions

Work done with Lisa Carloni, Karol Kampf, Stefan Lanz, Alexey A. Vladimirov
References


Leading logarithms (LL)

BEWARE: leading logarithms can mean very different things

- Take a quantity with a single scale: $F(M)$
- Subtraction scale in QFT (in dim. reg.) is logarithmic
  
  \[ L = \log \left( \frac{\mu}{M} \right) \]

\[ F = F_0 + (F_1^1 L + F_0^1) + (F_2^2 L^2 + F_1^2 L + F_0^2) + (F_3^3 L^3 + \cdots) + \cdots \]

- Leading Logarithms: The terms $F_m^m L^m$

  The $F_m^m$ can be more easily calculated than the full result

- $\mu \left( \frac{dF}{d\mu} \right) \equiv 0$
- Ultraviolet divergences in Quantum Field Theory are always local
Renormalizable theories

- Loop expansion $\equiv \alpha$ expansion
  \[
  F = \alpha + (f_1^1 \alpha^2 L + f_0^1 \alpha^2) + (f_2^2 \alpha^3 L^2 + f_1^2 \alpha^3 L + f_0^2 \alpha^3) \\
  + (f_3^3 \alpha^4 L^3 + \cdots) + \cdots
  \]

- $f_i^j$ are pure numbers

- $\frac{dF}{d\mu} \equiv F', \frac{d\alpha}{d\mu} \equiv \alpha', \frac{dL}{d\mu} = 1$

- $F' = \alpha' + (f_1^1 \alpha^2 + f_1^1 2\alpha' \alpha L + f_0^1 2\alpha \alpha')$
  \[
  + (f_2^2 \alpha^3 3L L + f_2^2 \alpha^3 \alpha^2 L^2 + f_1^2 \alpha^3 + 3f_1^2 \alpha' \alpha^2 L + 3f_0^2 \alpha' \alpha^2) \\
  + (f_3^3 \alpha^3 4\alpha' \alpha^3 L^3 + \cdots) + \cdots
  \]

- $\alpha' = \beta_1 \alpha^2 + \beta_2 \alpha^3 + \cdots$

- $0 = F' = (\beta_1 + f_1^1) \alpha^2 + (2\beta_1 f_1^1 + 2f_2^2) \alpha^3 L$
  \[
  + (2\beta_2 f_0^1 + f_1^2) \alpha^3 + (3\beta_1 f_2^2 + 3f_3^3) \alpha^4 L^2 + \cdots
  \]

- $f_1^1 = -\beta_1, \ f_2^2 = \beta_1^2, \ f_3^3 = -\beta_1^3, \ldots$
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  $F = \alpha + \left( f_1^1 \alpha^2 L + f_0^1 \alpha^2 \right) + \left( f_2^2 \alpha^3 L^2 + f_1^2 \alpha^3 L + f_0^2 \alpha^3 \right)$
  
  $\quad + \left( f_3^3 \alpha^4 L^3 + \cdots \right) + \cdots$

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  $F' = \alpha' + \left( f_1^1 \alpha^2 + f_1^1 2 \alpha' \alpha L + f_0^1 2 \alpha \alpha' \right)$
  
  $\quad + \left( f_2^2 \alpha^3 2 L + f_2^2 3 \alpha' \alpha^2 L^2 + f_1^2 \alpha^3 + 3 f_1^2 \alpha' \alpha^2 L + 3 f_0^2 \alpha' \alpha^2 \right)$
  
  $\quad + \left( f_3^3 \alpha^3 3 L^2 + f_3^3 4 \alpha' \alpha^3 L^3 + \cdots \right) + \cdots$

- $\alpha' = \beta_1 \alpha^2 + \beta_2 \alpha^3 + \cdots$

- $0 = F' = \left( \beta_1 + f_1^1 \right) \alpha^2 + \left( 2 \beta_1 f_1^1 + 2 f_2^2 \right) \alpha^3 L$
  
  $\quad + \left( \beta_2 + 2 \beta_1 f_0^1 + f_1^2 \right) \alpha^3 + \left( 3 \beta_1 f_2^2 + 3 f_3^3 \right) \alpha^4 L^2 + \cdots$

- $f_1^1 = -\beta_1, f_2^2 = \beta_1^2, f_3^3 = -\beta_1^3, \ldots$
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  \[ F = \alpha + (f_1^1 \alpha^2 L + f_0^1 \alpha^2) + (f_2^2 \alpha^3 L^2 + f_1^2 \alpha^3 L + f_0^2 \alpha^3) 
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  + (\beta_2 + 2\beta_1 f_0^1 + f_1^2) \alpha^3 + (3\beta_1 f_2^1 + 3f_3^1) \alpha^4 L^2 + \cdots$

- $f_1^1 = -\beta_1, \quad f_2^2 = \beta_1^2, \quad f_3^3 = -\beta_1^3, \ldots$
Renormalizable theories

- Leading logs known as soon as $\beta_1$ is known
  \[ F(M) = \alpha \left( 1 - \alpha \beta_1 L + (\alpha \beta_1 L)^2 + (\alpha \beta_1 L)^3 + \cdots \right) + \cdots \]
  \[ = \frac{\alpha}{1 + \alpha \beta_1 L} + \cdots \]
- Running coupling constant
- Generalizes to lower leading logarithms as well
- Multiscale problems: many other terms possible
Renormalization Group

- Can be extended to other operators as well
- Underlying argument always $\mu \frac{dF}{d\mu} = 0$.
- Gell-Mann–Low, Callan–Symanzik, Weinberg–’t Hooft
- In great detail: J.C. Collins, *Renormalization*
- Relies on the $\alpha$ the same in all orders
- LL one-loop $\beta_0$
- NLL two-loop $\beta_1$

- In effective field theories: different Lagrangian at each order
- **The recursive argument does not work**

http://en.wikipedia.org/wiki/Effective_field_theory

In physics, an effective field theory is an approximate theory (usually a quantum field theory) that contains the appropriate degrees of freedom to describe physical phenomena occurring at a chosen length scale, but ignores the substructure and the degrees of freedom at shorter distances (or, equivalently, higher energies).
Main Ideas:

- Use right degrees of freedom: essence of (most) physics
- If mass-gap in the excitation spectrum: neglect degrees of freedom above the gap.

