Symmetries in the multi-Higgs-doublet scalar sectors

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What this talk is about

N-Higgs-doublet model, NHDM is a broad class of EW symmetry breaking models beyond SM, in which in assume that doublets of Higgs fields come in N "generations".

There are several physics motivations behind NHDM, like

- richer scalar sector,
- possible solution of flavor puzzle,
- novel sources or forms of CP-violation,
- novel astroparticle effects.

My main motivation is pragmatic: this is a model which many people study but which still contains intricate mathematical issues. I am intrigued by these issues, and I want to contribute to this field by trying to resolve them.

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I am not going to:

- promote any specific beyond-SM model,
- or give detailed predictions for the LHC or astroparticle observables.

I will present some general results on what's possible, symmetry-wise, in the scalar sector of NHDM.

Even if you are not familiar with this subject at all — no problem! The main part of this talk is purely mathematical. Take it as an example of slightly unusual application of group theory to physics.

Sit back and have group-theoretical fun!

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Standard Model

We introduce an electroweak scalar doublet $\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$, which interacts with fundamental fermions and with gauge bosons, and also self-interacts via the scalar potential

$$V = -m^2(\phi^\dagger \phi) + \frac{\lambda}{2}(\phi^\dagger \phi)^2 .$$

This potential is invariant under the electroweak gauge group $G_{EW} = SU(2)_L \times U(1)_Y$ and under CP-transformation: $\phi(t, x) \mapsto \phi^\dagger(t, -x)$.

At positive m^2, λ, it develops a non-zero v.e.v., which can be rotated to

$$\langle \phi \rangle_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} , \quad v \approx 246 \text{ GeV} .$$

2HDM

Two electroweak doublets, ϕ_1 and ϕ_2, can interact via:

$$\begin{aligned} V &= -\frac{1}{2}\left[m_{11}^2(\phi_1^\dagger\phi_1) + m_{22}^2(\phi_2^\dagger\phi_2) + m_{12}^2(\phi_1^\dagger\phi_2) + m_{12}^{2\,*}(\phi_2^\dagger\phi_1)\right] \\ &\quad +\frac{\lambda_1}{2}(\phi_1^\dagger\phi_1)^2 + \frac{\lambda_2}{2}(\phi_2^\dagger\phi_2)^2 + \lambda_3(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + \lambda_4(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) \\ &\quad +\left\{\left[\frac{1}{2}\lambda_5(\phi_1^\dagger\phi_2) + \lambda_6(\phi_1^\dagger\phi_1) + \lambda_7(\phi_2^\dagger\phi_2)\right](\phi_1^\dagger\phi_2) + \mathrm{h.c.}\right\} \end{aligned}$$

V contains 14 free parameters: 4 free parameters m_{ab}^2 and 10 λ's.

The scalar sector of NHDM

In NHDM, we introduce $\phi_a = \begin{pmatrix} \phi_a^+ \\ \phi_a^0 \end{pmatrix}$, $a = 1, \dots, N$, and construct the general gauge-invariant and renormalizable potential:

$$V = Y_{ab}(\phi_a^\dagger \phi_b) + Z_{abcd}(\phi_a^\dagger \phi_b)(\phi_c^\dagger \phi_d)\,,$$

with $N^2(N^2+3)/2$ independent free parameters (e.g. 54 for 3HDM).

We want to qualitatively understand phase diagram of the model

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The scalar sector of NHDM

A good way to see structures in multi-parametric models is to understand possible symmetry classes.

Any NHDM potential is EW-symmetric. But upon certain choice of coefficients, it might be also invariant under additional ("accidental") symmetries, which form the group G.

We want to classify all possible groups G (with focus on finite groups).

Reparametrization transformation: any transformation of the doublets which keeps the generic form of the potentials but only change the values of free parameters. It a reparametrization transformation leaves the potential invariant, it is called reparametrization symmetry. We want to classify reparametrization symmetry groups G.

