Mass modes and secondary massive quark radiation
Based on our talks presented at SCET2013

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21st of March 2013, Vienna
Outline

1. SCET: basic ideas
2. Factorization theorem for thrust
3. Mass modes in SCET
4. Massive gauge boson radiation
5. From one-loop to two-loop
6. Soft function with mass modes
7. Effects on thrust
8. Summary & Outlook
Use of SCET

- SCET: Soft Collinear Effective Theory (top-down EFT)
- disentangling QCD vs other interactions requires a separation of scales → natural framework for EFTs
- for energetic hadrons $E_H \simeq Q \gg \Lambda_{\text{QCD}} \simeq m_H$
- for energetic jets $E_J \simeq Q \gg m_J$
- for (up to now) massless hard-soft-collinear interactions
- most of what can be done with SCET can be done with other techniques too. SCET can give new perspective on factorization, log-resummation, power corrections...

$B-$decays

\[ B \to X_s \gamma, \quad B \to \pi \ell \nu, \quad B \to \pi \pi, \quad B \to D \pi \quad \ldots \]

QCD processes with large $E$ transfer

\[ e^- p \to e^- X, \quad p\bar{p} \to X\ell^+ \ell^-, \quad e^+ e^- \to \text{jets} \quad \ldots \]
determine relevant scales for particles in initial and final states \( \Rightarrow \) expansion parameter \( \lambda \) is a ratio of such scales

construct momenta from these scales such that \( p^2 \leq Q^2 \lambda^2 \) \( \Rightarrow \) fields constructed from these sets of momenta

particles with \( p^2 \gg Q^2 \lambda^2 \) are integrated out

power counting: assign to the SCET fields a scaling in \( \lambda \)

construct Lagrangian (ensuring SCET gauge invariance)
Degrees of freedom: $e^+ e^- \rightarrow \text{jets}$

Relevant scales: $Q^2 \gg m_{\text{jet}}^2 \gg \Lambda_{\text{QCD}}^2 \Rightarrow \lambda \equiv m_{\text{jet}}/Q$

### Light-cone coordinates

$$n^\mu = (1, 0, 0, -1), \quad \bar{n}^\mu = (1, 0, 0, 1), \quad n^2 = \bar{n}^2 = 0$$

$$p^+ = n \cdot p, \quad p^- = \bar{n} \cdot p, \quad p^\perp = (p_x, p_y), \quad p^\mu = p^+ \frac{\bar{n}^\mu}{2} + p^- \frac{n^\mu}{2} + p^\perp$$

- retain particles with $p^2 \leq Q^2 \lambda^2 \Rightarrow$ collinear and ultrasoft
- jet constituents $p^\mu \sim (m_{\text{jet}}^2/Q, Q, m_{\text{jet}}) = Q(\lambda^2, 1, \lambda)$
- ultrasoft particles $p^\mu \sim (m_{\text{jet}}^2/Q, m_{\text{jet}}^2/Q, m_{\text{jet}}^2/Q) = Q(\lambda^2, \lambda^2, \lambda^2)$ and $p^2 \sim Q^2 \lambda^4$
- any other with $p^2 \gg Q^2 \lambda^2$ integrated out $\Rightarrow$ their dynamics present through matching coefficient $C(Q, \mu)$
\[
\begin{array}{|c|c|c|}
\hline
\text{mode} & p^\mu = (+, -, \perp) & p^2 & \text{fields} \\
\hline
\text{hard} & Q(1, 1, 1) & Q^2 & - \\
\text{n-collinear} & Q(\lambda^2, 1, \lambda) & Q^2\lambda^2 & \xi_n, A_n^{\mu} \\
\text{\bar{n}-collinear} & Q(1, \lambda^2, \lambda) & Q^2\lambda^2 & \xi_{\bar{n}}, A_{\bar{n}}^{\mu} \\
\text{ultrasoft} & Q(\lambda^2, \lambda^2, \lambda^2) & Q^2\lambda^4 & q_{us}, A_{us}^{\mu} \\
\hline
\end{array}
\]

- a field for each mode, but several fields for each particle
- always integrate out all modes above given hyperbola
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Thrust distribution

- thrust: \( \tau \equiv 1 - \max_t \frac{\sum_i \left| \hat{t} \cdot \vec{p}_i \right| }{\sum_i |\vec{p}_i|} \in [0, \frac{1}{2}] \)

  \[\text{back-to-back: } \tau \rightarrow 0 \quad \hat{t} \quad \text{isotropic: } \tau \rightarrow \frac{1}{2} \quad \hat{t} \quad \text{back-to-back: } \tau \rightarrow 0 \quad \hat{t} \quad \text{isotropic: } \tau \rightarrow \frac{1}{2} \quad \hat{t}\]

- thrust distribution from LEP data \((e^+ e^- \rightarrow \text{jets})\)

\[\frac{1}{\sigma} \frac{d\sigma}{d\tau} \]

\[\tau \rightarrow \text{peak region (} \tau \sim \Lambda_{QCD}/Q \): expansion parameter } \lambda = \sqrt{\Lambda_{QCD}/Q} \]

\[\rightarrow \text{tail region (} \tau \gg \Lambda_{QCD}/Q \): expansion parameter } \lambda = \sqrt{\tau} \]
Factorization theorem for thrust

Fleming, Hoang, Mantry, Stewart (2007)  

SCET result for \( \tau \ll 1 \):

\[
\frac{d\sigma}{d\tau} = Q^2 \sigma_0 H_0(Q, \mu) \int d\ell \ J_0(Q\ell, \mu) S_0(Q\tau - \ell, \mu)
\]

\( Q \rightarrow \) CM energy, \( \sigma_0 \rightarrow \) cross-section at LO

Ingredients:

- hard function: \( H_0(Q, \mu) = |C_V(Q, \mu)|^2 \)
- thrust jet function: \( J_0(s, \mu) = \int ds' J_n(s', \mu)J_{\bar{n}}(s - s', \mu) \)
- thrust soft function: \( S_0(\ell, \mu) \equiv \int dk_R dk_L \delta(\ell - k_R - k_L) S_0^{\text{hemisphere}}(k_R, k_L, \mu) \)
  \( \rightarrow \mu_s \sim Q\lambda^2 \sim \Lambda_{QCD} \): \( S_0 = S_0^{\text{model}} \): non-perturbative model
  \( \rightarrow \mu_s \sim Q\lambda^2 \gg \Lambda_{QCD} \): \( S_0 = S_0^{\text{partonic}} \otimes S_0^{\text{model}} \): \( S_0^{\text{partonic}} \) partonic piece (perturbative)
Profile functions

Profile functions: Parametrization of renormalization scales in terms of thrust
→ continuous transition between peak, tail and far-tail region

Abbate, Fickinger, Hoang, Mateu, Stewart (2011)

<table>
<thead>
<tr>
<th>region</th>
<th>$\mu_H$</th>
<th>$\mu_J$</th>
<th>$\mu_S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>peak</td>
<td>$\tau \sim \Lambda_{QCD}/Q$</td>
<td>$Q$</td>
<td>$\sqrt{Q\Lambda_{QCD}}$</td>
</tr>
<tr>
<td>tail</td>
<td>$\Lambda_{QCD}/Q \ll \tau \leq 1/3$</td>
<td>$Q$</td>
<td>$Q\sqrt{\tau}$</td>
</tr>
<tr>
<td>far-tail</td>
<td>$1/3 \leq \tau \leq 1/2$</td>
<td>$Q$</td>
<td>$Q$</td>
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$I. Jemos, P. Pietrulewicz (University of Vienna)$

MM and secondary quark radiation

Vienna, 21/03/2013
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Mass effects in QCD

in QCD: e.g. DIS

- mass effects via different schemes (FFNS, VFNS, ...)
  → correct limiting behavior, continuous description?

