JET BROADENING IN EFFECTIVE FIELD THEORY: WHEN DIMENSIONAL REGULARISATION FAILS

[ GUIDO BELL ]

           T. Becher, GB, JHEP 1211 (2012) 126
OUTLINE

EVENT-SHAPE VARIABLES

FACTORISATION
- REVIEW OF THRUST ANALYSIS
- FACTORISATION BREAKDOWN FOR BROADENING
- ANALYTIC REGULARISATION IN SCET

RESUMMATION
- COLLINEAR ANOMALY
- NNLL RESUMMATION
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Canonical event shape

Thrust:

\[ T = \frac{1}{Q} \max \left( \sum_i |\vec{p}_i \cdot \vec{n}_T| \right) \]

two-jet like: \( T \simeq 1 \)

spherical: \( T \simeq 1/2 \)

Thrust distribution precisely measured at LEP \((\tau = 1 - T)\)

in the two-jet region \( \tau \simeq 0 \)

\[
\frac{1}{\sigma_0} \frac{d\sigma}{d\tau} \sim \frac{\alpha_s C_F}{2\pi} \left[ \frac{-4 \ln \frac{\tau + 3}{\tau}}{\tau} + \ldots \right]
\]

\( \Rightarrow \) Sudakov logs require resummation
Motivation

Why $e^+e^-$ event shapes in 2013?

▶ clean environment to test understanding of QCD
  perturbation theory + resummation + non-perturbative effects
  same methods are applied at the LHC: soft gluon resummation ($\Rightarrow$ thrust)
  $p_T$ resummation ($\Rightarrow$ broadening)

$\alpha_s(M_Z) = 0.1202 \pm 0.0003^{(\text{stat})} \pm 0.0009^{(\text{exp})} \pm 0.0013^{(\text{had})} \pm 0.0047^{(\text{theo})}$ [LEP QCD working group 04]

$\alpha_s(M_Z) = 0.1224 \pm 0.0009^{(\text{stat})} \pm 0.0009^{(\text{exp})} \pm 0.0012^{(\text{had})} \pm 0.0035^{(\text{theo})}$ [Dissertori et al 09]

$\Rightarrow$ further improvements require to go beyond NLL resummation!
Motivation

Why $e^+ e^-$ event shapes in 2013?

- clean environment to test understanding of QCD
  perturbation theory + resummation + non-perturbative effects
  same methods are applied at the LHC: soft gluon resummation ($\Rightarrow$ thrust)
  $p_T$ resummation ($\Rightarrow$ broadening)

- precision determination of $\alpha_s$
  traditionally based on a fit to six event shapes ($T, \rho_H, B_T, B_W, C, y_3$)

NLO + NLL: $\alpha_s(M_Z) = 0.1202 \pm 0.0003 \text{ (stat)} \pm 0.0009 \text{ (exp)} \pm 0.0013 \text{ (had)} \pm 0.0047 \text{ (theo)}$

[LEP QCD working group 04]

NNLO + NLL: $\alpha_s(M_Z) = 0.1224 \pm 0.0009 \text{ (stat)} \pm 0.0009 \text{ (exp)} \pm 0.0012 \text{ (had)} \pm 0.0035 \text{ (theo)}$

[Dissertori et al 09]

$\Rightarrow$ further improvements require to go beyond NLL resummation!
Beyond NLL?

Traditional resummations are based on the coherent branching algorithm [Catani, Trentadue, Turnock, Webber 93]

- sums probabilities for independent gluon emissions
- apparently hard to extend beyond NLL

In SCET resummations are formulated in an operator language on the amplitude level [Becher, Schwartz 08]

- extension to higher orders requires standard EFT techniques
- thrust analysis extended by two orders to $N^3$LL accuracy

- field theoretical treatment of power corrections [Abbate, Fickinger, Hoang, Mateu, Stewart 10]

  two-dimensional fit to world thrust data

$$\text{NNLO + } N^3\text{LL: } \alpha_s(M_Z) = 0.1135 \pm 0.0002 \text{ (exp)} \pm 0.0005 \text{ (had)} \pm 0.0009 \text{ (pert)}$$
Beyond NLL?

Traditional resummations are based on the coherent branching algorithm \cite{Catani:1993ru} 
▶ sums probabilities for independent gluon emissions 
▶ apparently hard to extend beyond NLL

In SCET resummations are formulated in an operator language on the amplitude level 
▶ extension to higher orders requires standard EFT techniques 
▶ thrust analysis extended by \cite{Becher:2008ci} 
▶ field theoretical treatment of power corrections \cite{Abbate:2010ni} 
▶ two-dimensional fit to world thrust data

\[ \alpha_s(M_Z) = 0.1135 \pm 0.0002 \text{ (exp)} \pm 0.0005 \text{ (had)} \pm 0.0009 \text{ (pert)} \]

world average: \[ \alpha_s(M_Z) = 0.1184 \pm 0.0007 \]

almost 4\(\sigma\) below world average?
Precision thrust analysis

\[ \alpha_s(m_Z) \text{ from global thrust fits} \]

\begin{align*}
\alpha_s(m_Z) &= 0.1135 \pm 0.0002 \text{ (exp)} \pm 0.0005 \text{ (had)} \pm 0.0009 \text{ (pert)} \quad \text{[Abbate et al. 10]} \\
\alpha_s(m_Z) &= 0.1140 \pm 0.0004 \text{ (exp)} \pm 0.0013 \text{ (had)} \pm 0.0007 \text{ (pert)} \quad \text{[Abbate et al. 12]} \\
\text{NNLO + NNLL: } \alpha_s(m_Z) &= 0.1131 \pm 0.0028 \quad \text{[Monni, Gehrmann, Luisoni 12]} \\
\end{align*}
Event shape studies in SCET

