Finite flavour groups of fermions

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Review:

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Finite flavour groups of fermions
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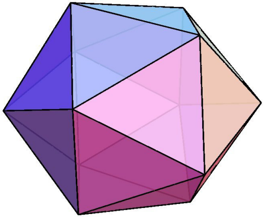


Groups

Definition: group $(G, \circ)$

$$\circ : \begin{array}{ccc} G \times G & \to & G \\ (g_1, g_2) & \mapsto & g_1 \circ g_2 \end{array}$$

1. Associative law:
   $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$
2. Neutral element:
   $\exists\, e \in G$ with $e \circ g = g \;\forall\, g \in G$
3. Inverse element:
   $\forall\, g \in G\; \exists\, g^{-1} \in G$ with $g^{-1} \circ g = e$

"Any set of $n \times n$ matrices, closed under multiplication and formation the inverse matrix, is a group."

Motivation and application

- Gauge group:
  - completely fixes gauge interactions
  - flavour-blind (Yukawa couplings free)
- Flavour group: determines flavour sector?
  - Lie group? Gauged?
  - Spontaneous symmetry breaking: Goldstone bosons?

Flavour group:

Discrete $\hookrightarrow$ avoid Goldstone bosons

There are no compelling argument in favour of discrete flavour groups!

Lepton mixing: until 2011 tri-bimaximal mixing and group A_4
TBM: Harrison, Perkins, Scott (2002)

$$U_{\rm PMNS} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

θ_{13} revolution:
Daya Bay, RENO exps. (2012): $\theta_{13} \neq 0$
Gonzalez-Garcia et al.: $\sin^2\theta_{13} = 0.023 \pm 0.0023$ or $\theta_{13} \simeq 9^\circ \pm 0.5^\circ$

Tri-bimaximal not a good approximation anymore!
No clue anymore for discrete flavour group!

Plan of the talk

1. General properties of discrete groups and their representations
   - Subgroups and normal subgroups
   - Semidirect products
   - Characters and character tables
2. Symmetric and alternating groups
   - A_4, S_4
3. The finite subgroups of $SU(3)$

Basic notions

Generators

Subset S of G such that every element of G can be written as a finite product of elements of S and their inverses

Presentation of a group

Set S of generators and a set R of relations among the generators

Examples:

- Cyclic group $\mathbb{Z}_n$ with one generator a and $a^n = e$
- Two generators a, b with $a^3 = b^2 = (ab)^2 = e \Rightarrow$ group with 6 elements (S_3) due to $ba = a^2 b$

General properties

Subgroup

Subset H of G which is closed under under multiplication and inverse
Proper subgroup: $H \neq \{e\}$ or G

Cosets

H subgroup
Left coset: $aH := \{ah|h \in H\}, \quad a \in G$
Right coset: $Hb := \{hb|h \in H\}, \quad b \in G$

$*$ Cosets aH, bH $\Rightarrow$ either $aH = bH$ or $aH \cap bH = \emptyset$ $*$

Order of a finite group

ord G = number of elements of G

Order of an element of G

Order of $a \in G$ is the smallest power ν such that $a^\nu = e$

Theorem (Lagrange)

* The number of elements of a subgroup is a divisor of ord G
* The order of an element of G is divisor of ord G

Proof:

a) Consider cosets $a_1 H, \ldots, a_k H \Rightarrow k \times \text{ord } H = \text{ord } G$

b) a generates $\mathbb{Z}_\nu$ Q.E.D.

General properties

Normal or invariant subgroup

N is a proper normal subgroup of G ($N \lhd G$) if $gNg^{-1} = N$ for all $g \in G$

Factor group

The cosets $gN = Ng$ of $N \lhd G$ with the multiplication rule $(aN)(bN) = (ab)N$ form a group called factor group G/N

Conjugate elements

- $a, b \in G$ are called conjugate ($a \sim b$) if there exists an element $g \in G$ such that $gag^{-1} = b$
- The equivalence relation $a \sim b$ allows to divide G into distinct "classes" C_k ($C_1 \cup \cdots \cup C_{n_c} = G$ with $C_k \cap C_l = \emptyset \, \forall \, k \neq l$)
- The class of an element $a \in G$ is defined as $C_a = \{gag^{-1} \,|\, g \in G\}$

Remarks: $C_e \equiv C_1 = \{e\}$
G Abelian $\Rightarrow$ every element is its own class

Theorem

- $N \lhd G$ iff it consists of complete conjugacy classes of G
- $N \lhd G$, $b \in G$ but $b \notin N$, C_k conjugacy class of N $\Rightarrow$ $bC_kb^{-1} = C_k$ or $bC_kb^{-1} \cap C_k = \emptyset$

Direct product

The set $G \times H$ with the multiplication law
$(g_1, h_1)(g_2, h_2) := (g_1 g_2, h_1 h_2) \quad g_1, g_2 \in G,\ h_1, h_2 \in H$ is a group
which is called the *direct product* of G and H.