Examples:

- **Solid state physics**: conductors: neglect the empty bands above the partially filled one
- **Atomic physics**: Blue sky: neglect atomic structure
Power Counting

- gap in the spectrum $\implies$ separation of scales
- with the lower degrees of freedom, build the most general effective Lagrangian

$\Rightarrow \infty \# \text{ parameters}$
$\Rightarrow \text{Where did my predictivity go?}$

$\Rightarrow \text{Need some ordering principle: power counting}$

- Taylor series expansion does not work (convergence radius is zero)
- Continuum of excitation states need to be taken into account
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Power Counting

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- with the lower degrees of freedom, build the most general effective Lagrangian
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Why is the sky blue?

System: Photons of visible light and neutral atoms
Length scales: a few 1000 Å versus 1 Å
Atomic excitations suppressed by $\approx 10^{-3}$

\[ \mathcal{L}_A = \Phi_v^\dagger \partial_t \Phi_v + \ldots \quad \mathcal{L}_{\gamma A} = G F^2_{\mu \nu} \Phi_v^\dagger \Phi_v + \ldots \]

Units with $\hbar = c = 1$: G energy dimension $-3$:

\[ \sigma \approx G^2 E_\gamma^4 \]

blue light scatters a lot more than red $\implies$ red sunsets $\implies$ blue sky

Higher orders suppressed by $1 \, \text{Å}/\lambda_\gamma$. 
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$\sigma \approx G^2 E^4$

blue light scatters a lot more than red \[ \Rightarrow \text{ red sunsets} \quad \Rightarrow \text{ blue sky} \]

Higher orders suppressed by $1 \text{ Å}/\lambda_\gamma$. 
Why Field Theory?

- Only known way to combine QM and special relativity
- Off-shell effects: there as new free parameters

**Drawbacks**
- Many parameters (but finite number at any order)
  - any model has few parameters but model-space is large
- expansion: it might not converge or only badly

**Advantages**
- Calculations are (relatively) simple
- It is general: model-independent
- **Theory** → errors can be estimated
- Systematic: ALL effects at a given order can be included
- Even if no convergence: classification of models often useful
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Examples of EFT

- Fermi theory of the weak interaction
- Chiral Perturbation Theory: hadronic physics
- NRQCD
- SCET
- General relativity as an EFT
- 2, 3, 4 nucleon systems from EFT point of view
Weinberg’s argument for leading logarithms

- Weinberg, Physica A96 (1979) 327

Two-loop leading logarithms can be calculated using only one-loop: **Weinberg consistency conditions**

- \( \pi \pi \) at 2-loop: Colangelo, hep-ph/9502285


- Proof at all orders
  - using \( \beta \)-functions: Büchler, Colangelo, hep-ph/0309049

- First mesonic case where loop-order = power-counting (or \( \hbar \)) order

- Later nucleon case where loop-order \( \neq \) power-counting (or \( \hbar \)) order
Weinberg’s argument

- $\mu$: dimensional regularization scale
- $d = 4 - w$
- loop-expansion $\equiv \hbar$-expansion
- $\mathcal{L}^{\text{bare}} = \sum_{n \geq 0} \hbar^n \mu^{-nw} \mathcal{L}^{(n)}$
- $\mathcal{L}^{(n)} = \sum_i c^{(n)}_i \mathcal{O}_i$
- $c^{(n)}_i = \sum_{k=0,n} c^{(n)}_{ki} \frac{w^k}{\mu}$
- $c^{(n)}_{0i}$ have a direct $\mu$-dependence
- $c^{(n)}_{ki}$ $k \geq 1$ only depend on $\mu$ through $c^{(m<n)}_{0i}$
Weinberg’s argument

- $L^n_{\ell}$-loop contribution at order $\hbar^n$
- Expand in divergences from the loops (not from the counterterms): $L^n_{\ell} = \sum_{k=0, l} \frac{1}{w^k} L^n_{k\ell}$
- Neglected positive powers: not relevant here, but should be kept in general

\[ \{ c \}^n_{\ell} \text{ all combinations } c_{k_1j_1}^{(m_1)} c_{k_2j_2}^{(m_2)} \ldots c_{k_rj_r}^{(m_r)} \text{ with } m_i \geq 1, \\text{such that } \sum_{i=1, r} m_i = n \text{ and } \sum_{i=1, r} k_i = \ell. \]

\[ \{ c \}^n_{2} \equiv \{ c_{ni}^{(n)} \}, \quad \{ c \}^2 = \{ c_{2i}^{(2)}, c_{1i}^{(1)} c_{1k}^{(1)} \} \]

\[ \mathcal{L}^{(n)} = \mathbf{n} \]
Weinberg’s argument

\[
\text{Mass} = 0 + 0 + 0 + 1 + 2 + 2 + 1 + \ldots
\]
Weinberg’s argument

\[ \mathcal{O}(N + 1)/\mathcal{O}(N) \]

\[ SU(N) \times SU(N)/SU(N) \]

\[ \text{Nucleon} \]

\[ \text{Conclusions} \]

- \( \mathcal{O}^0 \): \( L_0^0 \)
- \( \mathcal{O}^1 \): \( \frac{1}{w} (\mu^{-w} L_{00}^1(\{c\}_{11}^1) + L_{11}^1) + \mu^{-w} L_{00}^1(\{c\}_{0}^1) + L_{10}^1 \)

Expand \( \mu^{-w} = 1 - w \log \mu + \frac{1}{2} w^2 \log^2 \mu + \cdots \)

1/w must cancel: \( L_{00}^1(\{c\}_{11}^1) + L_{11}^1 = 0 \)

this determines the \( c_{1i}^1 \)

Explicit log \( \mu \): \( - \log \mu L_{00}^1(\{c\}_{11}^1) \equiv \log \mu L_{11}^1 \)
Weinberg's argument

\(\mathcal{L}_0^0: L_0^0\)

\(\mathcal{L}_1^0: \frac{1}{w} (\mu^{-w} L_{00}^1(\{c\}_{1}^1) + L_{11}^1) + \mu^{-w} L_{00}^1(\{c\}_{0}^1) + L_{10}^1\)

Expand \(\mu^{-w} = 1 - w \log \mu + \frac{1}{2} w^2 \log^2 \mu + \ldots\)