- ACOT scheme (VFNS): heavy quark production (3 scales: $Q, m, \Lambda_{QCD}$)
  Aivazis, Collins, Olness, Tung (1994)
  factorization theorem interpolating between the regions
  → $m \gg Q$: full decoupling
  → $m \sim Q$: exact kinematics, mass effects in Wilson coefficients (FFNS)
  → $m \ll Q$: Log-resummation, massless kinematics, mass effects in pdf’s

⇒ setup for additional scales? e.g. in endpoint region $x \rightarrow 1$: $Q^2(1 - x)$
Mass effects in SCET

in SCET: event shapes for $e^+ e^- \to jets$

- factorization and resummation for production of primary massive $\bar{t}t$-pairs
  Fleming, Hoang, Mantry, Stewart (2008)

$\Rightarrow$ still missing: systematic treatment of secondary massive quarks
  $\to$ nontrivial setup with dynamical thresholds
introduce particle species with mass $m$
→ several scale hierarchies in one single event shape spectrum (for $Q > m$)

\[ \mu_H > \mu_m > \mu_J > \mu_S \]
introduce particle species with mass \( m \)
→ several scale hierarchies in one single event shape spectrum (for \( Q > m \))

Scenario III: \( \mu_H > \mu_J > \mu_m > \mu_S \)
introduce particle species with mass $m$
→ several scale hierarchies in one single event shape spectrum (for $Q > m$)

Scenario IV: $\mu_H > \mu_J > \mu_S > \mu_m$
Mass modes

- new degrees of freedom: "mass modes"
- additional scaling parameter: $\lambda_m = m/Q$

\[ p^+ p^- \sim m^2 \]

\[ m^2/Q \]

\[ m \]

\[ Q \]

\[ p^+ p^- = \text{const.} \]

\[ \sim m^2 \]

\[ n\text{-coll. MM} \]

\[ \bar{n}\text{-coll. MM} \]

\[ \text{soft MM} \]

\[ \text{hard} \]

\[ \text{invariant mass} \]

<table>
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<td>$n$-coll MM</td>
<td>$Q(\lambda_m^2, 1, \lambda_m)$</td>
<td>$m^2$</td>
</tr>
<tr>
<td>soft MM</td>
<td>$Q(\lambda_m, \lambda_m, \lambda_m)$</td>
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$\Rightarrow$ How to include mass modes into massless setup?
Mass mode setup

- construct a sequence of EFTs depending on $\lambda_m \leftrightarrow \lambda$
  
  I. $\lambda_m > 1 > \lambda > \lambda^2$
  II. $1 > \lambda_m > \lambda > \lambda^2$
  III. $1 > \lambda > \lambda_m > \lambda^2$
  IV. $1 > \lambda > \lambda^2 > \lambda_m$

- aims:
  - continuity between scaling situations (“scenarios”)
  - mass-independent UV divergences
  - decoupling for $m \to \infty$
  - correct IR-finite massless limit for $m \to 0$
I. $\lambda_m > 1 > \lambda > \lambda^2$

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<td>$Q(1, 1, 1)$</td>
<td>$Q^2$</td>
</tr>
<tr>
<td>$n$-coll ML</td>
<td>$Q(\lambda^2, 1, \lambda)$</td>
<td>$Q^2\lambda^2$</td>
</tr>
<tr>
<td>usoft ML</td>
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\[ \text{MM and secondary quark radiation} \]

\[ \text{Vienna, 21/03/2013} \]
II. $1 > \lambda_m > \lambda > \lambda^2$


diagram with phases and momenta

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III. $1 > \lambda > \lambda_m > \lambda^2$
IV. $1 > \lambda > \lambda^2 > \lambda_m$
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Motivation

Setup for massive gauge boson (mass $M$) with vector coupling (group factors denoted as in $SU(3)$)

- dispersive technique:

$$\frac{q^2}{\pi} \int_{4m^2}^{\infty} \frac{dM^2}{M^2} \left( \frac{q}{M} \right) \times \text{Im} \left[ \frac{m}{q^2 \rightarrow M^2} \right]$$

- separate mass mode concept from technical issues at two-loop
- in principle applicable for EW effects, ...
Full theory result

\[
\frac{1}{\sigma_0} \frac{d\sigma}{d\tau} = \frac{\alpha_s C_F}{4\pi} \left\{ \delta(\tau) \delta \tilde{F}_m^{QCD} \left( \frac{M^2}{Q^2} \right) + \theta(\tau - \tau_{\text{min}}) \left[ A \left( \tau, \frac{M^2}{Q^2} \right) + B \left( \tau, \frac{M^2}{Q^2} \right) \right] + \theta(\tau - \bar{\tau}) C \left( \tau, \frac{M^2}{Q^2} \right) \right\}
\]

\[
\tau_{\text{min}} = \frac{M^2}{Q^2} \rightarrow \text{threshold for real jet radiation}
\]

\[
\bar{\tau} = \frac{M}{Q} \rightarrow \text{threshold for real soft radiation}
\]
Mass mode setup: RG evolution

Top-down evolution: evolve to $\mu \sim \mu_S$

$I$  
$\mu_M \sim M$  
$\mu_H \sim Q$  
$\mu_J \sim Q\lambda$  
$\mu_S \sim Q\lambda^2$

coll.ML

$II$  
$\mu_M \sim M$  
$\mu_H \sim Q$  
$\mu_J \sim Q\lambda$  
$\mu_S \sim Q\lambda^2$

coll.ML

$III$  
$\mu_M \sim M$  
$\mu_H \sim Q$  
$\mu_J \sim Q\lambda$  
$\mu_S \sim Q\lambda^2$

coll.ML

$IV$  
$\mu_M \sim M$  
$\mu_H \sim Q$  
$\mu_J \sim Q\lambda$  
$\mu_S \sim Q\lambda^2$

coll.ML

MM = mass-mode, ML = massless, M = massive
Scenario I: $\lambda_M > 1 > \lambda > \lambda^2$

integrate out mass modes at QCD level
$\rightarrow$ modification of hard matching coefficient, otherwise like in massless SCET

\[
\frac{d\sigma}{d\tau} \sim \left| C^I(\mu_H) \right|^2 U_H^{(0)}(\mu_H, \mu_S) \\
\times \int d\ell \int ds J_0(s) U_J^{(0)}(Q\ell - s, \mu_S, \mu_J) S_0(Q\tau - \ell, \mu_S)
\]