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Technical point 1: $PSU(N)$

Here we focus only on Higgs-family transformations:
unitary transformations in the space of N doublets. *A priori*, they form the group $U(N)$. The symmetry group must be $G \subset U(N)$.

$U(N)$ contains the subgroup of overall phase rotations, which is already included in $U(1)_Y \subset G_{EW}$. We want to classify structural symmetries of the NHDM potentials, so we disregard transformations $\in U(1)$.

This leads us to the group $U(N)/U(1) \simeq PSU(N)$.

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Technical point 1: $PSU(N)$

Note that people often consider $SU(N)$. But $SU(N)$ still contains overall phase rotations: $\mathrm{diag}(e^{2\pi i/N} \dots, e^{2\pi i/N})$. They form the center of the group, $Z(SU(N)) \simeq \mathbb{Z}_N \in U(1)_Y$. Therefore, if we want to study structural properties of NHDM, we need to consider the factor group

$$SU(N)/Z(SU(N)) = PSU(N)\,.$$

All reparametrization symmetry groups we describe below are subgroups of $PSU(N)$, not $SU(N)$.

Technical point 2: Realizable symmetry groups

Classification of (finite) subgroups of $SU(N)$, with a particular emphasis on $SU(3)$, and its particle physics applications have attracted much attention:

Fairbairn, Fulton, Klink, *J. Math. Phys.* **5**, 1038 (1964);
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Here, we study a different problem: we need a classification not of all possible (finite) subgroups of $SU(N)$, but only those subgroups which can be symmetry groups of some NHDM potential.

Searching for automorphism groups of some potentials strongly reduces the list of possibilities.

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Definition:

we call a symmetry group $G \subset PSU(N)$ realizable if there exists a G-symmetric potential which is not symmetric under a larger group $\tilde{G}$ such that $G \subset \tilde{G} \subset PSU(N)$.

Good points about realizable groups:

- it represents the full symmetry content of the potential;
- the symmetry group of the vacuum is guaranteed to be a subgroup of the symmetry group of the potential.

Finding realizable groups is rather challenging: whenever I claim that a group is realizable, I must prove that there exists a potential with no other symmetries!

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Symmetries in NHDM

In 2HDM, all questions regarding symmetries have been answered. In particular, the only realizable finite Higgs-family symmetry groups are $\mathbb{Z}_2$ and $(\mathbb{Z}_2)^2$.

For any $N > 2$, despite several attempts, the classification was still missing.

We recently made two steps forward:

- We characterized all abelian symmetry groups for any N,
- We classified all finite symmetry groups for $N = 3$.
- In all cases, we also gave examples of potentials symmetric under each group.

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Outline of the strategy

Three steps towards classification of finite symmetry groups in 3HDM:

- Abelian groups are basic building blocks of any group, so we first find all relevant finite abelian symmetry groups in 3HDM.
- Group-theoretic part: prove that any finite symmetry group G must satisfy $G/A \subseteq Aut(A)$, where A is one of the abelian groups found previously. So, G can be constructed from A by extension.
- Calculational part: check all possible A's and extensions and see whether the potential supports this group.

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Outline of the strategy

Note differences with the "standard" way the group theory is used in particle physics:

- Usually, we first choose the group, then assign fields to some representations, then construct group-invariant interactions.
- In this way we do not know *a priori* which groups can be used. Even if some groups can be guessed, it is not clear how to prove that no other groups can be implemented → lack of a completeness criterion.
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Step 1: Abelian symmetry groups

All realizable abelian groups for any N were characterized in *Ivanov, Keus, Vdovin, J.Phys.A45, 215201 (2012)*.

The strategy is:

- find maximal abelian subgroups in $PSU(N)$,
- find which of their subgroups can be realizable symmetry groups.

"maximal abelian" = "not contained in a larger abelian".

Step 1: Abelian symmetry groups

In $SU(N)$, all maximal abelian subgroups are maximal tori $T_0 = [U(1)]^{N-1}$. All of them are conjugate to the subgroup of diagonal matrices (Cartan subgroup).
All abelian subgroups of $SU(N) \simeq$ groups of rephasing transformations.