Heavy jet mass:

\[ \rho_H = \frac{1}{Q^2} \max (M_L^2, M_R^2) \]

hemisphere jet masses \( M_{L/R}^2 = \left( \sum_{i \in L/R} p_i \right)^2 \)

- similar to thrust \( \Rightarrow \) again N^3LL resummation

- non-perturbative effects more involved (hadron masses, . . . )
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hémisphere jet masses

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- similar to thrust ⇒ again N^3LL resummation
- non-perturbative effects more involved (hadron masses, \ldots )

Total and wide jet broadening:

\[ b_T = b_L + b_R \]

\[ b_W = \max (b_L, b_R) \]

hémisphere jet broadenings

\[ b_{L/R} = \frac{1}{2} \sum_{i \in L/R} |\vec{p}_i \times \vec{n}_T| \]

- orthogonal to thrust (measure transverse momentum distribution)
- different type of factorisation formula ⇒ aim at NNLL resummation
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In the two-jet limit $\tau \to 0$ the thrust distribution factorises as

$$\frac{1}{\sigma_0} \frac{d\sigma}{d\tau} = H(Q^2, \mu) \int dp_L^2 \int dp_R^2 J(p_L^2, \mu) J(p_R^2, \mu) S\left(\tau Q - \frac{p_L^2 + p_R^2}{Q}, \mu\right)$$

multi-scale problem: $Q^2 \gg p_L^2 \sim p_R^2 \sim \tau Q^2 \gg \tau^2 Q^2$

- hard
- collinear
- soft
Thrust in SCET

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hard collinear soft

Hard function:

▸ on-shell vector form factor of a massless quark

$$H(Q^2) = \left| \begin{array}{c} \psi_0 \\bar{\psi}_0 \end{array} \right|$$

▸ known to three-loop accuracy

▸ also enters Drell-Yan and DIS in the endpoint region
Thrust in SCET

In the two-jet limit \( \tau \to 0 \) the thrust distribution factorises as

\[
\frac{1}{\sigma_0} \frac{d\sigma}{d\tau} = H(Q^2, \mu) \int dp_L^2 \int dp_R^2 \, J(p_L^2, \mu) \, J(p_R^2, \mu) \, S\left(\tau Q - \frac{p_L^2 + p_R^2}{Q}, \mu\right)
\]

multi-scale problem:

\[ Q^2 \gg p_L^2 \sim p_R^2 \sim \tau Q^2 \gg \tau^2 Q^2 \]

hard \quad \text{collinear} \quad \text{soft}

Jet function:

- imaginary part of quark propagator in light-cone gauge

\[
J(p^2) \sim \text{Im} \left[ \text{F.T.} \left\langle 0 \left| \frac{n\bar{n}}{4} W^\dagger(0) \psi(0) \bar{\psi}(x) W(x) \frac{n\bar{n}}{4} \right| 0 \right\rangle \right]
\]

- known to two-loop accuracy (anomalous dimension to three-loop)

- also enters inclusive \( B \) decays and DIS in the endpoint region

[Fleming, Hoang, Mantry, Stewart 07; Schwartz 07]
Thrust in SCET

In the two-jet limit $\tau \to 0$ the thrust distribution factorises as

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\frac{1}{\sigma_0} \frac{d\sigma}{d\tau} = H(Q^2, \mu) \int dp_L^2 \int dp_R^2 \ J(p_L^2, \mu) \ J(p_R^2, \mu) \ S(\tau Q - \frac{p_L^2 + p_R^2}{Q}, \mu)
$$

multi-scale problem: $Q^2 \gg p_L^2 \sim p_R^2 \sim \tau Q^2 \gg \tau^2 Q^2$

hard collinear soft

Soft function:

- matrix element of Wilson lines along the directions of energetic quarks

$$
S(\omega) = \sum_X \left| \langle X | S_n^\dagger(0) \ S_{\bar{n}}(0) | 0 \rangle \right|^2 \delta(\omega - n \cdot p_{Xn} - \bar{n} \cdot p_{\bar{Xn}}) \quad S_n(x) = \mathbf{P} \ \exp \left( ig_s \int_{-\infty}^{0} ds \ n \cdot A_s(x + sn) \right)
$$

- known to two-loop accuracy (anomalous dimension to three-loop)

[Fleming, Hoang, Mantry, Stewart 07; Schwartz 07]
How does resummation work (roughly)?