$U_H^{(0)}$, $U_J^{(0)}$: massless evolution factors
$C^I(\mu_H) = C_0(\mu_H) + \delta F^{QCD}_m$

$C_0$: massless matching coefficient
$\delta F^{QCD}_m$: massive full theory contribution (OS)
$\rightarrow$ decoupling for $M/Q \rightarrow \infty$
Scenario II: $1 > \lambda_M > \lambda > \lambda^2$

mass modes enter SCET, but integrated out before the jet scale
→ modification of the matching coefficient at $\mu_H$
→ additional matching contribution at $\mu_M$
→ massless jet & soft function

\[
\frac{d\sigma}{d\tau} \sim \left| C_H^{II}(\mu_H) \right|^2 U_H^{(1)}(\mu_H, \mu_M) \left| M_H(\mu_M) \right|^2 U_H^{(0)}(\mu_M, \mu_S) \\
\times \int d\ell \int ds J_0(s, \mu_J) U_J^{(0)}(Q\ell - s, \mu_S, \mu_J) S_0(Q\tau - \ell, \mu_S)
\]

$U_H^{(1)}$: evolution factor ($\gamma_H^{(1)} = 2\gamma_H^{(0)}$)

$C_H^{II}(\mu_H) = C_H(\mu_H) - \delta F^\text{eff}_m(\mu_H)$

$\delta F^\text{eff}_m$: massive SCET contribution
Hard function contribution $\delta F_{m}^{\text{eff}}$

\[
\delta F_{m}^{\text{eff}} (Q, M, \mu) = \frac{\alpha_s C_F}{4\pi} \left\{ \ln \left( \frac{M^2}{\mu^2} \right) \left[ 2 \ln \left( \frac{-Q^2}{\mu^2} \right) - \ln \left( \frac{M^2}{\mu^2} \right) - 3 \right] - \frac{5\pi^2}{6} + \frac{9}{2} \right\}
\]

Chiu, Golf, Kelley, Manohar (2008)
Chiu, Fuhrer, Hoang, Kelley, Manohar (2009)

double counting of mass mode effects!
→ subtraction of collinear diagrams with soft scaling necessary = soft-bin subtraction
→ allows to obtain regulator-independent, gauge-invariant result

correct massless limit for matching coefficient:

\[
C_{\text{II}} (Q, M, \mu_H) = C_{\text{I}} (\mu_H) - \delta F_{m}^{\text{eff}} (\mu_H) \xrightarrow{M \to 0} 2C_0 (Q, \mu_H)
\]
Scenario II: $1 > \lambda_M > \lambda > \lambda^2$

\[ \frac{d\sigma}{d\tau} \sim |C^H(\mu_H)|^2 U_H^{(1)}(\mu_H, \mu_M) |M_H(\mu_M)|^2 U_H^{(0)}(\mu_M, \mu_S) \times \int d\ell \int ds J_0(s, \mu_J) U_J^{(0)}(Q\ell - s, \mu_S, \mu_J) S_0(Q\tau - \ell, \mu_S) \]

$U_H^{(1)}$: evolution factor ($\gamma_H^{(1)} = 2\gamma_H^{(0)}$)

$C^H(\mu_H) = C^I(\mu_H) - \delta F_m^{\text{eff}}(\mu_H)$

$M_H(\mu_M) = 1 + \delta F_m^{\text{eff}}(\mu_M)$

$\delta F_m^{\text{eff}}$: massive SCET contribution

continuity to scenario I for $\mu_M = \mu_H$:

$|C^H(\mu_H)|^2 |M_H(\mu_M)|^2 = |C^I(\mu_H)|^2$

ML = massless
MM = mass mode
Scenario II: Matching to full theory

Expansion of the most singular terms, i.e. for $\tau \sim M^2/Q^2 \ll 1$ and $\tau \sim M/Q \ll 1$

$$\frac{1}{\sigma_0} \left. \frac{d\sigma^\text{full th.}}{d\tau} \right|_{\text{FO}} = \delta(\tau) + 2 \text{Re} \left[ \delta F_m^{QCD} \left( \frac{M^2}{Q^2} \right) \right] \delta(\tau) + Q^2 \theta \left( \tau - \frac{M^2}{Q^2} \right) \delta J_m^{\text{real}} (Q^2 \tau) + Q \theta \left( \tau - \frac{M}{Q} \right) \delta S_m^{\text{real}} (Q \tau)$$
Scenario II: Matching to full theory

- expansion of the most singular terms, i.e. for \( \tau \sim M^2/Q^2 \ll 1 \) and \( \tau \sim M/Q \ll 1 \)

\[
\left. \frac{1}{\sigma_0} \frac{d\sigma^{\text{full th.}}}{d\tau} \right|_{\text{FO}} = \delta(\tau) + 2 \text{Re} \left[ \delta F_m^{\text{QCD}} \left( \frac{M^2}{Q^2} \right) \right] \delta(\tau) + Q^2 \theta \left( \tau - \frac{M^2}{Q^2} \right) \delta J_m^{\text{real}} (Q^2 \tau) + Q \theta \left( \tau - \frac{M}{Q} \right) \delta S_m^{\text{real}} (Q \tau)
\]

- matching with SCET result at fixed order gives mass matching functions in scenario II, \( \tau < M^2/Q^2 \):

\[
\left. \frac{1}{\sigma_0} \frac{d\sigma^{\text{SCET}}}{d\tau} \right|^{\text{II}} = \delta(\tau) + 2 \text{Re} \left[ \delta F_m^{\text{QCD}} \left( \frac{M}{Q} \right) \right] \delta(\tau) + 2 \text{Re} \left[ \delta F_m^{\text{eff}} (Q, M, \mu) + \mathcal{M}_m^{(1)} (Q, M, \mu) \right] \delta(\tau)
\]

\[
\rightarrow \text{Re} \left[ \mathcal{M}_m^{(1)} (Q, M, \mu) \right] = - \text{Re} \left[ \delta F_m^{\text{eff}} (Q, M, \mu) \right]
\]
Scenario III: $1 > \lambda > \lambda_M > \lambda^2$

Massive and massless collinear modes fluctuate over comparable scales ($\lambda_M \leq \lambda$)

→ assign collinear massless scaling (keep $M \neq 0$)

→ modification of the jet function at $\mu_J$

→ additional jet matching contribution at $\mu_M$

→ massless soft function

\[
\frac{d\sigma}{d\tau} \sim |C^H(\mu_H)|^2 U_H^{(1)}(\mu_H, \mu_m) |M_H(\mu_M)|^2 U_H^{(0)}(\mu_M, \mu_S) \times \int d\ell \int ds \int ds' \int ds'' J_{0+m}(s, \mu_J) U_J^{(1)}(s' - s, \mu_M, \mu_J) \times M_J(s'' - s', \mu_M) U_J^{(0)}(s'' - Q\ell, \mu_S, \mu_M) S_0(Q\tau - \ell, \mu_S)
\]

\[
J_{0+m}(s, \mu_J) = J_0(s, \mu_J) + \delta J_m^{\text{virt}}(s, \mu_J) + \theta(s - M^2) \delta J_m^{\text{real}}(s)
\]

$\delta J_m^{\text{virt}}$: virtual piece of jet function (distributive structure)