In $PSU(N)$, there are two sorts of maximal abelian subgroups:

- maximal tori $T = [U(1)]^{N-1}$, which are images of T_0 under

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- certain finite groups, whose full preimage in $SU(N)$ is not abelian but is a nilpotent group of class 2.

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Step 1: Abelian symmetry groups

For general N, we developed an algorithm that gives all subgroups of maximal tori which can be realizable symmetry groups of some potential. In short, it associates an integer matrix to any potential, whose Smith normal form gives the subgroup of the maximal torus T.

In what concerns finite abelian groups, the answer is particularly simple for $N < 6$: all abelian groups with order $\leq 2^{N-1}$ are realizable.

Finite maximal abelian subgroups of $PSU(N)$ need to be studied separately for each N.

Step 1: $N = 3$ case

For $N = 3$ we get the following finite abelian groups:

$$\mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_3 \times \mathbb{Z}_3 \,.$$

This list is complete: imposing any other finite abelian symmetry group on the potential unavoidably leads to continuous symmetry group.

Note that the orders of these groups have only two prime factors: 2 and 3.

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Step 2: Group-theoretic part

- Any finite (non-abelian) G must contain only these abelian subgroups,
- $\Rightarrow$ by Cauchy's theorem, its order $|G| = 2^a 3^b$,
- $\Rightarrow$ by Burnside's $p^a q^b$ theorem, G is solvable.
- $\Rightarrow$ it contains a normal abelian subgroup A

$$g^{-1}Ag = A \quad \forall g \in G\,.$$

- $\Rightarrow$ we can consider G/A, but so far, we don't have any restriction on the size and structure of G/A.
- For a generic solvable group, we cannot go further. But in our case we have a solvable subgroup of $PSU(3)$, and in this case a stronger statement holds: G contains a normal maximal abelian subgroup. And this has remarkable consequences.

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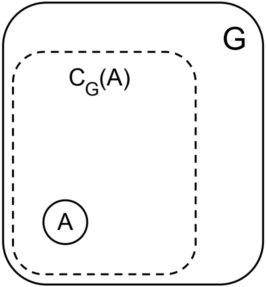
Consequences of a normal maximal abelian subgroup



Consider A, abelian subgroup of G. Centralizer of A in G is the subgroup of all elements $g \in G$ which commute with all elements $x \in A$. We get

$$A \subseteq C_G(A) \subset G\,.$$

If $A = C_G(A)$, then A is self-centralizing.

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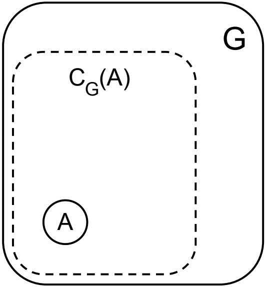

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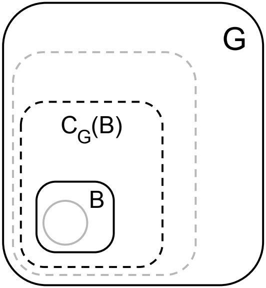

If $A \subset C_G(A)$, pick up some $b \in C_G(A)$, $b \notin A$ and consider $B = \langle A, b\rangle$, which is also an abelian subgroup of G.
We then get:

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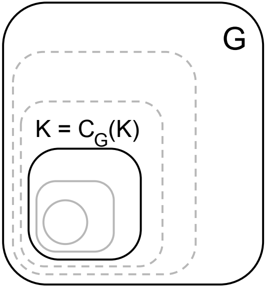

If $B \subset C_G(B)$, pick up some $c \in C_G(B)$, $c \notin B$ and consider $C = \langle B, c \rangle$, which is also an abelian subgroup of G.
Repeat until we hit a self-centralizing (maximal) abelian subgroup:

$$A \subset B \subset \cdots \subset K = C_G(K) \subseteq \cdots \subseteq C_G(B) \subseteq C_G(A) \subset G\,.$$

Consequences of a normal maximal abelian subgroup

What happens if a maximal abelian (=self-centralizing) subgroup A is normal in G?