Let us have a closer look at the one-loop expressions

\[ H(Q^2, \mu) = 1 + \frac{\alpha_s C_F}{4\pi} \left[ -2 \ln^2 \frac{Q^2}{\mu^2} + 6 \ln \frac{Q^2}{\mu^2} - 16 + \frac{7\pi^2}{3} \right] \]

\[ J(p^2, \mu) = \delta(p^2) + \frac{\alpha_s C_F}{4\pi} \left[ \left( \frac{4 \ln(p^2/\mu^2) - 3}{p^2} \right) \mu^2 + (7 - \pi^2) \delta(p^2) \right] \]

\[ S(\omega, \mu) = \delta(\omega) + \frac{\alpha_s C_F}{4\pi} \left[ \left( -16 \frac{\ln(\omega/\mu)}{\omega} \right) \mu \right] + \frac{\pi^2}{3} \delta(\omega) \]
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\]

General structure:

- **logarithms** ⇔ **divergences**

anomalous dimensions of EFT operators ⇒ resum logs via RG techniques

\[
\frac{d}{d \ln \mu} H(Q^2, \mu) = \left[ 2\Gamma_{\text{cusp}}(\alpha_s) \ln \frac{Q^2}{\mu^2} + 4\gamma^q(\alpha_s) \right] H(Q^2, \mu)
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- finite terms ⇒ accounted for in matching calculations
How does resummation work (roughly)?

Let us have a closer look at the one-loop expressions

\[ H(Q^2, \mu) = 1 + \frac{\alpha_s C_F}{4\pi} \left[ -2 \ln^2 \frac{Q^2}{\mu^2} + 6 \ln \frac{Q^2}{\mu^2} - 16 + \frac{7\pi^2}{3} \right] \]

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- finite terms ⇒ accounted for in matching calculations

Notice: there is no large log when each function is evaluated at its natural scale!
Angularities

Interesting class of event shape variables

\[ \tau_a = \frac{1}{Q} \sum_i E_i (\sin \theta_i)^a (1 - |\cos \theta_i|)^{1-a} \]

- interpolates between thrust \((a = 0)\) and broadening \((a = 1)\)
- infrared safe for \(a < 2\), but standard factorisation only for \(a < 1\)

SCET analysis

- relevant scales:
  \[ \mu_H^2 \sim Q^2 \quad \gg \quad \mu_J^2 \sim Q^2 \tau_a^{\frac{2}{2-a}} \quad \gg \quad \mu_S^2 \sim Q^2 \tau_a^2 \]
  
  thrust:
  \[ \mu_H^2 \sim Q^2 \quad \gg \quad \mu_J^2 \sim Q^2 \tau \quad \gg \quad \mu_S^2 \sim Q^2 \tau^2 \] (SCET_I)

  broadening:
  \[ \mu_H^2 \sim Q^2 \quad \gg \quad \mu_J^2 \sim Q^2 B^2 \quad \sim \quad \mu_S^2 \sim Q^2 B^2 \] (SCET_{II})

\[ \Rightarrow \] factorisation formula for broadening will be different (and more complicated)
Jet broadening

In the two-jet limit $b_L \sim b_R \to 0$ expect that the broadening distribution factorises as

$$
\frac{1}{\sigma_0} \frac{d^2\sigma}{db_L db_R} = H(Q^2, \mu) \int db_L^s \int db_R^s \int d^{d-2}p_L^\perp \int d^{d-2}p_R^\perp 
\mathcal{J}_L(b_L - b_L^s, p_L^\perp, \mu) \mathcal{J}_R(b_R - b_R^s, p_R^\perp, \mu) S(b_L^s, b_R^s, -p_L^\perp, -p_R^\perp, \mu)
$$

two-scale problem: $Q^2 \gg b_L^2 \sim b_R^2$

- relevant modes have $p_{\text{coll}}^\perp \sim p_{\text{soft}}^\perp \sim b_{L,R}$
- jet recoils against soft radiation
Jet broadening

In the two-jet limit $b_L \sim b_R \to 0$ expect that the broadening distribution factorises as

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Hard function:

- precisely the same object as for thrust
- recall the RG equation

$$\frac{d}{d \ln \mu} H(Q^2, \mu) = \left[ 2 \Gamma_{\text{cusp}}(\alpha_s) \ln \frac{Q^2}{\mu^2} + 4 \gamma^q(\alpha_s) \right] H(Q^2, \mu)$$

$\Rightarrow$ there is a hidden $Q$-dependence in the second line!

$$\text{thrust} \quad \frac{\mu_J^2}{\mu_S} = \frac{\tau Q^2}{\tau Q} = Q \quad \Leftrightarrow \quad \text{broadening} \quad \frac{\mu_J^2}{\mu_S} = \frac{b^2}{b} = b$$
Jet broadening

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$$\frac{1}{\sigma_0} \frac{d^2 \sigma}{db_L db_R} = H(Q^2, \mu) \int db_L^s \int db_R^s \int d^{d-2}p_L^\perp \int d^{d-2}p_R^\perp J_L(b_L - b_L^s, p_L^\perp, \mu) J_R(b_R - b_R^s, p_R^\perp, \mu) S(b_L^s, b_R^s, -p_L^\perp, -p_R^\perp, \mu)$$

Some manipulations:

- Laplace transform $b_{L,R} \to \tau_{L,R}$
- Fourier transform $p_{L,R}^\perp \to x_{L,R}^\perp$
- Define dimensionless variable $z_{L,R} = \frac{2|x_{L,R}^\perp|}{\tau_{L,R}}$

$\Rightarrow$ the naive factorisation theorem takes the form

$$\frac{1}{\sigma_0} \frac{d^2 \sigma}{d\tau_L d\tau_R} = H(Q^2, \mu) \int_0^\infty dz_L \int_0^\infty dz_R \overline{J}_L(\tau_L, z_L, \mu) \overline{J}_R(\tau_R, z_R, \mu) \overline{S}(\tau_L, \tau_R, z_L, z_R, \mu)$$
Jet function