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Explicit log \(\mu\): \(-\log \mu L_{00}^1(\{c\}_{1}^1) \equiv \log \mu L_{11}^1\)
Weinberg’s argument

- $h^2:\quad \frac{1}{w^2} \left( \mu^{-2w} L_{00}^2(\{c\}^2_2) + \mu^{-w} L_{11}^2(\{c\}_1^1) + L_{22}^2 \right) + \frac{1}{w} \left( \mu^{-2w} L_{00}^2(\{c\}_1^1) + \mu^{-w} L_{11}^2(\{c\}_0^1) + \mu^{-w} L_{10}^2(\{c\}_1^1) + L_{21}^2 \right) + \left( \mu^{-2w} L_{00}^2(\{c\}_0^1) + \mu^{-w} L_{10}^2(\{c\}_0^1) + L_{20}^2 \right)$

- $1/w^2$ and log $\mu/w$ must cancel
  
  $L_{00}^2(\{c\}_2^2) + L_{11}^2(\{c\}_1^1) + L_{22}^2 = 0$
  
  $2L_{00}^2(\{c\}_2^2) + L_{11}^2(\{c\}_1^1) = 0$

- Solution: $L_{00}^2(\{c\}_2^2) = -\frac{1}{2} L_{11}^2(\{c\}_1^1)$
  
  $L_{11}^2(\{c\}_1^1) = -2L_{22}^2$

- Explicit log $\mu$ dependence (one-loop is enough)
  
  $\frac{1}{2} \log^2 \mu \left( 4L_{00}^2(\{c\}_2^2) + L_{11}^2(\{c\}_1^1) \right) = -\frac{1}{2} L_{11}^2(\{c\}_1^1) \log^2 \mu$. 

Weinberg’s argument

- \( h^2: \)
  \[
  \frac{1}{w^2} \left( \mu^{-2w} L_{00}^2(\{c\}_2^2) + \mu^{-w} L_{11}^2(\{c\}_1^1) + L_{22}^2 \right) \\
  + \frac{1}{w} \left( \mu^{-2w} L_{00}^2(\{c\}_1^2) + \mu^{-w} L_{11}^2(\{c\}_0^1) + \mu^{-w} L_{10}^2(\{c\}_1^1) + L_{21}^2 \right) \\
  + \left( \mu^{-2w} L_{00}^2(\{c\}_0^2) + \mu^{-w} L_{10}^2(\{c\}_0^1) \right) + L_{20}^2 
  \]

- 1/w² and log \( \mu/w \) must cancel
  \[
  L_{00}^2(\{c\}_2^2) + L_{11}^2(\{c\}_1^1) + L_{22}^2 = 0 \\
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- \( \frac{1}{w^2} \left( \mu^{-2w} L_{00}^2 \{c\}_{2}^2 + \mu^{-w} L_{11}^2 \{c\}_{1}^1 + L_{22}^2 \right) \)
- \( \left( \mu^{-2w} L_{00}^2 \{c\}_{0}^2 + \mu^{-w} L_{10}^2 \{c\}_{1}^1 + L_{20}^2 \right) \)

1/\(w^2\) and \(\log \mu/w\) must cancel

\[
L_{00}^2 \{c\}_{2}^2 + L_{11}^2 \{c\}_{1}^1 + L_{22}^2 = 0
\]

\[
2L_{00}^2 \{c\}_{2}^2 + L_{11}^2 \{c\}_{1}^1 = 0
\]

Solution: \(L_{00}^2 \{c\}_{2}^2 = -\frac{1}{2} L_{11}^2 \{c\}_{1}^1\)

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Explicit log \(\mu\) dependence (one-loop is enough)

\[
\frac{1}{2} \log^2 \mu \left( 4L_{00}^2 \{c\}_{2}^2 + L_{11}^2 \{c\}_{1}^1 \right) = -\frac{1}{2} L_{11}^2 \{c\}_{1}^1 \log^2 \mu.
\]
All orders

\[ \frac{1}{w^n} \left( \mu^{-nw} L_{00}^n(\{c\}_n) + \mu^{-(n-1)w} L_{11}^n(\{c\}_{n-1}) + \cdots \right. \]
\[ + \mu^{-w} L_{n-1}^n n_{-1}(\{c\}_{11}) + L_{nn}^n \left) + \frac{1}{w^{n-1}} \cdots \right. \]

1/w^n, log \mu/w^{n-1}, \ldots, log^{n-1} \mu/w cancel:

\[ \sum_{i=0}^n \binom{n}{j} L_{n-i}^n n_{-i}(\{c\}_i) = 0 \quad j = 0, \ldots, n - 1. \]

Solution: \[ L_{n-i}^n n_{-i}(\{c\}_i) = (-1)^i \binom{n}{i} L_{nn}^n \]

explicit leading log \mu dependence and divergence

\[ \log^n \mu \frac{(-1)^{n-1}}{n} L_{11}^n(\{c\}_{n-1}) \]

\[ L_{00}^n(\{c\}_n) = -\frac{1}{n} L_{11}^n(\{c\}_{n-1}) \]
All orders

- $\hbar^n:\quad \frac{1}{w^n} \left( \mu^{-nw} L_{00}^n(\{c\}_n^n) + \mu^{-(n-1)w} L_{11}^n(\{c\}_{n-1}^{n-1}) + \cdots \right.
\left. + \mu^{-w} L_{n-1}^n(\{c\}_1^1) + L_{nn}^n \right) + \frac{1}{w^{n-1}} \cdots$

- $1/w^n, \log \mu/w^{n-1}, \ldots, \log^{n-1} \mu/w$ cancel:
$$\sum_{i=0}^{n} j^i L_{n-i}^n(\{c\}_{i}^{i}) = 0 \quad j = 0, \ldots, n - 1.$$

- Solution: $L_{n-i}^n(\{c\}_{i}^{i}) = (-1)^{i} \binom{n}{i} L_{nn}^n$

- Explicit leading log $\mu$ dependence and divergence
$$\log^n \mu \frac{(-1)^{n-1}}{n} L_{11}^n(\{c\}_{n-1}^{n-1})$$
$$L_{00}^n(\{c\}_n^n) = -\frac{1}{n} L_{11}^n(\{c\}_{n-1}^{n-1})$$
All orders

- $\frac{\hbar^n}{w^n} \left( \mu^{-nw} L_{00}^n(\{c\}_n^n) + \mu^{-(n-1)w} L_{11}^n(\{c\}_n^{n-1}) + \cdots \right.
  