- If A is normal in G, then $g^{-1}Ag = A$, so g acts on elements of A by some group-preserving permutation (automorphism of A).
- So, for every $g \in G$ we get an automorphism $\in Aut(A)$. We get a map $f : G \to Aut(A)$.
- Note that $Ker\, f = C_G(A)$. Indeed, $Ker\, f$ contains all elements g which induce the trivial permutation on A: $g^{-1}ag = a$ for all $a \in A$.
- If A is self-centralizing, $Ker\, f = A$. Therefore, map $\tilde{f} : G/A \to Aut(A)$ is injective: different elements of G/A map to different elements of $Aut(A)$.
- Thus, $G/A \subseteq Aut(A)$, and G can be constructed as an extension of A by a subgroup of $Aut(A)$.

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- If A is normal in G, then $g^{-1}Ag = A$, so g acts on elements of A by some group-preserving permutation (automorphism of A).
- So, for every $g \in G$ we get an automorphism $\in Aut(A)$. We get a map $f : G \to Aut(A)$.
- Note that $Ker\, f = C_G(A)$. Indeed, $Ker\, f$ contains all elements g which induce the trivial permutation on A: $g^{-1}ag = a$ for all $a \in A$.
- If A is self-centralizing, $Ker\, f = A$. Therefore, map $\tilde{f} : G/A \to Aut(A)$ is injective: different elements of G/A map to different elements of $Aut(A)$.
- Thus, $G/A \subseteq Aut(A)$, and G can be constructed as an extension of A by a subgroup of $Aut(A)$.

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Automorphism groups

$$G = \text{extension of } A \text{ by } P\,, \quad P \subseteq Aut(A)\,.$$

Overview of possibilities:

A	$Aut(A)$	"usable" subgroups P
$\mathbb{Z}_2$	$\{1\}$	—
$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$GL_2(2) \simeq S_3$	$\mathbb{Z}_2,\, \mathbb{Z}_3,\, S_3$
$\mathbb{Z}_3 \times \mathbb{Z}_3$	$GL_2(3)$	$\mathbb{Z}_2,\, \mathbb{Z}_4$

Step 3: Constructing G by extensions, $\mathbb{Z}_4$ example

Example: $A = \mathbb{Z}_4$. Then $Aut(\mathbb{Z}_4) = \mathbb{Z}_2$, so G is extension of $\mathbb{Z}_4$ by $\mathbb{Z}_2$.

There are several possibilities.
(1) extensions which lead to larger abelian groups ($\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$) are immediately excluded;

(2) dihedral group D_8, the symmetry group of the square.

$$D_8 = \langle a, b \rangle \text{ with conditions } a^4 = 1,\ b^2 = 1,\ ab = ba^3\,.$$

If $a = \mathrm{diag}(i, -i, 1)$, then

$$b = \begin{pmatrix} 0 & e^{i\delta} & 0 \\ e^{-i\delta} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{with arbitrary } \delta.$$

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A generic $\mathbb{Z}_4$ potential can be brought to the form $V_0 + V_{\mathbb{Z}_4}$, where

$$V_0 = -\sum_a m_a^2(\phi_a^\dagger\phi_a) + \sum_{a,b}\lambda_{ab}(\phi_a^\dagger\phi_a)(\phi_b^\dagger\phi_b) + \sum_{a\neq b}\lambda'_{ab}(\phi_a^\dagger\phi_b)(\phi_b^\dagger\phi_a)\,,$$

and

$$V_{\mathbb{Z}_4} = \lambda_1(\phi_3^\dagger\phi_1)(\phi_3^\dagger\phi_2) + \lambda_2(\phi_1^\dagger\phi_2)^2 + h.c.$$

The λ_1 term is invariant under any b, while the λ_2 term transforms as

$$(\phi_1^\dagger\phi_2)^2 \mapsto e^{-4i\delta}(\phi_2^\dagger\phi_1)^2\,.$$

If we restrict parameters of V_0 ($m_{11}^2 = m_{22}^2$, $\lambda_{11} = \lambda_{22}$, $\lambda_{13} = \lambda_{23}$, $\lambda'_{13} = \lambda'_{23}$) then the potential is symmetric under one particular D_8 group in which the value of $\delta = \arg\lambda_2/2$.