The quark jet function for broadening reads

\[ J(b, p^\perp) \sim \sum_X \delta(\vec{n} \cdot p_X - Q) \delta^{d-2}(p_X^\perp - p^\perp) \delta \left( b - \frac{1}{2} \sum_{i \in X} |p_i^\perp| \right) \left| \left< X \mid \bar{\psi}(0) W(0) \frac{n}{4} \mid 0 \right> \right|^2 \]

- delta-functions ensure that jet has given energy, \( p^\perp \) and \( b \)
- tree level: \( J(b, p^\perp) = \delta \left( b - \frac{1}{2} |p^\perp| \right) \Rightarrow J(\tau, z) = \frac{z}{(1 + z^2)^{3/2}} + O(\epsilon) \)
The quark jet function for broadening reads

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- delta-functions ensure that jet has given energy, $p^\perp$ and $b$
- tree level: $\mathcal{J}(b, p^\perp) = \delta \left( b - \frac{1}{2} |p^\perp| \right) \Rightarrow \mathcal{J}(\tau, z) = \frac{z}{(1+2z)^{3/2}} + \mathcal{O}(\epsilon)$

At one-loop the calculation involves

- Wilson-line diagrams are not well-defined in dimensional regularisation!

$$\int_0^Q \frac{dk_-}{k_-}$$ diverges in the soft limit \quad (DR regularises $d^{d-2}k_\perp$)

- this does not happen for thrust or any SCET$_1$ problem
Momentum modes

Thrust (SCET\textsubscript{I})

- thrust: \( p_s^2 \ll p_c^2 \)

Broadening (SCET\textsubscript{II})

- broadening: \( p_s^2 \sim p_c^2 \)

\[ \Rightarrow \] cannot distinguish soft mode from collinear mode when radiated into jet direction

\[ \Rightarrow \] need additional regulator that distinguishes modes by their rapidities
Regularisation in SCET$_{\Pi}$

The regularisation of individual diagrams is largely arbitrary, one could use e.g.

\[
\frac{1}{p^2 + i\varepsilon} \rightarrow \frac{1}{p^2 - \Delta + i\varepsilon}, \quad \frac{(\nu^2)^\alpha}{(p^2 + i\varepsilon)^{1+\alpha}}, \quad \ldots
\]

▶ trivial for QCD, but regularises ill-defined EFT diagrams

▶ spoils gauge-invariance and eikonal structure of Wilson line emissions
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\[
\frac{1}{p^2 + i\varepsilon} \rightarrow \frac{1}{p^2 - \Delta + i\varepsilon}, \quad \frac{(\nu^2)^\alpha}{(p^2 + i\varepsilon)^{1+\alpha}}, \quad \ldots
\]

- trivial for QCD, but regularises ill-defined EFT diagrams
- spoils gauge-invariance and eikonal structure of Wilson line emissions

In a massless theory it is sufficient to regularise phase space integrals

\[
\int d^d k \, \delta(k^2) \, \theta(k^0) \ \Rightarrow \ \int d^d k \left( \frac{\nu_+}{k_+} \right)^\alpha \delta(k^2) \, \theta(k^0)
\]

- does not modify SCET at all \ \Rightarrow \ keeps gauge-invariance and eikonal structure
- analytic, minimal and adopted to the problem (LC propagators)
Why does it work?

Our new prescription amounts to

\[
\int d^d k \, \delta(k^2) \, \theta(k^0) \implies \int d^d k \left( \frac{\nu_+}{k_+} \right)^\alpha \delta(k^2) \, \theta(k^0)
\]

- virtual corrections do not need regularisation
- matrix elements of Wilson lines in QCD \( \Rightarrow \) the same for thrust and broadening
- technical reason: \( \int d^{d-2} k_\perp \, f(k_\perp, k_+) \sim k_+^{-\epsilon} \)
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$$
\int d^d k \, \delta(k^2) \, \theta(k^0) \ \Rightarrow \ \int d^d k \, \left( \frac{\nu_+}{k_+} \right)^\alpha \delta(k^2) \, \theta(k^0)
$$

- virtual corrections do not need regularisation
- matrix elements of Wilson lines in QCD \(\Rightarrow\) the same for thrust and broadening
- technical reason: \(\int d^{d-2} k_\perp \ f(k_\perp, k_+) \sim k_+^{-\epsilon}\)
- required for observables sensitive to transverse momenta
  \(f(k_\perp, k_+) \sim \delta^{d-2}(k_\perp - p_\perp) \ \Rightarrow \ \text{factor } k_+^{-\epsilon} \text{ absent} \ \Rightarrow \ \text{reinstalled as } k_+^{-\alpha}\)
- can show that the prescription regularises all LC singularities in SCET [Becher, GB 11]
- not sufficient for cases where virtual corrections are ill-defined
  examples: electroweak Sudakov corrections, Regge limits
Jet function revisited

With the additional regulator in place, the jet functions can be evaluated

\[ \mathcal{J}_L(b, p^\perp = 0) = \delta(b) + \frac{C_F \alpha_s}{2\pi} \frac{e^{\epsilon\gamma E}}{\Gamma(1 - \epsilon)} \frac{1}{b} \left( \frac{\mu}{b} \right)^{2\epsilon} \left[ 1 - \epsilon + \frac{4 \Gamma(2 + \alpha) \Gamma(\alpha)}{\Gamma(2 + 2\alpha)} \left( \frac{Q\nu_+}{b^2} \right)^\alpha \right] \]

\[ \mathcal{J}_R(b, p^\perp = 0) = \delta(b) + \frac{C_F \alpha_s}{2\pi} \frac{e^{\epsilon\gamma E}}{\Gamma(1 - \epsilon)} \frac{1}{b} \left( \frac{\mu}{b} \right)^{2\epsilon} \left[ 1 - \epsilon + \frac{4 \Gamma(-\alpha) \Gamma(2 - \alpha)}{\Gamma(2 - \alpha)} \left( \frac{\nu_+}{Q} \right)^\alpha \right] \]