  + \mu^{-w} L_{n-1}^n(\{c\}_1^1) + L_{nn}^n \right) + \frac{1}{w^{n-1}} \cdots$

  1/$w^n$, log $\mu/w^{n-1}$, $\ldots$, log$^{n-1} \mu/w$ cancel:

  $$\sum_{i=0}^n j^i L_{n-i}^n(\{c\}_i^i) = 0 \quad j = 0, \ldots, n - 1.$$ 

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  explicit leading log $\mu$ dependence and divergence

  $$\log^n \mu \frac{(-1)^{n-1}}{n} L_{11}^n(\{c\}_n^{n-1})$$

  $$L_{00}^n(\{c\}_n^n) = -\frac{1}{n} L_{11}^n(\{c\}_n^{n-1})$$
Mass to $\hbar^2$

- $\hbar^1$: $0 \rightarrow 1$

- $\hbar^2$: $1 \rightarrow 0 \rightarrow 2$

- but also needs $\hbar^1$: $0 \rightarrow 1$
Mass to order $\hbar^3$
Mass to order $\hbar^6$
Calculate the divergence
- rewrite it in terms of a local Lagrangian
  - Luckily: symmetry kept: we know result will be symmetrical, hence do not need to explicitly rewrite the Lagrangians in a nice form
  - Luckily: we do not need to go to a minimal Lagrangian
  - So everything can be computerized
  - Thank Jos Vermaseren for FORM

We keep all terms to have all 1PI (one particle irreducible) diagrams finite
Massive $O(N)$ sigma model

- $O(N+1)/O(N)$ nonlinear sigma model
  \[ \mathcal{L}_{n\sigma} = \frac{F^2}{2} \partial_\mu \Phi^T \partial^\mu \Phi + F^2 \chi^T \Phi. \]
  - $\Phi$ is a real $N+1$ vector; $\Phi \to O\Phi$; $\Phi^T \Phi = 1$.
  - Vacuum expectation value $\langle \Phi^T \rangle = (1 \ 0 \ldots 0)$
  - Explicit symmetry breaking: $\chi^T = (M^2_0 \ldots 0)$
  - Both spontaneous and explicit symmetry breaking
  - $N$-vector $\phi$

- $N$ (pseudo-)Nambu-Goldstone Bosons
- $N = 3$ is two-flavour Chiral Perturbation Theory
Massive $O(N)$ sigma model

- $O(N+1)/O(N)$ nonlinear sigma model
- $\mathcal{L}_{n\sigma} = \frac{F^2}{2} \partial_\mu \Phi^T \partial^\mu \Phi + F^2 \chi^T \Phi$.
- $\Phi$ is a real $N+1$ vector; $\Phi \rightarrow O\Phi$; $\Phi^T \Phi = 1$.
- Vacuum expectation value $\langle \Phi^T \rangle = (1 \ 0 \ldots 0)$
- Explicit symmetry breaking: $\chi^T = (M^2_0 \ldots 0)$
- Both spontaneous and explicit symmetry breaking
- $N$-vector $\phi$
- $N$ (pseudo-)Nambu-Goldstone Bosons
- $N = 3$ is two-flavour Chiral Perturbation Theory
Massive $O(N)$ sigma model: $\Phi$ vs $\phi$

\[ \Phi_1 = \begin{pmatrix} \sqrt{1 - \frac{\phi^T \phi}{F^2}} \\ \frac{\phi^1}{F} \\ \vdots \\ \frac{\phi^N}{F} \end{pmatrix} = \begin{pmatrix} \sqrt{1 - \frac{\phi^T \phi}{F^2}} \\ \frac{\phi}{F} \end{pmatrix} \]

Gasser, Leutwyler

\[ \Phi_5 = \frac{1}{1 + \frac{\phi^T \phi}{4F^2}} \left( 1 - \frac{\phi^T \phi}{2F^2} \right) \]

Weinberg

\[ \Phi_3 = \begin{pmatrix} 1 - \frac{1}{2} \frac{\phi^T \phi}{F^2} \\ \frac{\phi}{\sqrt{\phi^T \phi}} \end{pmatrix} \]

only mass term

\[ \Phi_4 = \begin{pmatrix} \cos \sqrt{\frac{\phi^T \phi}{F^2}} \\ \sin \sqrt{\frac{\phi^T \phi}{F^2}} \end{pmatrix} \]

CCWZ
Massive $O(N)$ sigma model: Checks

Need (many) checks:

- use many different parametrizations
- compare with known results:
  \[ M_{\text{phys}}^2 = M^2 \left( 1 - \frac{1}{2} L_M + \frac{17}{8} L_M^2 + \cdots \right), \]
  \[ L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2} \]
  Usual choice $\mathcal{M} = M$.

- large $N$ but massive results more hidden
  Coleman, Jackiw, Politzer 1974

- JB, Carloni, : mass to 5 loops
\[ M_{\text{phys}}^2 = M^2 \left( 1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \ldots \right) \]

\[ L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2} \]

<table>
<thead>
<tr>
<th>( i )</th>
<th>( a_i, \ N = 3 )</th>
<th>( a_i ) for general ( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-\frac{1}{2})</td>
<td>( 1 - \frac{N}{2} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{17}{8} )</td>
<td>( \frac{7}{4} - \frac{7N}{4} + \frac{5N^2}{8} )</td>
</tr>
<tr>
<td>3</td>
<td>(-\frac{103}{24})</td>
<td>( \frac{37}{12} - \frac{113N}{24} + \frac{15N^2}{4} - N^3 )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{24367}{1152})</td>
<td>( \frac{839}{144} - \frac{1601N}{144} + \frac{695N^2}{48} - \frac{135N^3}{16} + \frac{231N^4}{128} )</td>
</tr>
<tr>
<td>5</td>
<td>(-\frac{8821}{144})</td>
<td>( \frac{33661}{2400} - \frac{1151407N}{43200} + \frac{197587N^2}{4320} - \frac{12709N^3}{300} + \frac{6271N^4}{320} - \frac{7N^5}{2} )</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{1922984667}{6220800})</td>
<td>( \frac{158393809}{3888000} - \frac{182792131}{2592000} \cdot N ) + ( \frac{1046805817}{7776000} \cdot N^2 - \frac{17241967}{103680} \cdot N^3 ) + ( \frac{70046633}{576000} \cdot N^4 - \frac{23775}{512} \cdot N^5 + \frac{7293}{1024} \cdot N^6 )</td>
</tr>
</tbody>
</table>
\[ F_{\text{phys}} = F(1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \ldots) \]