$\delta J_m^{\text{real}}$: real radiation piece of jet function (function)
Jet function

\[
J_{0+m}(s, M, \mu) = J_0(s, \mu) + \delta J^\text{virt}_m(s, M, \mu) + \theta(s - M^2) \delta J^\text{real}_m(s, M)
\]

\[
\mu^2 \delta J^\text{virt}_m(s, M, \mu) = \frac{\alpha_s C_F}{4\pi} \left\{ \delta(\bar{s}) \left[ -4 \ln^2 \left( \frac{M^2}{\mu^2} \right) - 6 \ln \left( \frac{M^2}{\mu^2} \right) + 9 - 2\pi^2 \right] + 8 \ln \left( \frac{M^2}{\mu^2} \right) \left[ \frac{\theta(\bar{s})}{\bar{s}} \right] \right\}
\]

\[
\delta J^\text{real}_m(s, M) = \frac{\alpha_s C_F}{4\pi} \left\{ \frac{2(M^2 - s)(3s + M^2)}{s^3} + \frac{8}{s} \ln \left( \frac{s}{M^2} \right) \right\}
\]

\[
\delta J^\text{virt}_m = \text{virtual radiation} \quad (\bar{s} \equiv s/\mu^2)
\]

\[
\delta J^\text{real}_m = \text{real radiation for} \ s > M^2, \text{continuous:} \quad \delta J^\text{real}_m(s = M^2, M) = 0
\]

\[
\text{correct massless limit:} \quad J_{0+m}(s, M, \mu_J) \xrightarrow{M \to 0} 2J_0(s, \mu_J)
\]
Scenario III: $1 > \lambda > \lambda_M > \lambda^2$

\[
\frac{d\sigma}{d\tau} \sim |C^H(\mu_H)|^2 U_H^{(1)}(\mu_H, \mu_m) |\mathcal{M}_H(\mu_M)|^2 U_H^{(0)}(\mu_M, \mu_S) \\
\times \int d\ell \int ds \int ds' \int ds'' J_{0+m}(s, \mu_J) U_J^{(1)}(s' - s, \mu_M, \mu_J) \\
\times \mathcal{M}_J(s'' - s', \mu_M) U_J^{(0)}(s'' - Q\ell, \mu_S, \mu_M) S_0 (Q\tau - \ell, \mu_S)
\]

\[
J_{0+m}(s, \mu_J) = J_0(s, \mu_J) + \delta J_m^{\text{virt}}(s, \mu_J) + \theta(s - M^2) \delta J_m^{\text{real}}(s)
\]

\[
\mathcal{M}_J(s, \mu_M) = \delta(s) - \delta J_m^{\text{virt}}(s, \mu_M)
\]

\[
\delta J_m^{\text{virt}}: \text{virtual piece of jet function (distributive structure)}
\]

\[
\delta J_m^{\text{real}}: \text{real radiation piece of jet function (function)}
\]

continuity to scenario II for $\mu_M = \mu_J$ ($\mu_M \leq M$):

\[
J_{0+m}(s, \mu_J) \mathcal{M}_J(s, \mu_J) = J_0(s, \mu_J)
\]

ML = massless
MM = mass mode
M = massive
Scenario III: Matching to full theory

- expansion of the most singular terms, i.e. for $\tau \sim M^2/Q^2 \ll 1$ and $\tau \sim M/Q \ll 1$

$$\frac{1}{\sigma_0} \frac{d\sigma^{\text{full th.}}}{d\tau} \bigg|_{\text{FO}} = \delta(\tau) + 2 \text{Re} \left[ \delta F_m^{\text{QCD}} \left( \frac{M^2}{Q^2} \right) \right] \delta(\tau)$$

$$+ Q^2 \theta \left( \tau - \frac{M^2}{Q^2} \right) \delta J_m^{\text{real}}(Q^2 \tau) + Q \theta \left( \tau - \frac{M}{Q} \right) \delta S_m^{\text{real}}(Q\tau)$$

- matching with SCET result at fixed order gives mass matching functions
  e.g. in scenario III, $M/Q > \tau > M^2/Q^2$:

$$\frac{1}{\sigma_0} \frac{d\sigma^{\text{SCET}}}{d\tau} \bigg|_{\text{III}} = \delta(\tau) + 2 \text{Re} \left[ \delta F_m^{\text{QCD}} \left( \frac{M}{Q} \right) \right] \delta(\tau)$$

$$+ 2 \text{Re} \left[ \delta F_m^{\text{eff}}(Q, M, \mu) + \mathcal{M}_H^{(1)}(Q, M, \mu) \right] \delta(\tau)$$

$$+ Q^2 \theta \left( \tau - \frac{M^2}{Q^2} \right) \delta J_m^{\text{real}}(Q^2 \tau, M)$$

$$+ Q^2 \left[ \delta J_m^{\text{virt}}(Q^2 \tau, M, \mu) + \mathcal{M}_j^{(1)}(Q^2 \tau, M, \mu) \right]$$

$$\to \mathcal{M}_j^{(1)}(Q^2 \tau, M, \mu) = -\delta J_m^{\text{virt}}(Q^2 \tau, M, \mu)$$ (integrate out virtual contributions)

$$\to$$ no real radiation appearing in mass matching functions
**Scenario IV: $1 > \lambda > \lambda^2 > \lambda_M$**

- Massive soft and massless usoft modes fluctuate over comparable scales ($\lambda_M \leq \lambda^2$)
- $\rightarrow$ assign usoft massless scaling (keep $M \neq 0$)!
- $\rightarrow$ all structures get massive contributions
- $\rightarrow$ massive modes stay in the game to the end

$$\frac{d\sigma}{d\tau} \sim |C_{II}(\mu_H)|^2 U_{H}^{(1)}(\mu_H, \mu_S) \times \int d\ell \int ds J_{0+m}(s, \mu_J) U_{J}^{(1)}(Q\ell - s, \mu_S, \mu_J) S_{0+m}(Q\tau - \ell, \mu_S)$$

$$S_{0+m}(\ell, \mu_S) = S_0(\ell, \mu_S) + \delta S_m^{\text{virt}}(\ell, \mu_S) + \theta(\ell - M) \delta S_m^{\text{real}}(\ell)$$