► ordered limit \( \alpha \to 0, \epsilon \to 0 \) generates a pole in the analytic regulator

► note the characteristic scaling \( \left( \frac{\nu_+}{K_+} \right)^\alpha \) in each region

For \( p^\perp \neq 0 \) the computation is considerably more involved (\( \to \) later)

\[ \overline{\mathcal{J}}_L(\tau, z) = \overline{\mathcal{J}}_L^{(0)}(\tau, z) \left[ 1 - \frac{C_F \alpha_s}{\pi} \frac{1}{\alpha} \left( \frac{1}{\epsilon} + \ln \left( \mu^2 \tau^2 \right) + 2 \ln \frac{\sqrt{1 + z^2} + 1}{4} \right) \left( Q\nu_+ \tau^2 \right)^\alpha + \ldots \right] \]

► divergent term has non-trivial \( z \)-dependence
The soft function for broadening reads

\[
S(b_L, b_R, p^+_L, p^+_R) \sim \sum_{X_L, X_R} \delta^{d-2}(p^+_X - p^+_L) \delta^{d-2}(p^+_X - p^+_R) \\
\delta \left( b_L - \frac{1}{2} \sum_{i \in X_L} |p^+_{L,i}| \right) \delta \left( b_R - \frac{1}{2} \sum_{j \in X_R} |p^+_{R,j}| \right) \left| \langle X_L X_R | S^\dagger_n(0) S_n(0) | 0 \rangle \right|^2
\]

- split final state into left and right-moving particles
- tree level:  \( S(b_L, b_R, p^+_L, p^+_R) = \delta(b_L) \delta(b_R) \delta^{d-2}(p^+_L) \delta^{d-2}(p^+_R) \Rightarrow S(\tau_L, \tau_R, z_L, z_R) = 1 \)

At one-loop the calculation involves

\[
\Rightarrow S(\tau_L, \tau_R, z_L, z_R) = 1 + \frac{C_F \alpha_s}{\pi} \left\{ \frac{1}{\epsilon} \left( \frac{1}{\epsilon} + \ln (\mu^2 \tau_L^2) \right) + 2 \ln \left( \frac{\sqrt{1 + z_L^2} + 1}{4} \right) \left( \nu + \tau_L \right)^\alpha - (L \leftrightarrow R) + \ldots \right\}
\]
Anomalous $Q$ dependence

Let us now put the jet and soft functions together

$$\mathcal{J}_L(\tau_L, z_L) \mathcal{J}_R(\tau_R, z_R) \mathcal{S}(\tau_L, \tau_R, z_L, z_R) = \mathcal{J}_L^{(0)}(\tau_L, z_L) \mathcal{J}_R^{(0)}(\tau_R, z_R)$$

$$\left\{ 1 + \frac{C_F \alpha_s}{\pi} \left[ \left( -\frac{1}{\alpha} - \ln \left( Q\nu_+ \bar{\tau}_L^2 \right) + \frac{1}{\alpha} + \ln (\nu_+ \bar{\tau}_L) \right) \left( \frac{1}{\epsilon} + \ln \left( \mu^2 \bar{\tau}_L^2 \right) + 2 \ln \frac{\sqrt{1 + z_L^2} + 1}{4} \right) 
+ \left( + \frac{1}{\alpha} + \ln \left( \frac{\nu_+}{Q} \right) - \frac{1}{\alpha} - \ln (\nu_+ \bar{\tau}_R) \right) \left( \frac{1}{\epsilon} + \ln \left( \mu^2 \bar{\tau}_R^2 \right) + 2 \ln \frac{\sqrt{1 + z_R^2} + 1}{4} \right) + \ldots \right] \right\}$$

- well-defined without additional regulators
Anomalous $Q$ dependence

Let us now put the jet and soft functions together

$$
\tilde{J}_L(\tau_L, z_L) \tilde{J}_R(\tau_R, z_R) \tilde{S}(\tau_L, \tau_R, z_L, z_R) = \tilde{J}_L^{(0)}(\tau_L, z_L) \tilde{J}_R^{(0)}(\tau_R, z_R)
$$

$$
\left\{ 1 + \frac{C_F \alpha_s}{\pi} \left[ \left( - \ln (Q \nu_+ \bar{\tau}_L^2) + \ln (\nu_+ \bar{\tau}_L) \right) \left( \frac{1}{\epsilon} + \ln (\mu^2 \bar{\tau}_L^2) + 2 \ln \frac{\sqrt{1 + z_L^2} + 1}{4} \right) 
\right.
\right.
\left.
\left. + \left( + \ln \left( \frac{\nu_+}{Q} \right) - \ln (\nu_+ \bar{\tau}_R) \right) \left( \frac{1}{\epsilon} + \ln (\mu^2 \bar{\tau}_R^2) + 2 \ln \frac{\sqrt{1 + z_R^2} + 1}{4} \right) + \ldots \right] \right\}
$$