<table>
<thead>
<tr>
<th>i</th>
<th>( a_i ) for ( N = 3 )</th>
<th>( a_i ) for general ( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>(-\frac{1}{2} + \frac{1}{2} N)</td>
</tr>
<tr>
<td>2</td>
<td>(-\frac{5}{4})</td>
<td>(-\frac{1}{2} + 7/8 N - 3/8 N^2)</td>
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<tr>
<td>3</td>
<td>(\frac{83}{24})</td>
<td>(-\frac{7}{24} + 21/16 N - 73/48 N^2 + 1/2 N^3)</td>
</tr>
<tr>
<td>4</td>
<td>(-\frac{3013}{288})</td>
<td>(47/576 + 1345/864 N - 14077/3456 N^2 + 625/192 N^3)</td>
</tr>
<tr>
<td></td>
<td>(\frac{2060147}{51840})</td>
<td>(-105/128 N^4)</td>
</tr>
<tr>
<td>5</td>
<td>(-\frac{69228787}{466560})</td>
<td>(-23087/64800 + 459413/172800 N - 189875/20736 N^2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(+546941/43200 N^3 - 1169/160 N^4 + 3/2 N^5)</td>
</tr>
<tr>
<td>6</td>
<td>(-\frac{277079063}{93312000} + 1680071029/186624000 , N)</td>
<td>(-686641633/31104000 N^2 + 813791909/20736000 N^3)</td>
</tr>
<tr>
<td></td>
<td>(-128643359/3456000 N^4 + 260399/15360 N^5 - 3003/1024 N^6)</td>
<td></td>
</tr>
</tbody>
</table>
Results

\[ \langle \bar{q}_i q_i \rangle = -BF^2(1 + c_1 L_M + c_2 L_M^2 + c_3 L_M^3 + \ldots) \]

\[ M^2 = 2B \hat{m} \quad \chi^T = 2B(s 0 \ldots 0) \]

\( s \) corresponds to \( \bar{u}u + \bar{d}d \) current

<table>
<thead>
<tr>
<th>i</th>
<th>( c_i ) for ( N = 3 )</th>
<th>( c_i ) for general ( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3/2</td>
<td>( \frac{N}{2} )</td>
</tr>
<tr>
<td>2</td>
<td>(-\frac{9}{8})</td>
<td>( \frac{3N}{4} - \frac{3N^2}{8} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{9}{2} )</td>
<td>( \frac{3N}{2} - \frac{3N^2}{2} + \frac{N^3}{2} )</td>
</tr>
<tr>
<td>4</td>
<td>(-\frac{1285}{128})</td>
<td>( \frac{145N}{48} - \frac{55N^2}{12} + \frac{105N^3}{32} - \frac{105N^4}{128} )</td>
</tr>
<tr>
<td>5</td>
<td>46</td>
<td>( \frac{3007N}{480} - \frac{1471N^2}{120} + \frac{557N^3}{40} - \frac{1191N^4}{160} + \frac{3N^5}{2} )</td>
</tr>
</tbody>
</table>

Anyone recognize any funny functions?
**Power counting:** pick $\mathcal{L}$ extensive in $N \Rightarrow F^2 \sim N, \ M^2 \sim 1$

\[
\begin{align*}
\text{2n legs} & \quad \iff F^{2-2n} \sim \frac{1}{N^{n-1}} \\
\text{N} & \quad \iff N
\end{align*}
\]

**1PI diagrams:**
\[
\begin{align*}
N_L &= N_I - \sum_n N_{2n} + 1 \\
2N_I + N_E &= \sum_n 2nN_{2n}
\end{align*}
\]
\[
\Rightarrow N_L = \sum_n (n - 1)N_{2n} - \frac{1}{2}N_E + 1
\]

**Diagram suppression factor:**
\[
\frac{N_L^{N_L}}{N_E^{N_E/2-1}}
\]
Large $N$

- diagrams with shared lines are suppressed
- each new loop needs also a new flavour loop
- in the large $N$ limit only “cactus” diagrams survive:

\[ \frac{O(N+1)}{O(N)} \]

Other expansions/Numerics
Other work
$\pi\pi$-scattering

Anomaly
$SU(N) \times SU(N) / SU(N)$
Nucleon
Conclusions
large N: propagator

Generate recursively via a Gap equation

\[(\cdots)^{-1} = (\cdots)^{-1} + \frac{0}{0} + \frac{0}{0} + \frac{0}{0} + \frac{0}{0} + \cdots\]

⇒ resum the series and look for the pole

\[
\bar{A}(m^2) = \frac{m^2}{16\pi^2} \log \frac{\mu^2}{m^2}.
\]

Solve recursively, agrees with other result

Note: can be done for all parametrizations
large N: Decay constant

\[ F_{\text{phys}} = F \sqrt{1 + \frac{N}{F^2} \overline{A}(M_{\text{phys}}^2)} \]

⇒ and include wave-function renormalization

Solve recursively, agrees with other result

Note: can be done for all parametrizations
large N: Vacuum Expectation Value

\[ \langle \bar{q}q \rangle_{\text{phys}} = \langle \bar{q}q \rangle_0 \sqrt{1 + \frac{N}{F^2} \overline{A}(M_{\text{phys}}^2)} \]

Comments:
- These are the full* leading $N$ results, not just leading log
- But depends on the choice of $N$-dependence of higher order coefficients
- Assumes higher LECs zero ($< N^{n+1}$ for $\hbar^n$)
- Large $N$ as in $O(N)$ not large $N_c$
Alternative expansions

- $L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2}$

- $\tilde{L}_M = \frac{M_{\text{phys}}^2}{16\pi^2 F_{\text{phys}}^2} \log \frac{\mu^2}{M_{\text{phys}}^2}$