- $\delta S_m^{\text{virt}}$: virtual piece of massive soft function (distributive structure)
- $\delta S_m^{\text{real}}$: real radiation piece of massive soft function (function)
\[ S_{0+m}(\ell, M, \mu) = S_0(\ell, \mu) + \delta S_m^\text{virt}(\ell, M, \mu) + \theta(\ell - M) \delta S_m^\text{real}(\ell, M) \]

\[ \mu \delta S_m^\text{virt}(\ell, M, \mu) = \frac{\alpha_s C_F}{4\pi} \left\{ \delta(\bar{\ell}) \left[ 2 \ln^2 \left( \frac{M^2}{\mu^2} \right) + \frac{\pi^2}{3} \right] - 8 \ln \left( \frac{M^2}{\mu^2} \right) \left[ \frac{\theta(\bar{\ell})}{\bar{\ell}} \right] \right\} \]

\[ \delta S_m^\text{real}(\ell, M) = \frac{\alpha_s C_F}{4\pi} \left\{ -8 \frac{\ell}{\ell} \ln \left( \frac{\ell^2}{M^2} \right) \right\} \]

\[ \rightarrow \delta S_m^\text{virt} = \text{virtual radiation} (\bar{\ell} \equiv \ell / \mu) \]

\[ \rightarrow \delta S_m^\text{real} = \text{real radiation for } \ell > M, \text{ continuous: } \delta S_m^\text{real}(\ell = M, M) = 0 \]

\[ \rightarrow \text{correct massless limit: } S_{0+m}(\ell, M, \mu_S) \xrightarrow{M \to 0} 2S_0(\ell, \mu_S) \]
Scenario IV: $1 > \lambda > \lambda^2 > \lambda_M$

\[ \frac{d\sigma}{d\tau} \sim |C^{II}(\mu_H)|^2 U^{(1)}_{H}(\mu_H, \mu_S) \times \int d\ell \int ds J_{0+m}(s, \mu_J) U^{(1)}_J(Q\ell - s, \mu_S, \mu_J) S_{0+m}(Q\tau - \ell, \mu_S) \]

\[ S_{0+m}(\ell, \mu_S) = S_0(\ell, \mu_S) + \delta S^\text{virt}_m(\ell, \mu_S) + \theta(\ell - M) \delta S^\text{real}_m(\ell) \]

$\delta S^\text{virt}_m$: virtual piece of massive soft function (distributive structure)
$\delta S^\text{real}_m$: real radiation piece of massive soft function (function)

agreement with expanded full theory result at fixed order

continuity to scenario III for $\mu_M = \mu_J$:
$\rightarrow$ consistency relation between virtual modes:
\[ 2 \text{ Re} \left[ \delta F^\text{eff}_m(Q, \mu) \right] \delta(\tau) - Q^2 \delta J^\text{virt}_m(Q^2\tau, \mu) - Q \delta S^\text{virt}_m(Q\tau, \mu) = 0 \]
Summary so far

- EFT setup for massive gauge boson radiation in thrust distribution for different scale hierarchies
- Patch four scenarios continuously (up to $O(\alpha_s^2)$)

several scenarios needed for one single thrust distribution
Summary so far

- EFT setup for massive gauge boson radiation in thrust distribution for different scale hierarchies
- Patch four scenarios \textbf{continuously} (up to $O(\alpha_s^2)$)
- Same anomalous dimension as in $M \to 0$ limit $\Rightarrow$ evolution factors affected only by number of massive gauge bosons
- Decoupling limit: $C^I(Q, M, \mu_H) \xrightarrow{M \to \infty} C_0(Q, \mu_H)$
- Massless limit for hard, jet and soft functions:
  \begin{align*}
    C^{II}(Q, M, \mu_H) & \xrightarrow{M \to 0} 2C_0(Q, \mu_H) \\
    J_{0+m}(s, M, \mu_J) & \xrightarrow{M \to 0} 2J_0(s, \mu_J) \\
    S_{0+m}(\ell, M, \mu_S) & \xrightarrow{M \to 0} 2S_0(\ell, \mu_S)
  \end{align*}
- Interesting consistency condition $\Rightarrow$ possible to rearrange mass-mode contributions into different functions depending on the choice of $\mu$
  \begin{equation*}
    2 \Re \left[ \delta F_{\text{eff}}(Q, M, \mu) \right] \delta(\tau) - Q^2 \delta J_m^{\text{virt}}(Q^2 \tau, M, \mu) - Q \delta S_m^{\text{virt}}(Q \tau, M, \mu) = 0
  \end{equation*}
Outline

1. SCET: basic ideas
2. Factorization theorem for thrust
3. Mass modes in SCET
4. Massive gauge boson radiation
5. From one-loop to two-loop
6. Soft function with mass modes
7. Effects on thrust
8. Summary & Outlook
Aim

- **PART I**: technique for mass modes in SCET, massive gauge boson example  
  \[ \text{arxiv:1302.4743} \]

- **PART II**: two-loop extension \(\Rightarrow\) *secondary* heavy quark radiation

- *primary* quarks massless

- to be specific \(n_m = 1\)

- factorization theorems at two-loop (same structure as massive gauge boson case, but some complications)

- quark bubble \((C_F T_F n_f)\) contributions to \(C(Q, m, \mu), J(s, m, \mu)\) and \(S(\ell, m, \mu)\)

massless case \(\rightarrow\) Moch et al. (2005), Becher et al. (2007), Kelley et al. (2011), Hornig et al. (2011)
Dispersion relation

**Observation**: interpret fermion bubble insertion as “gluon” with massive propagator

\[
\Pi_{\mu\nu}(q^2) \equiv \frac{(-i)^2 g_{\mu\rho} \Pi^{\rho\sigma}(q^2) g_{\sigma\nu}}{(q^2 + i\epsilon)^2} = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{dM^2}{M^2} \text{Im} \left[ \Pi(M^2) \right] \frac{-i \left( g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2} \right)}{q^2 - M^2 + i\epsilon}
\]

- calculate one-loop massive gauge boson result
- convolute with \text{Im} \left[ \Pi(M^2) \right]
- use same dispersive integral for full theory, collinear and soft sector
- convolution in \( d = 4 - 2\epsilon \) !!! (but for finite parts \( d = 4 \))

\[
q \rightarrow m \quad = \quad \frac{q^2}{\pi} \int_{4m^2}^{\infty} \frac{dM^2}{M^2} \left( \frac{q}{M} \right) \times \text{Im} \left[ \frac{q}{q} \right] \quad \pi \int_{4m^2}^{\infty} \frac{dM^2}{M^2} \text{Im} \left[ \Pi(M^2) \right] \frac{-i \left( g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2} \right)}{q^2 - M^2 + i\epsilon}
\]
Dispersive technique allows to address difficulties separately:

1. separation of dynamical modes depending on $m$ vs $\mu_H$, $\mu_J$, $\mu_S \Rightarrow$ treated in massive “gluon” context
2. secondary quark radiation and its influence on RG evolutions $\Rightarrow$ taken care of after final convolution

**Part I:** Soft-bin subtraction can be carried out completely in the massive “gluon” context at $\mathcal{O}(\alpha_s)$. 

\[ \Downarrow \]

No further soft-bin subtraction needed at $\mathcal{O}(\alpha_s^2)$. Convolution already regulated in dim. reg.