- well-defined without additional regulators
- similarly the artificial scale $\nu_+$ drops out
Anomalous $Q$ dependence

Let us now put the jet and soft functions together

$$\overline{J}_L(\tau_L, z_L) \overline{J}_R(\tau_R, z_R) \overline{S}(\tau_L, \tau_R, z_L, z_R) = \overline{J}_L^{(0)}(\tau_L, z_L) \overline{J}_R^{(0)}(\tau_R, z_R)$$

$$\left\{ 1 + \frac{C_F \alpha_s}{\pi} \left[ \left( - \ln(Q \bar{\tau}_L) \right) \left( \frac{1}{\epsilon} + \ln(\mu^2 \bar{\tau}_L^2) + 2 \ln \frac{\sqrt{1 + z^2} + 1}{4} \right) 
+ \left( - \ln(Q \bar{\tau}_R) \right) \left( \frac{1}{\epsilon} + \ln(\mu^2 \bar{\tau}_R^2) + 2 \ln \frac{\sqrt{1 + z^2} + 1}{4} \right) + \ldots \right] \right\}$$

- well-defined without additional regulators
- similarly the artificial scale $\nu_+$ drops out
- the hidden $Q$ dependence shows up!

⇒ the naive factorisation formula does not achieve a proper scale separation

How can we resum a logarithm that appears in a matching calculation?
OUTLINE

EVENT-SHAPE VARIABLES

FACTORISATION
- REVIEW OF THRUST ANALYSIS
- FACTORISATION BREAKDOWN FOR BROADENING
- ANALYTIC REGULARISATION IN SCET

RESUMMATION
- COLLINEAR ANOMALY
- NNLL RESUMMATION
Collinear anomaly

Can show that the $Q$ dependence exponentiates using and extending arguments from

- electroweak Sudakov resummation [Chiu, Golf, Kelley, Manohar 07]
- $p_T$ resummation in Drell-Yan production [Becher, Neubert 10]

Start from the logarithm of the product of jet and soft functions

$$
\ln P = \ln \mathcal{J}_L \left( \ln (Q \nu + \tau^2_L); \tau_L, z_L \right) + \ln \mathcal{J}_R \left( \ln \left( \frac{\nu^2}{Q} \right); \tau_R, z_R \right) + \ln \mathcal{S} \left( \ln (\nu^2 + \tau_L); \tau_L, \tau_R, z_L, z_R \right)
$$

\[
\begin{align*}
\text{collinear: } k_+ & \sim \frac{b^2}{Q} \\
\text{anticollinear: } k_+ & \sim Q \\
\text{soft: } k_+ & \sim b
\end{align*}
\]
Collinear anomaly

Can show that the $Q$ dependence exponentiates using and extending arguments from

- electroweak Sudakov resummation [Chiu, Golf, Kelley, Manohar 07]
- $p_T$ resummation in Drell-Yan production [Becher, Neubert 10]

Start from the logarithm of the product of jet and soft functions

$$
\ln P = \ln J_L \left( \ln \left( \frac{Q \nu + \bar{\tau}_L^2}{\nu} ; \tau_L, z_L \right) \right) + \ln J_R \left( \ln \left( \frac{\nu + \bar{\tau}_R^2}{Q} ; \tau_R, z_R \right) \right) + \ln S \left( \ln \left( \nu + \bar{\tau}_L^2 ; \tau_L, \tau_R, z_L, z_R \right) \right)
$$

\[
\text{collinear: } k_+ \sim \frac{b^2}{Q} \quad \text{anticollinear: } k_+ \sim Q \quad \text{soft: } k_+ \sim b
\]

- use that product does not depend on $\nu_+$ and that it is LR symmetric

$$
\Rightarrow \ln P = \frac{k_2(\mu)}{4} \ln^2 (Q^2 \bar{\tau}_L \bar{\tau}_R) - F_B(\tau_L, z_L, \mu) \ln (Q^2 \bar{\tau}_L^2) - F_B(\tau_R, z_R, \mu) \ln (Q^2 \bar{\tau}_R^2) + \ln W(\tau_L, \tau_R, z_L, z_R, \mu)
$$

- RG invariance implies $k_2(\mu) = 0$ to all orders

$$
\Rightarrow \quad P(Q^2, \tau_L, \tau_R, z_L, z_R, \mu) = (Q^2 \bar{\tau}_L^2) - F_B(\tau_L, z_L, \mu) \quad (Q^2 \bar{\tau}_R^2) - F_B(\tau_R, z_R, \mu) \quad W(\tau_L, \tau_R, z_L, z_R, \mu)
$$
Final factorisation formula

The corrected all-order generalisation of the naive factorisation formula becomes

\[
\frac{1}{\sigma_0} \frac{d^2 \sigma}{d\tau_L d\tau_R} = H(Q^2, \mu) \int_0^\infty dz_L \int_0^\infty dz_R \left( Q^2 \tau_L^2 - F_B(\tau_L, z_L, \mu) \right) \left( Q^2 \tau_R^2 - F_B(\tau_R, z_R, \mu) \right) W(\tau_L, \tau_R, z_L, z_R, \mu)
\]