- $L_{\text{phys}} = \frac{M_{\text{phys}}^2}{16\pi^2 F_{\text{phys}}^2} \log \frac{\mu^2}{M_{\text{phys}}^2}$

For masses expansion in $\tilde{L}_M$ best, but no general obvious choice
Numerical results

Left: \( \frac{M_{\text{phys}}^2}{M^2} = 1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \cdots \) \n
\( F = 90 \text{ MeV}, \mu = 0.77 \text{ GeV} \)

Right: \( \frac{M_{\text{phys}}^2}{M^2} = 1 + c_1 L_{\text{phys}} + c_2 L_{\text{phys}}^2 + c_3 L_{\text{phys}}^3 + \cdots \) \n
\( F_\pi = 92 \text{ MeV}, \mu = 0.77 \text{ GeV} \)
Numerical results

Left: \( \frac{F_{\text{phys}}}{F} = 1 + a_1 L M + a_2 L^2 M + a_3 L^3 M + \cdots \)

\( F = 90 \text{ MeV}, \mu = 0.77 \text{ GeV} \)

Right: \( \frac{F_{\text{phys}}}{F} = 1 + c_1 L_{\text{phys}} + c_2 L^2_{\text{phys}} + c_3 L^3_{\text{phys}} + \cdots \)

\( F_\pi = 92 \text{ MeV}, \mu = 0.77 \text{ GeV} \)
Other results

- Bissegger, Fuhrer, hep-ph/0612096  Dispersive methods, massless $\Pi_S$ to five loops
  - In the massless case tadpoles vanish
  - hence the number of external legs needed does not grow
  - All 4-meson vertices via Legendre polynomials
  - can do divergence of all one-loop diagrams analytically
  - algebraic (but quadratic) recursion relations
  - massless $\pi\pi$, $F_V$ and $F_S$ to arbitrarily high order
  - large $N$ agrees with Coleman, Wess, Zumino
  - large $N$ is not a good approximation
Large $N$: $\pi\pi$-scattering

- Semiclassical methods, Coleman, Jackiw, Politzer 1974
- Diagram resummation, Dobado, Pelaez 1992
  
  $$ A(\phi^i \phi^j \rightarrow \phi^k \phi^l) = A(s, t, u)\delta^{ij}\delta^{kl} + A(t, u, s)\delta^{ik}\delta^{jl} + A(u, s, t)\delta^{il}\delta^{jk} $$
  
  $$ A(s, t, u) = A(s, u, t) $$
  
  Proof same as Weinberg’s for $O(4)/O(3)$, group theory and crossing
Large $N$: $\pi\pi$-scattering

- Cactus diagrams for $A(s, t, u)$
  - Branch with no momentum: resummed by
  - Branch starting at vertex: resum by

\[
\begin{align*}
\text{\ding{152}} & = \text{\ding{153}} + \text{\ding{154}} + \text{\ding{155}} + \text{\ding{156}} + \cdots \\
\text{\ding{157}} & = \text{\ding{158}} + \text{\ding{159}}
\end{align*}
\]

- The full result is then
  - Can be summarized by a recursive equation
Large $N$: $\pi\pi$ scattering

\[ y = \frac{N}{F^2} \bar{A}(\mathcal{M}_{\text{phys}}^2) \]

\[ A(s, t, u) = \frac{s F^2(1+y)}{F^2(1+y)^{3/2}} - \frac{M^2}{F^2(1+y)^{3/2}} \right) = \frac{B(\mathcal{M}_{\text{phys}}^2, \mathcal{M}_{\text{phys}}^2, s)}{1 - \frac{1}{2} \left( \frac{s}{F^2(1+y)} - \frac{M^2}{F^2(1+y)^{3/2}} \right) B(\mathcal{M}_{\text{phys}}^2, \mathcal{M}_{\text{phys}}^2, s) \]

or

or

\[ A(s, t, u) = \frac{s - \mathcal{M}_{\text{phys}}^2}{F_{\text{phys}}} \right) B(\mathcal{M}_{\text{phys}}^2, \mathcal{M}_{\text{phys}}^2, s) \]

- $M^2 \to 0$ agrees with the known results
- Agrees with our 4-loop results
Anomaly for $O(4)/O(3)$


$$\mathcal{L}_{WZW} = -\frac{N_c}{8\pi^2} \epsilon_{\mu \nu \rho \sigma} \left\{ \epsilon^{abc} \left( \frac{1}{3} \Phi^0 \partial_\mu \Phi^a \partial_\nu \Phi^b \partial_\rho \Phi^c - \partial_\mu \Phi^0 \partial_\nu \Phi^a \partial_\rho \Phi^b \Phi^c \right) \nu^0_\sigma \\ + (\partial_\mu \Phi^0 \Phi^a - \Phi^0 \partial_\mu \Phi^a) v_\nu^a \partial_\rho v_\nu^0 + \frac{1}{2} \epsilon^{abc} \Phi^0 \Phi^a v_\nu^b v_\rho^c \partial_\sigma v_\mu^0 \right\}.$$

$$A(\pi^0 \rightarrow \gamma(k_1)\gamma(k_2)) = \epsilon_{\mu \nu \alpha \beta} \varepsilon_{1}^{*\mu}(k_1)\varepsilon_{2}^{*\nu}(k_2) k_1^{\alpha} k_2^{\beta} F_{\pi\gamma\gamma}(k_1^2, k_2^2)$$

$$F_{\pi\gamma\gamma}(k_1^2, k_2^2) = \frac{e^2}{4\pi^2 F_{\pi}} \hat{F} F_{\gamma}(k_1^2) F_{\gamma}(k_2^2) F_{\gamma\gamma}(k_1^2, k_2^2)$$

$\hat{F}$: on-shell photon; $F_{\gamma}(k^2)$: formfactor; $F_{\gamma\gamma}$ nonfactorizable part
Anomaly for $O(4)/O(3)$

- Done to six-loops
  
  \[ \hat{F} = 1 + 0 - 0.000372 + 0.000088 + 0.000036 + 0.000009 + 0.0000002 + \ldots \]

- Really good convergence

- $F_{\gamma\gamma}$ only starts at three-loop order (could have been two)

- $F_{\gamma\gamma}$ in the chiral limit only starts at four-loops.