**Beware:** setup is general, but dispersive treatment suitable for observables depending on invariant mass of secondary fermion pair
fermion mass ⇒ additional scale parameter, distinguishes four scenarios

\[ \mu_m \sim m \]
\[ \mu_H \sim Q \]
\[ \mu_J \sim Q \lambda \]
\[ \mu_S \sim Q \lambda^2 \]

MM = mass-mode, ML = massless, M = massive
Issues at two-loop

- Decoupling of $\alpha_s$ in evolution

\[ \begin{align*}
\text{hard} & \quad \sim Q \\
\text{jet} & \quad \sim Q\lambda \\
\text{soft} & \quad \sim Q\lambda^2 \\
\text{MM} & = \text{mass-mode}, \text{ML} = \text{massless}, \text{M} = \text{massive}
\end{align*} \]

- at scale $\mu_m$ integrate out massive flavours
  \[ \Rightarrow \alpha_s^{n_{\text{light}}+1} \rightarrow \alpha_s^{n_{\text{light}}} \]

- terms of the form
  \[ \frac{4}{3} \frac{T_F}{4\pi} \log\left(\frac{m^2}{\mu_m^2}\right) \times \text{one loop} \]

appear in factorization theorem

- Subtracted vs unsubtracted dispersive relation
- fixed order full QCD results not calculated yet
Issues at two-loop

- Decoupling of $\alpha_s$ in evolution
  -\- 
- Subtracted vs unsubtracted dispersive relation
  - unsubtracted and unrenormalized ($\overline{MS}$ scheme)
    \[
    \Pi(q^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} dM^2 \frac{1}{M^2 - q^2 - i\epsilon} \text{Im} \left[ \Pi(M^2) \right]
    \]

  - use: $\Pi(q^2) - \frac{4}{3} \frac{1}{\epsilon}$

  - heavy quark contributes to RGE
  - subtracted (on-shell scheme)
    \[
    \Pi^{os}(q^2) = \Pi(q^2) - \Pi(0) = \frac{q^2}{\pi} \int_{4m^2}^{\infty} \frac{dM^2}{M^2} \frac{1}{M^2 - q^2 - i\epsilon} \text{Im} \left[ \Pi(M^2) \right]
    \]

  - heavy quark does not contribute to any RGE

Use of unsubtracted relations crucial to cancel unwanted mass terms

- fixed order full QCD results not calculated yet
- Soft function with hemisphere (thrust) prescription is complicated
Issues at two-loop

- Decoupling of $\alpha_s$ in evolution
- Subtracted vs unsubtracted dispersive relation
- fixed order full QCD results not calculated yet
  - virtual terms given by vertex function $F_{QCD}(Q^2, m^2)$
  - and SCET gives all the singular terms
- Soft function with hemisphere (thrust) prescription is complicated

Kniehl (1990), Hoang (1995)
Issues at two-loop

- Decoupling of $\alpha_s$ in evolution
- Subtracted vs unsubtracted dispersive relation
- fixed order full QCD results not calculated yet
- Soft function with hemisphere (thrust) prescription is complicated
  - $S(k_L, k_R, m, \mu)$ not constrained by secondary quark invariant mass
  - $\Rightarrow$ dispersive approach only give correct UV and log structure
Scenario I: $\lambda_m > 1 > \lambda > \lambda^2$

heavy quark integrated out at QCD level $\Rightarrow \mathcal{C}^I(Q, m, \mu_H)$

\[
\frac{d\sigma}{d\tau} = Q\sigma_0 \left| \mathcal{C}^I(Q, m, \mu_H) \right|^2 U_H^{(n_f)}(Q, \mu_H, \mu_S) \int ds \int ds' \times U_J^{(n_f)}(s - s', \mu_S, \mu_J) J_0(s', \mu_J) S_0 \left( \frac{Q\tau - s}{Q}, \mu_S \right)
\]

\[
U_i^{(n_f)}(Q, \mu_H, \mu_S) = \text{massless RG factors, } n_f \text{ light flavours}
\]

\[
\mathcal{C}^I(Q, m, \mu) = 1 + \alpha_s^{(n_f)} C_0^{(1)} + (\alpha_s^{(n_f)})^2 \left( C_0^{(2)} + F_{\text{QCD}}^{(2)}(Q, m) \right)^{\text{OS}}
\]

Hoang (1995)

- $F_{\text{QCD}}^{(2)}(Q, m \to \infty)^{\text{OS}} \to 0$ (decoupling)
- $F_{\text{QCD}}^{(2)}(Q, m \to 0)^{\text{OS}} \to a_1 \ln^3(-x) + a_2 \ln^2(-x) + a_3 \ln(-x) + \ldots$
  \[ x = \frac{m^2}{Q^2} \]
Scenario II : $1 > \lambda_m > \lambda > \lambda^2$

\[
\frac{d\sigma}{d\tau} \sim |C^{II}(Q, m, \mu_H)|^2 U^{(n_f+1)}_{H_Q}(Q, \mu_H, \mu_m) |M_{H_Q}(Q, m, \mu_m)|^2 \\
\times U_{H_Q}^{(n_f)}(Q, \mu_m, \mu_S) \int ds \int ds' U_{J}^{(n_f)}(s - s', \mu_S, \mu_J) \\
\times J_{0, \tau}(s', \mu_J)S_{0, \tau} \left(Q\tau - \frac{s}{Q}, \mu_S\right)
\]

ML = massless
MM = mass mode
Scenario II: $1 > \lambda_m > \lambda > \lambda^2$

\[ C^{II}(Q, m, \mu) = 1 + C_0^{(1)} \alpha_s^{(n_f+1)} + (\alpha_s^{(n_f+1)})^2 \times \left( C_0^{(2)} + C_m^{(2)}(Q, m, \mu) \right) \]

\[ \mathcal{M}_{HQ}(Q, m, \mu_m) = 1 + (\alpha_s^{(n_f+1)})^2 \left[ F_{\text{SCET}}^{(2)}(Q, m, \mu_m) \right]^\text{MS} + \alpha_s \text{ dec.} \]

- $C_m^{(2)}(Q, m, \mu)$ → mass modes contributions
- in $C_m^{(2)}(Q, m, \mu)$ cancellation of divergences takes place
- sum and subtract same terms at different scales ($\mu_H$ vs $\mu_m$)
- continuous transition scenario I ↔ II
Scenario III: $1 > \lambda > \lambda_m > \lambda^2$

\[
\frac{d\sigma}{d\tau} \sim |C_{\mu}^\mu(Q, m, \mu_H)|^2 \ U_{HQ}^{(n_f+1)}(Q, \mu_H, \mu_m) \left| \mathcal{M}_{HQ}(Q, m, \mu_m) \right|^2 \\
\times U_{HQ}^{(n_f)}(Q, \mu_m, \mu_S) \int ds \, ds' \, ds'' \, dt \, J_{0+m}(s'', m, \mu_J) \\
\times U_J^{(n_f+1)}(s' - s'', \mu_m, \mu_J) \mathcal{M}_{J}(t, m, \mu_m) U_J^{(n_f)}(s - s', \mu_S, \mu_m) \\
\times S_0(Q\tau - s/Q - t/Q, \mu_S)
\]

\[
J_{0+m}(s, m, \mu) = 1 + \alpha_s^{(n_f+1)} J_0^{(1)} + (\alpha_s^{(n_f+1)})^2 \left( J_0^{(2)} + J_m^{\text{virt},(2)}(s, m, \mu) \right) \\
\times \theta(s - 4m^2) J_m^{\text{real},(2)}(s, m^2)
\]

\[
\mathcal{M}_J(s, m, \mu_m) = \delta(s) + (\alpha_s^{(n_f+1)})^2 \left[ -J_m^{\text{virt},(2)}(s, m, \mu_m) + \alpha_s \text{ dec.} \right]
\]