[Becher, GB, Neubert 11]
The corrected all-order generalisation of the naive factorisation formula becomes

\[
\frac{1}{\sigma_0} \frac{d^2\sigma}{d\tau_L d\tau_R} = H(Q^2, \mu) \int_0^\infty dz_L \int_0^\infty dz_R \left( Q^2 \frac{\tau_L^2}{\tau_L} - F_B(\tau_L, z_L, \mu) \right) \left( Q^2 \frac{\tau_R^2}{\tau_R} - F_B(\tau_R, z_R, \mu) \right) W(\tau_L, \tau_R, z_L, z_R, \mu)
\]

To NLL the Mellin inversion can be performed analytically

\[
\frac{1}{\sigma_0} \frac{d\sigma}{db_T} = H(Q^2, \mu) \frac{e^{-2\gamma_E \eta}}{\Gamma(2\eta)} \frac{1}{b_T} \left( \frac{b_T}{\mu} \right)^{2\eta} I^2(\eta) \quad \eta = \frac{C_F \alpha_s(\mu)}{\pi} \ln \frac{Q^2}{\mu^2} = \mathcal{O}(1)
\]

\[
\frac{1}{\sigma_0} \frac{d\sigma}{db_W} = H(Q^2, \mu) \frac{2\eta e^{-2\gamma_E \eta}}{\Gamma^2(1+\eta)} \frac{1}{b_W} \left( \frac{b_W}{\mu} \right)^{2\eta} I^2(\eta)
\]

The non-trivial \(z\)-dependence of the anomaly coefficient is encoded in

\[
l(\eta) = \int_0^\infty dz \frac{z}{(1+z^2)^{3/2}} \left( \frac{\sqrt{1+z^2} + 1}{4} \right)^{-\eta} = \frac{4\eta}{1+\eta} \binom{2}{1+\eta, 2+\eta, -1}
\]
Comparison with literature

Traditional resummation

- pioneering work missed quark recoil effects ⇒ valid to LL [Catani, Turnock, Webber 92]
- first NLL resummation by Dokshitzer et al [Dokshitzer, Lucenti, Marchesini, Salam 98]
  we find complete analytical agreement with this work

Resummation using SCET [Chiu, Jain, Neill, Rothstein 11,12]

- start from same naive factorisation formula
- modify Wilson-line propagators to regularise rapidity divergences
- treat additional divergences in a ”rapidity renormalization group”
- 2011 paper missed quark recoil effects ⇒ valid only to LL
  2012 paper in agreement with Dokshitzer et al
Beyond NLL

The extension to NNLL requires three ingredients

▶ one-loop soft function
▶ one-loop jet function
▶ two-loop anomaly coefficient

The calculation of the one-loop soft function is straight-forward

\[ S(\tau_L, \tau_R, z_L, z_R) = \frac{1}{4\pi} \left\{ \left( \mu^2 \bar{\tau}_L^2 \right)^\varepsilon \left( \nu + \bar{\tau}_L \right)^\alpha \left[ \frac{4}{\alpha} \left( \frac{1}{\varepsilon} + 2 \ln \left( \frac{1 + \sqrt{1 + z^2_L}}{4} \right) \right) - \frac{2}{\varepsilon^2} \right. \right. \]
\[ + 8 \text{Li}_2 \left( - \frac{\sqrt{1 + z^2_L} - 1}{\sqrt{1 + z^2_L} + 1} \right) + 4 \ln^2 \left( \frac{1 + \sqrt{1 + z^2_L}}{4} \right) + \frac{5\pi^2}{6} \left. \right\} - (L \leftrightarrow R) \]
One-loop jet function

The calculation of the one-loop jet function is surprisingly complicated

\[ \sim \int d^d q \delta(q^2) \theta(q^0) \int d^d k \left( \frac{\nu_+}{k_+} \right)^{\alpha} \delta(k^2) \theta(k^0) \frac{\bar{n} q (\bar{n} k + \bar{n} q)}{\bar{n} k (q + k)^2} \]

\[ \times \delta(Q - \bar{n} q - \bar{n} k) \delta^{d-2}(p_\perp - q_\perp - k_\perp) \delta(b - \frac{1}{2} |q_\perp| - \frac{1}{2} |k_\perp|) \]

\[ \sim \int_0^1 d\eta \eta (1 - \eta)^{-1+\alpha} \int_{1-y}^{1+y} d\xi \frac{\xi (2 - \xi)^{1-2\alpha}(\xi(2 - \xi) - 1 + y^2)^{-\frac{1}{2}-\epsilon}}{(\xi - 2y\eta)^2 + 4\eta(1 - y)(1 + y - \xi)} \]

- non-trivial angle complicates calculation
- expansion in \( \alpha \) and \( \epsilon \) is subtle
  \[ \Rightarrow \text{have to keep } (2b - p)^{-1-\epsilon}, (2b - p)^{-1-2\epsilon}, \ldots \text{ to all orders} \]
- computed the integrals in closed form without expanding in \( \epsilon \)
  \[ \Rightarrow \text{hypergeometric functions of half-integer parameters} \]
- performed Laplace + Fourier transformations analytically
Two-loop anomaly coefficient