The leading logarithms thus predict this part to be fairly small.

- $F_{\gamma}(k^2)$: plot
Anomaly for $O(4)/O(3)$

Leading logs small, converge fast
\( \gamma 3\pi \)

- Experiment 1: \( \bar{F}_{3\pi}^{\exp} = 12.9 \pm 0.9 \pm 0.5 \text{ GeV}^{-3} \)
- Experiment 2: \( F_{0,\exp}^{3\pi} = 9.9 \pm 1.1 \text{ GeV}^{-3} \)
- Theory lowest order: \( F_0^{3\pi} = 9.8 \text{ GeV}^{-3} \)
- Theory (LL only): \( F_0^{3\pi LL} = (9.8 - 0.3 + 0.04 + 0.02 + 0.006 + 0.001 + \ldots) \text{ GeV}^{-3} \)
- good convergence
$SU(N) \times SU(N)/SU(N)$

- $SU(N) \times SU(N)/SU(N)$ (vector and scalar)
- Mass, Decay constants, Form-factors
- Meson-Meson, $\gamma\gamma \rightarrow \pi\pi$
- No luck with guess for general $N$-dependence either
- For different parametrizations of a unitary matrix used
### Mass

<table>
<thead>
<tr>
<th>$i$</th>
<th>$a_i$ for $N = 2$</th>
<th>$a_i$ for $N = 3$</th>
<th>$a_i$ for general $N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-1/2$</td>
<td>$-1/3$</td>
<td>$-N^{-1}$</td>
</tr>
<tr>
<td>2</td>
<td>$17/8$</td>
<td>$27/8$</td>
<td>$9/2 N^{-2} - 1/2 + 3/8 N^2$</td>
</tr>
<tr>
<td>3</td>
<td>$-103/24$</td>
<td>$-3799/648$</td>
<td>$-89/3 N^{-3} + 19/3 N^{-1} - 37/24 N - 1/12 N^3$</td>
</tr>
<tr>
<td>4</td>
<td>$24367/1152$</td>
<td>$146657/2592$</td>
<td>$2015/8 N^{-4} - 773/12 N^{-2} + 193/18 + 121/288 N^2 + 41/72 N^4$</td>
</tr>
<tr>
<td>5</td>
<td>$-8821/144$</td>
<td>$-27470059/186624$</td>
<td>$-38684/15 N^{-5} + 6633/10 N^{-3} - 59303/1080 N^{-1}$</td>
</tr>
<tr>
<td>6*</td>
<td>$1922964667/6220800$</td>
<td>$12902773163/9331200$</td>
<td>$7329919/240 N^{-6} - 1652293/240 N^{-4}$ $-4910303/15552 N^{-2} + 205365409/972000$ $-69368761/777600 N^2 + 14222209/2592000 N^4 + 3778133/3110400 N^6$</td>
</tr>
</tbody>
</table>
Meson-meson scattering

\[ M(s, t, u) = \left[ \text{tr} T^a T^b T^c T^d + \text{tr} T^a T^d T^c T^b \right] B(s, t, u) + \left[ \text{tr} T^a T^c T^d T^b + \text{tr} T^a T^b T^d T^c \right] B(t, u, s) + \left[ \text{tr} T^a T^d T^b T^c + \text{tr} T^a T^c T^b T^d \right] B(u, s, t) + \delta^{ab} \delta^{cd} C(s, t, u) + \delta^{ac} \delta^{bd} C(t, u, s) + \delta^{ad} \delta^{bc} C(u, s, t). \]

- Two functions needed
- Leading logs done to five loops
- 7 different channels (\(\pi\pi\) has I=0,1,2)
- No obvious pattern, not even large \(N\)
Nucleon Lagrangian

- We use the heavy-baryon approach, explicit powercounting
- LO Lagrangian is order $p$ (mesons $p^2$):
  \[ \mathcal{L}^{(0)}_{N\pi} = \bar{N} \left( i\nu^\mu D_\mu + g_A S^\mu u_\mu \right) N \]
- Propagator is order $1/p$ (mesons $1/p^2$)
- Loops add $p^2$ just as for mesons
- Different parametrizations for mesons

Two different $p^2$ Lagrangians:

\[ \mathcal{L}^{(1)}_{N\pi} = \bar{N} \nu \left[ \left( (\nu \cdot D)^2 - D \cdot D - ig_A \{ S \cdot D, \nu \cdot u \} \right) \frac{1}{2M} \right] + c_1 \text{tr} (\chi_+) + \left( c_2 - \frac{g_A^2}{8M} \right) (\nu \cdot u)^2 + c_3 u \cdot u + \left( c_4 + \frac{1}{4M} \right) i\nu^{\mu\nu\rho\sigma} u_\mu u_\nu v_\rho S_\sigma \]

\[ \mathcal{L}^{(1)}_{N\pi} = \bar{N} \nu \left[ -\frac{1}{2} (D_\mu D^\mu + ig_A \{ S_\mu D^\mu, \nu \cdot u_\nu \}) + A_1 \text{tr} (u_\mu u^\mu) + A_2 \text{tr} ((\nu_\mu u^\mu)^2) + A_3 \text{tr} (\chi_+) + A_5 i\nu^{\mu\nu\rho\sigma} v_\mu S_\nu u_\rho u_\sigma \right] N \]
Nucleon loops

- set $\hbar^n \sim p^{n+1}$ for meson-nucleon
- set $\hbar^n \sim p^{n+2}$ for mesons
- Introduce a RGO renormalization group order $\approx \max$ power of $1/w$
- same $p$-order can be different RGO, e.g. both $p^5$, left RGO 1, right RGO 2
- Note: same equations, if no tree level contribution next-to-leading log also calculable
- For nucleon can have fractional powers of quark masses
Nucleon loops

- set $\hbar^n \sim p^{n+1}$ for meson-nucleon
- set $\hbar^n \sim p^{n+2}$ for mesons
- Introduce a RGO renormalization group order $\approx \max \text{power of } 1/w$
- same $p$-order can be different RGO, e.g.

both $p^5$, left RGO 1, right RGO 2

- Note: same equations, if no tree level contribution next-to-leading log also calculable
- For nucleon can have fractional powers of quark masses
Results