- match before real radiation sets in (i.e. $\tau_m < 4m^2/Q^2$)
- $J_m^{\text{real},(2)}(s, m^2) = 0$ at threshold
Scenario IV: $1 > \lambda > \lambda^2 > \lambda_m$

\[
\frac{d\sigma}{d\tau} \sim |C''(Q, m, \mu_H)|^2 U^{(n_f+1)}_{HQ}(Q, \mu_H, \mu_S) \int ds\, ds' U_j^{(n_f+1)}(s - s', \mu_S, \mu_J) J_{0+m}(s', m, \mu_J) \\
\times S_{0+m} \left( Q \tau - \frac{s}{Q}, m, \mu_S \right)
\]

\[
S_m^{(2)}(\ell, m, \mu) = S_m^{\text{virt}, (2)}(\ell, m, \mu) + \theta(\ell - 2m) S_m^{\text{real}, (2)}(\ell, m)
\]

- **consistency relation:**

\[
0 = 2\text{Re} \left[ F_{\text{SCET}}^{\text{MS}} \right] \delta(\tau) - Q^2 J_m^{\text{virt}, (2)} \mid_{\text{MS}} - QS_m^{\text{virt}, (2)} \mid_{\text{MS}} \\
+ \frac{4}{3} \ln \left( \frac{m^2}{\mu_m^2} \right) \left\{ 2\text{Re} \left[ C_0^{(1)} \right] \delta(\tau) + Q^2 J_0^{(1)} + QS_0^{(1)} \right\}
\]

- $\Rightarrow$ continuous transition III $\leftrightarrow$ IV
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Calculation for the soft function

- $m = 0$ calculation

Kelley et al. (2011), Hornig et al. (2011)

Remember: dispersive technique suitable for observables depending on fermion pair invariant mass $(k + q)^2$

“quark hemisphere” prescription $\Rightarrow k$ and $q$ treated separately if both are in different hemispheres: $S^{(QQ)}(k_L, k_R, m, \mu)$
Prescriptions

First approximation: “gluon hemisphere” prescription $\Rightarrow k + q$: 

\[ S^{(g)}(k_L, k_R, m, \mu) \]

\[ k + q \rightarrow \delta(\ell - k^- - q^-) \]

\[ k \rightarrow \delta(\ell - k^- - q^-) \]

\[ q \rightarrow \delta(\ell - k^- - q^-) \]

same hemisphere: $k^+ > k^-$, $q^+ > q^-$

opposite hemisphere: $k^- > k^+$, $q^+ > q^-$

- quarks in same hemisphere $\Rightarrow S^{(g)}(k_L, k_R, m, \mu)|_{\text{same}} = S^{(QQ)}(k_L, k_R, m, \mu)|_{\text{same}}$
- quarks in opposite hemispheres $\Rightarrow S^{(g)}(k_L, k_R, m, \mu)|_{\text{opp}} \neq S^{(QQ)}(k_L, k_R, m, \mu)|_{\text{opp}}$
Method

- \( S^{(g)}(k_L, k_R, m, \mu) \): same UV divergences of \( S^{(QQ)}(k_L, k_R, m, \mu) \) (RG consistency)

\[
S^{(QQ)}(k_L, k_R, m, \mu) = S^{(g)}(k_L, k_R, m, \mu) + \\
\left( S^{(QQ)}(k_L, k_R, m, \mu) \big|_{\text{opp}} - S^{(g)}(k_L, k_R, m, \mu) \big|_{\text{opp}} \right)
\]

finite = \( \Delta S(k_L, k_R, m) \)

- \( \Delta S(k_L, k_R, m) \) not easy analytically
- \( \Delta S(k_L, k_R, m) \) finite! so numerically doable in 4 dimensions
Thrust soft function at two-loop

\[
S_\tau(\ell, m, \mu) = \int dk_L dk_R \ S(k_L, k_R, m, \mu) \ \delta(\ell - k_L - k_R)
\]

\[
S^{(QQ), (2)}_\tau(\ell, m, \mu) \sim \alpha_s^2 \left\{ \delta(\ell) \left[ \frac{1}{18} L_m^3 - \frac{5}{18} L_m^2 + \left( \frac{28}{27} - \frac{\pi^2}{18} \right) L_m + \ldots \right] + \frac{1}{\mu} \left[ \frac{\mu \theta(\ell)}{\ell} \right] + \left( \frac{1}{3} L_m^2 - \frac{10}{9} L_m + \frac{28}{27} \right) \right. \\
+ \frac{1}{\mu} \left[ \frac{\mu \theta(\ell) \ln(\ell/\mu)}{\ell} \right] + \left. \frac{4}{3} L_m \right\}
\]

+ \theta(\ell - 2m)S^{\text{real}, (2)}(\ell, m) + \Delta S_\tau(\ell, m)

- \( L_m \equiv \ln(m/\mu) \)

- \( S^{(QQ), (2)}_\tau(\ell, m, \mu) \) reaches the massless limit
- same anomalous dimension as massless function
- massive quark radiation described by \( \theta(\ell - 2m)S^{\text{real}, (2)}(\ell, m) \) (as in jet function)
- also \( \Delta S_\tau(\ell, m) \) corresponds to real radiation
The correction $\Delta S_\tau(\ell, m)$

- crucial to find correct massless limit
  \[
  \Delta S_\tau(\ell, m) \xrightarrow{m \to 0} \delta(\ell) \times \left\{ -\frac{4}{9} + \frac{13\pi^2}{54} - \frac{4\zeta(3)}{3} \right\}
  \]
  checked numerically and analytically!
- contains no thresholds even though describes real radiation
- numerically small $< 5\%$ of $S_\tau(\ell, m, \mu)$
- analytical asymptotic expansions
fit to a Breit Wigner type function (4 free parameters)
7 parameters constrained with asymptotic expansions and normalization

\[ \Delta S(x) \sim \frac{1}{m} \frac{(ax)^\alpha}{(1 + (ax)^\beta)^{\gamma/\beta}} \left[ b \log^2 \left( 1 + Ax + Bx^2 \right) + c \log \left( 1 + Cx + Dx^2 \right) + d \right] \]

\[ x \equiv \ell/m \]
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Plots for $Q = 14, 35$ GeV

- numerical code for thrust distribution at $N^3$LL
- preliminary, so far no non-perturbative physics!
- $Q = 14, 35$ GeV ↔ data from PETRA
- determination of $\alpha_s(M_Z)$: $Q = 35...207$ GeV

Abbate et al. (2011, 2012)

- massless: 5 light flavours vs. massive: 4 light + 1 massive $b$ ($m_b = 4.2$ GeV)

Thrust distribution: massive ($Q = 14$ GeV, $\alpha_s(M_Z) = 0.118$)
\( Q = 14 \text{ GeV} \)

Thrust distribution: massive vs. massless

\[ \frac{1}{\sigma_0} \frac{d\sigma}{d\tau} \]

relative mass effects
$Q = 35$ GeV: possible effect on $\alpha_s$

massive vs. massless

$\alpha_s(M_z) = 0.119$ vs. $\alpha_s(M_z) = 0.118$
\( Q = 500 \text{ GeV} \)