Most easily extracted from the two-loop soft function

- again two particles in final state
- but requires to go one order higher in $\epsilon$-expansion
- encounter Nielsen polylogs and elliptic integrals

\[
d_B^2(z) = C_A \left\{ - \frac{1 + z^2}{9} h_1(z) + \frac{67 + 2z^2}{9} h_2(z) - 8 h_3(z) + 32 S_{1,2} \left( - \frac{z_-}{z_+} \right) - 8 \text{Li}_3 \left( - \frac{z_-}{z_+} \right) \\
+ 8 S_{1,2}(-w) - 24 \text{Li}_3(-w) - 24 S_{1,2}(1 - w) + 8 \text{Li}_3(1 - w) + 24 S_{1,2} \left( \frac{1 - w}{2} \right) \\
+ \text{a few more lines} \right\}
\]

\[
+ T_F n_f \left\{ \frac{2(1 + z^2)}{9} h_1(z) - \frac{2(13 + 2z^2)}{9} h_2(z) - \frac{4}{3} \ln^2 z_+ - \frac{20}{9} \ln z_+ + \frac{4}{9} z^2 - \frac{82}{27} \\
+ \frac{4w(5 - z^2)}{9} \ln \left( \frac{1 + w}{w} \right) + \frac{2w(11 + 2z^2)}{9} \right\}
\]

with $w = \sqrt{1 + z^2}$ and $z_\pm = (w \pm 1)/4$
A glimpse at the data

theory uncertainty significantly reduced in fit region for $\alpha_s$ extraction

- without matching to fixed-order calculation
- without estimate of non-perturbative corrections
Compare with fixed order

Confront with output of fixed-order MC generators (EVENT2, EERAD3)

$\Rightarrow$ we obtain the right logarithmic terms for small values of $L = \ln B_T$
Conclusions

Resummation beyond standard RG techniques via collinear anomaly

▶ we proposed an analytic phase space regularisation for SCET_{II} problems

\[ \int d^d k \ \delta(k^2) \ \theta(k^0) \Rightarrow \int d^d k \ \left( \frac{\nu_+}{k_+} \right)^\alpha \delta(k^2) \ \theta(k^0) \]

▶ respects symmetries of EFT, well-suited for efficient calculations

We determined all ingredients to perform NNLL resummation for jet broadening

▶ allows for precision determinations of \( \alpha_s \) from \( b_T \) and \( b_W \) distributions

The formalism is relevant for many interesting LHC observables

▶ Higgs production, \( \bar{t}t \), jet vetoes, jet substructure, . . .
Transverse momentum-dependent PDFs

Central ingredient for $p_T$ resummation at hardon colliders

$$B_{q/N}(z, x_T; \mu) = \frac{1}{2\pi} \int dt \ e^{-iz\vec{n} \cdot \vec{p}} \sum_{\chi, \chi'} \frac{\tilde{n}_{\alpha \beta}}{2} \langle N(p) | \tilde{\chi}_\alpha (t\vec{n}+x_{\perp}) | \chi \rangle \langle N(p) | \chi_{\beta} (0) | N(p) \rangle$$

⇒ ill-defined in DimReg because of unregularized rapidity divergences
Transverse momentum-dependent PDFs

Central ingredient for $p_T$ resummation at hardon colliders

\[
\mathcal{B}_{q/N}(z, x_T; \mu) = \frac{1}{2\pi} \int dt \ e^{-izt\cdot p} \sum_{\chi} \frac{\bar{\eta}_{\alpha\beta}}{2} \langle N(p) | \bar{\chi}_\alpha(t\bar{n} + x_\perp) | X \rangle \langle X | \chi_\beta(0) | N(p) \rangle
\]

⇒ ill-defined in DimReg because of unregularized rapidity divergences

Many attempts to find an optimal definition, e.g.

\[
\mathcal{B}_{q/N}(z, x_T; \zeta_A; \mu) = \lim_{y_{1,2} \to \pm \infty} \mathcal{B}_{q/N}^{\text{unsub}}(z, x_T; y_p - y_2) \sqrt{\frac{\tilde{S}(x_T, y_1, y_n)}{\tilde{S}(x_T, y_1, y_2) \tilde{S}(0)(x_T, y_n, y_2)}}
\]

"This definition seems unexpectedly complicated"
Transverse momentum-dependent PDFs

Central ingredient for $p_T$ resummation at hadron colliders

$$B_{q/N}(z, x_T; \mu) = \frac{1}{2\pi} \int dt \ e^{-izt\vec{n}\cdot p} \sum_{X, \text{reg}} \frac{\bar{n}_{\alpha\beta}}{2} \langle N(p) | \bar{\chi}_\alpha (t\vec{n} + x_\perp) | X \rangle \langle X | \chi_\beta (0) | N(p) \rangle$$

⇒ ill-defined in DimReg because of unregularized rapidity divergences

Many attempts to find an optimal definition, e.g.

$$B_{q/N}(z, x_T; \zeta_A; \mu) = \lim_{y_1, 2 \rightarrow \pm \infty} B_{q/N}^{\text{unsub}} (z, x_T; y_1 - y_2) \sqrt{\frac{\tilde{S}(x_T, y_1, y_n)}{\tilde{S}(x_T, y_1, y_2) \tilde{S}(0)(x_T, y_n, y_2)}}$$

"This definition seems unexpectedly complicated"

We propose a minimal modification of the naive definition

$$B_{q/N}(z, x_T; \mu) = \frac{1}{2\pi} \int dt \ e^{-izt\vec{n}\cdot p} \sum_{X, \text{reg}} \frac{\bar{n}_{\alpha\beta}}{2} \langle N(p) | \bar{\chi}_\alpha (t\vec{n} + x_\perp) | X \rangle \langle X | \chi_\beta (0) | N(p) \rangle$$

⇒ the only definition that has shown to work at two-loop order