\( M: \) nucleon mass, \( m: \) pion mass, \( L = \frac{m^2}{4\pi F} \log \frac{\mu}{m^2} \)

\[
M_{\text{phys}} = M + k_2 \frac{m^2}{M} + k_3 \frac{\pi m^3}{(4\pi F)^2} + k_4 \frac{m^4}{(4\pi F)^2 M} \ln \frac{\mu^2}{m^2} \\
+ k_5 \frac{\pi m^5}{(4\pi F)^4} \ln \frac{\mu^2}{m^2} + \ldots
\]

\[
= M + \frac{m^2}{M} \sum_{n=1}^{\infty} k_{2n} L^{n-1} + \pi m \frac{m^2}{(4\pi F)^2} \sum_{n=1}^{\infty} k_{2n+1} L^{n-1},
\]
### Results

<table>
<thead>
<tr>
<th>$k_2$</th>
<th>$-4c_1M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_3$</td>
<td>$-\frac{3}{2}g_A^2$</td>
</tr>
<tr>
<td>$k_4$</td>
<td>$\frac{3}{4}(g_A^2 + (c_2 + 4c_3 - 4c_1)M) - 3c_1M$</td>
</tr>
<tr>
<td>$k_5$</td>
<td>$\frac{3g_A^2}{8}(3 - 16g_A^2)$</td>
</tr>
<tr>
<td>$k_6$</td>
<td>$-\frac{3}{4}(g_A^2 + (c_2 + 4c_3 - 4c_1)M) + \frac{3}{2}c_1M$</td>
</tr>
<tr>
<td>$k_7$</td>
<td>$g_A^2\left(-18g_A^4 + \frac{35g_A^2}{4} - \frac{443}{64}\right)$</td>
</tr>
<tr>
<td>$k_8$</td>
<td>$\frac{27}{8}(g_A^2 + (c_2 + 4c_3 - 4c_1)M) - \frac{9}{2}c_1M$</td>
</tr>
<tr>
<td>$k_9$</td>
<td>$\frac{g_A^2}{3}\left(-116g_A^6 + \frac{2537g_A^4}{20} - \frac{3569g_A^2}{24} + \frac{55609}{1280}\right)$</td>
</tr>
<tr>
<td>$k_{10}$</td>
<td>$-\frac{257}{32}(g_A^2 + (c_2 + 4c_3 - 4c_1)M) + \frac{257}{32}c_1M$</td>
</tr>
<tr>
<td>$k_{11}$</td>
<td>$\frac{g_A^2}{2}\left(-95g_A^8 + \frac{5187407g_A^6}{20160} - \frac{449039g_A^4}{945} + \frac{16733923g_A^2}{60480} - \frac{298785521}{1935360}\right)$</td>
</tr>
</tbody>
</table>

- $g_A \leftrightarrow -g_A$: only even powers
- $k_{2n}$ peculiar structure
- Drop $g_A^3$ then can calculate $k_{12}$
Results

| \( r_2 \) | \(-4c_1 M\) |
| \( r_3 \) | \(-\frac{3}{2}g_A^2\) |
| \( r_4 \) | \(\frac{3}{4}(g_A^2 + (c_2 + 4c_3 - 4c_1)M) - 5c_1 M\) |
| \( r_5 \) | \(-6g_A^2\) |
| \( r_6 \) | \(5c_1 M\) |
| \( r_7 \) | \(\frac{g_A^2}{4}(-8 + 5g_A^2 - 72g_A^4)\) |
| \( r_8 \) | \(\frac{25}{3}c_1 M\) |
| \( r_9 \) | \(\frac{g_A^2}{3}(-116g_A^6 + \frac{647g_A^4}{20} - \frac{457g_A^2}{12} + \frac{17}{40})\) |
| \( r_{10} \) | \(\frac{725}{36}c_1 M\) |
| \( r_{11} \) | \(\frac{g_A^2}{2}(95g_A^8 - \frac{1679567g_A^6}{20160} + \frac{451799g_A^4}{3780} - \frac{320557g_A^2}{15120} + \frac{896467}{60480})\) |
| \( r_{12} \) | \(\frac{175}{4}c_1 M\) |

- everything rewritten in terms of physical pion mass
- Simpler expression
Results

Conjecture:

\[
M = M_{\text{phys}} + \frac{3}{4} m_{\text{phys}}^4 \frac{\log \frac{\mu^2}{m_{\text{phys}}^2}}{(4\pi F)^2} \left( \frac{g_A^2}{M_{\text{phys}}} - 4c_1 + c_2 + 4c_3 \right) \\
- \frac{3c_1}{(4\pi F)^2} \int_{m_{\text{phys}}^2}^{\mu^2} m_{\text{phys}}^4(\mu') \frac{d\mu'^2}{\mu'^2}.
\]

Take now known result for pion mass, \(k_{14}\) and \(k_{16}\) calculable
Numerics

\[ M = 938 \text{ MeV}, \]
\[ c_1 = -0.87 \text{ GeV}^{-1}, \]
\[ c_2 = 3.34 \text{ GeV}^{-1}, \]
\[ c_3 = -5.25 \text{ GeV}^{-1}, \]
\[ g_A = 1.25, \]
\[ \mu = 0.77 \text{ GeV}, \]
\[ F = 92.4 \text{ MeV}. \]
Numerics

The graph shows the relationship between $M_{\text{phys}} - M$ (in GeV) and $n$. The y-axis represents the difference in mass, ranging from $10^{-8}$ to 1 GeV, while the x-axis represents the variable $n$. The data points are marked with different symbols, indicating distinct categories or groups.

Key points:
- The mass difference decreases as $n$ increases.
- The graph makes use of a logarithmic scale for both axes, allowing for a wide range of values to be plotted on a single graph.
- The data points are spread across the range of $n$ from 2 to 16, showing a clear trend.

Related concepts:
- Weinberg’s anomaly
- $SU(N) \times SU(N) / SU(N)$
- Conclusions
Conclusions

- Leading logarithms can be calculated using only one-loop diagrams
- Results for a large number of quantities for mesons
- Look at the (very) many tables, we would be very interested in all-order conjectures
- Nucleon mass as the first result in the nucleon sector