- \( Q = 500 \text{ GeV} \leftrightarrow \text{ILC} \)
- massless: 6 light flavours vs. massive: 5 light + 1 massive \( t (m_t = 175 \text{ GeV}) \)

Thrust distribution: massive vs. massless

\[
\begin{align*}
\frac{1}{\sigma_0} \frac{d\sigma}{d\tau} & \quad \text{II} \\
\frac{\Delta}{d\tau} & \quad \text{III} \\
\frac{d\sigma}{d\tau} & \quad \text{IV}
\end{align*}
\]
$Q = 500$ GeV (decoupling)

- $Q = 500$ GeV ↔ ILC
- massless: 5 light flavours vs. massive: 5 light + 1 massive $t$ ($m_t = 175$ GeV)

Thrust distribution: massive vs. massless
$Q = 500$ GeV: Consistency check

Thrust distribution: $\tau = 0.15$ fixed, vary mass $m$
- massless limit (6 flavours): dashed
- decoupling limit (5 flavours): dotted

Continuous transitions and correct limiting behaviour!
Extreme approach

Thrust distribution: $\tau = 0.15$, $m_t = 175$ GeV fixed, vary $Q$

massless limit (6 flavours): dashed
decoupling limit (5 flavours): dotted
Extreme approach vs ours

- \( Q = 500 \text{ GeV} \)
- \( \tau \approx m_t^2/Q^2 \)

Thrust distribution: \( m_t = 175 \text{ GeV} \) fixed, vary \( Q \)
- massless limit (6 flavours): dashed
- decoupling limit (5 flavours): dotted

deviation from extreme approach can be up to \( \approx 4\% \)
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Summary & Outlook

- inclusion of heavy quark masses important for high precision collider physics
- setup for secondary massive quarks in terms of mass modes
- calculation of all ingredients for thrust distribution at $N^3LL$

NEXT:

- double hemisphere soft function $S^{(QQ)}(k_L, k_R, m, \mu)$
- renormalon analysis
- possible applications
  1. bottom mass effects in $\alpha_s$ determination at $N^3LL$
  2. analysis of LEP data (at $Q = 14$ GeV) and data from B factories
  4. massive effects for parton distribution functions in heavy quark production
  5. hard photoproduction with a heavy quark jet
  6. ...
Degrees of freedom: $B \rightarrow D\pi$

\[ p_\pi^{\mu} \simeq (2\text{GeV}, 0, 0, -2\text{GeV}) = Qn^{\mu} \quad (E_\pi \simeq Q \gg \Lambda_{\text{QCD}} \simeq m_\pi) \]

- $\pi$ rest frame: $\pi$ constituents $p^{\mu}_\pi \simeq (\Lambda_{\text{QCD}}, \Lambda_{\text{QCD}}, \Lambda_{\text{QCD}})$
- $B$ rest frame: $\pi$ boosted along $z$, $p^{\mu}_\pi \simeq (\Lambda_{\text{QCD}}^2/Q, Q, \Lambda_{\text{QCD}})$ collinear $\pi$
- $B$ and $D$ constituents $p^{\mu} \simeq (\Lambda_{\text{QCD}}, \Lambda_{\text{QCD}}, \Lambda_{\text{QCD}})$
- $\lambda \equiv \Lambda_{\text{QCD}}/Q \Rightarrow$ collinear $p^{\mu} = Q(\lambda^2, 1, \lambda)$, soft $p^{\mu} = Q(\lambda, \lambda, \lambda)$
Soft-bin subtractions

- hard function

\[ \delta F^\text{eff}_m \sim V_n - V_{n,0M} + \bar{V}_{n} - \bar{V}_{n,0M} + V_s - W_f \]
\[ \sim V_n + \bar{V}_n - V_s - W_f \]

Idilbi, Mehen (2007)
no regulator required if suitable combinations of integrals taken

- jet function

\[ \delta J_m \sim J_a - J_{a,0M} + J_b + J_c \]

→ remark: no soft diagrams appearing here
Soft-bin subtractions to self-energy diagrams

- SCET reproduces QCD wave-function renormalization
- in scenario II: $Q < M < Q\lambda$
  \[ s = (p + k)^2 \sim M^2 \]
  \[ \rightarrow \text{off-shellness for interactions with collinear mass mode gauge bosons:} \]
  \[ \Rightarrow \text{soft-bin subtractions to (collinear) self-energy power-suppressed by } M/Q \]
additional divergences in the lightcone components (not regularized by DIMREG)
→ $S_a$ and $S_b$ individually ill-defined
→ we use analytic $\alpha$-regulator (DIMREG in lightcone components)

Becher, Bell (2011)

\[
\int dk^- \int dk^+ \rightarrow \int dk^- \left( \frac{\nu_1}{k^-} \right)^{\alpha_1} \int dk^+ \left( \frac{\nu_2}{k^+} \right)^{\alpha_2}
\]

→ $S_a = 0$, $\delta S_m = S_b$
Use of two different dispersive relations \( \Rightarrow F_{QCD}(Q, m, \mu)|_{\overline{MS}} \) vs \( F_{QCD}(Q, m, \mu)|_{\text{OS}} \)

\[
F_{QCD}^{(2)}(Q, m, \mu)|_{\overline{MS}} = F_{QCD}^{(2)}(Q, m)|_{\text{OS}} - \left( \Pi(0) - \frac{4}{3} \frac{1}{\epsilon} \right) \times (1\text{-loop QCD (in d-dim!))}
\]

- \( (\text{OS} \leftrightarrow \overline{\text{MS}}) \) contains IR divergences which exactly cancel those from SCET diagrams
"quark prescription":

\[
F(k_R, k_L) = (-2\pi i)^2 \delta(k^2) \delta(q^2) \Theta(k^+ + k^-) \Theta(q^+ + q^-) \\
\times \left[ \Theta(k^+ - k^-) \Theta(q^- - q^+) \delta(q^+ - k_R) \delta(k^- - k_L) \\
+ \Theta(k^- - k^+) \Theta(q^+ - q^-) \delta(k^+ - k_R) \delta(q^- - k_L) \\
+ \Theta(k^- - k^+) \Theta(q^- - q^+) \delta(k^+ + q^- - k_R) \delta(k_L) \\
+ \Theta(k^+ - k^-) \Theta(q^+ - q^-) \delta(k^- + q^- - k_L) \delta(k_R) \right]
\]

"gluon prescription"

\[
F^{(g)}(k_R, k_L) = (-2\pi i)^2 \delta(k^2) \delta(q^2) \Theta(k^+ + k^-) \Theta(q^+ + q^-) \\
\times \left[ \Theta(k^+ + q^+ - k^- - q^-) \delta(k_R) \delta(k^- + q^- - k_L) \\
+ \Theta(k^- + q^- - k^+ - q^+) \delta(k_L) \delta(k^+ + q^+ - k_R) \right]
\]

\[k_L, k_R = \text{light cone components of left, right hemispheres momenta}\]
$Q = 91$ GeV: possible effect on $\alpha_s$

massive vs. massless

$\alpha_s(M_z) = 0.119$ vs. $\alpha_s(M_z) = 0.118$

relative mass effects

relative correction