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(3) quaternion group Q_8:

$$Q_8 = \langle a, b\rangle \text{ with conditions } a^4 = 1,\ b^2 = a^2,\ ab = ba^3\,.$$

If $a = \mathrm{diag}(i, -i, 1)$, then

$$b(Q_8) = \begin{pmatrix} 0 & e^{i\delta} & 0 \\ -e^{-i\delta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Step 3: Constructing G by extensions, $\mathbb{Z}_4$ example

Again, the $\mathbb{Z}_4$ part of the potential:

$$V_{\mathbb{Z}_4} = \lambda_1(\phi_3^\dagger\phi_1)(\phi_3^\dagger\phi_2) + \lambda_2(\phi_1^\dagger\phi_2)^2 + h.c.$$

Upon this b, the λ_1 term changes its sign. The only way to impose Q_8 is to set $\lambda_1 = 0$. But then the potential becomes invariant under a continuous transformation: $\mathrm{diag}(e^{i\alpha}, e^{i\alpha}, 1)$.

We conclude that Q_8 cannot be a symmetry group of potential.

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Finite symmetry groups for $N = 3$

We performed this kind of analysis for all abelian groups we have.

Results:

$$\begin{gathered} \mathbb{Z}_2\,, \quad \mathbb{Z}_3\,, \quad \mathbb{Z}_4\,, \quad \mathbb{Z}_2 \times \mathbb{Z}_2\,, \\ D_6 \simeq S_3\,, \quad D_8\,, \quad T \simeq A_4\,, \quad O \simeq S_4\,, \\ (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 = \Delta(54)/\mathbb{Z}_3\,, \quad (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4 = \Sigma(36)\,. \end{gathered}$$

This list is complete: trying to impose any other finite symmetry group will lead to a potential symmetric under a continuous group.

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Generalized-CP symmetries

We also extended this method to groups which include generalized-CP transformations. Again, all realizable groups are found.

Some particular observations:

- unlike 2HDM, it is possible in 3HDM to have a Higgs-family symmetry without explicit CP-conservation;
- but a sufficiently high Higgs-family symmetry nevertheless guarantees explicit CP-conservation;
- in particular, $\mathbb{Z}_4$ symmetry automatically leads to explicit CP-conservation.
- In fact, family-symmetry groups A_4, S_4, Σ_{36} are imcompatible not only with explicit, but also with spontaneous CP-violation.

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Interplay between scalar and flavor symmetries

How are these scalar symmetries G related to the flavor symmetries?

- One can extend G to flavor sector, check all possible vev alignments and G-breaking patterns, find the generic fermion mass matrix compatible with the symmetry, and check if it reproduces the mass/mixing observables.
- General result: sufficiently large G in 3HDM are not compatible with data. Pure 3HDM does not offer any neat solution of flavor puzzle $\rightarrow$ either we give up a high symmetry group or we add extra fields.
- Another possibility: G is continuous, but the Yukawa sector selects out a finite subgroup of G. This is an standard approach followed by many, but general understanding of what's possible here is still insufficient. We are now trying to develop a systematic treatment of this situation for rephasing symmetry groups.

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Astroparticle issues

- Examples of scalar dark matter models based on group $\mathbb{Z}_p$ rather than $\mathbb{Z}_2$ with desired microscopic dynamics can be easily constructed [*Ivanov, Keus, PRD86, 016004 (2012)*].
- Multi-doublet scalar potential can have several coexisting minima, either degenerate or not $\rightarrow$ issues of the vacuum metastability even at the tree level, and of possible thermal phase transitions in the early hot Universe. Note that exotic (e.g. charge-breaking) intermediate phases in early Universe become possible.