Theorem (structure of Abelian groups)

* A Abelian, ord $A = p_1^{a_1} \cdots p_n^{a_n}$ (p_i distinct primes) $\Rightarrow$
  $A \cong A_1 \times \cdots \times A_n$ with ord $A_i = p_i^{a_i}$
* A' Abelian, ord $A' = p^b$ (p prime) $\Rightarrow$
  $\exists\ b_1 + \cdots + b_m = b$ with $A' \cong \mathbb{Z}_{b_1} \times \cdots \times \mathbb{Z}_{b_m}$

Example:
A Abelian with four elements $\Rightarrow$
$A = \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ (Klein four-group)
Note $\mathbb{Z}_4 \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$

Semidirect product

Automorphisms

* $\mathrm{Aut}(G)$ = group of isomorphisms $f : G \to G$
* Inner automorphisms: for every $g \in G$ $f_g(a) = gag^{-1}$

Semidirect product $G \rtimes_\phi H$

Homomorphism $\phi : H \to \mathrm{Aut}(G)$ ("H acts on G")
$G \times H$ with the multiplication law
$(g_1, h_1)(g_2, h_2) := (g_1\phi(h_1)g_2, h_1h_2)$
forms a group, the semidirect product

Note: generalization of direct product with $\phi(h) = \mathrm{id}\ \forall\, h \in H$
* $G \times \{e'\}$ is a normal subgroup, $\{e\} \times H$ a subgroup of $G \rtimes_\phi H$ *

Decomposition of a group into a semidirect product

Group S, G normal subgroup of S, H subgroup of S with following properties:

1. $G \cap H = \{e\}$,
2. every element $s \in S$ can be written as $s = gh$ with $g \in G$, $h \in H$.

Then the following holds:

- $S \cong G \rtimes_\phi H$ with $\phi(h)g = hgh^{-1}$,
- decomposition $s = gh$ is unique,
- $S/G \cong H$.

Semidirect product structure of S simply given by
$* \; (g_1 h_1)(g_1 h_1) = (g_1 h_1 g_2 h_1^{-1})(h_1 h_2) \; *$
Note: $G \lhd S$ alone does in general *not* result in a semidirect product on S.

General properties

Semidirect products are ubiquitous!

$S_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$, $A_4 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$, $S_4 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes S_3$, etc.

Discussion of $\mathbb{Z}_n \rtimes \mathbb{Z}_m$

Relations: $a^n = b^m = e$, $\phi(b)a = bab^{-1} = a^r$

Determination of r:

$b^2ab^{-2} = a^{r^2}, \quad b^3ab^{-3} = a^{r^3}, \ldots, \quad b^m ab^{-m} = a^{r^m} = a$

$\Rightarrow$ consistency relation $*$ $r^m = 1 \bmod n$ $*$

$r = 1 \Rightarrow$ direct product

In general several solutions r with $2 \leq r \leq n-1 \Rightarrow$ inequivalent semidirect products

Example: $\mathbb{Z}_3 \rtimes \mathbb{Z}_2 \Rightarrow r^2 = 1 \bmod 3 \Rightarrow$ unique solution $r = 2$

$a^3 = b^2 = e$, $bab^{-1} = a^2$ ($\Leftrightarrow (ab)^2 = e$)

a: cyclic permutation of order 3, b: transposition $\Rightarrow S_3$

General properties

Are there groups without normal subgroups?

Simple group

A group is called simple if it has no non-trivial normal subgroups

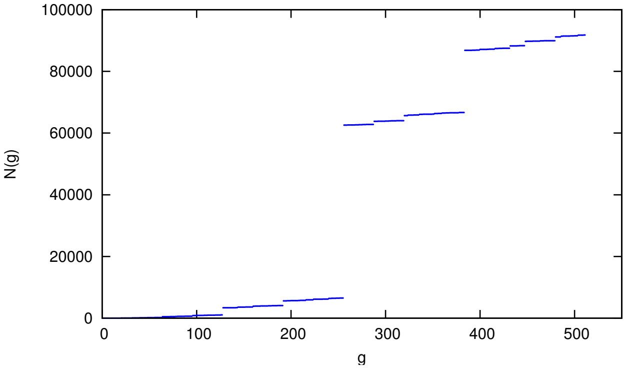
Finite Abelian simple groups: $\mathbb{Z}_p$ with p prime

Finite non-Abelian simple groups: *All* groups have been classified!
Though infinitely many, they are "rare":
Orders below 1000 are 60 (A_5), 168, 360, 504, 660.

1. Alternating groups A_n with $n \geq 5$
2. 16 series of Lie type
3. 26 sporadic groups

Order of largest sporadic group $\simeq 8 \times 10^{53}$

General properties: number of finite groups



P.O. Ludl (2010)

$g \equiv \text{ord}\, G$, $N(g)$ = number of non-Abelian groups with ord $G \leq g$

Remarks: jumps in $N(g)$ at $g = 2^8 = 256$ and $3 \times 2^7 = 384$

Jump at $g = 512$: $N(511) = 91774 \to N(512) = 10494193$

General properties

There are groups without normal subgroups, however, every group except $\mathbb{Z}_p$ with p prime has subgroups!

First theorem of Sylow

ord $G = p_1^{a_1} \cdots p_n^{a_n}$ (prime factor decomposition) $\Rightarrow$ G possesses subgroups of all orders $p_i^{s_i}$ with $0 \leq s_i \leq a_i$ $(i = 1, \ldots, n)$

General properties

Representation of group G

$\mathcal{V}$ vector space over $\mathbb{C}$
Homorphism $D : G \to \mathrm{Lin}(\mathcal{V})$ with $D(e) = \mathbb{1}$

All representations of finite groups are equivalent to unitary representations $\Rightarrow$ If D is reducible, there exists a basis such that

$$D(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix}$$

Irreducible representations (irreps) are basic building blocks of representations.

General properties

Character of a representation

The character $\chi_D : G \to \mathbb{C}$ is defined by
$\chi_D(a) := \mathrm{Tr}\, D(a), \quad a \in G$.

Properties:

- Equivalent representations have the same character
- $a \sim b \Rightarrow \chi_D(a) = \chi_D(b)$
- $\chi_D(a^{-1}) = \chi_D^*(a)$
- $\chi_{D \oplus D'}(a) = \chi_D(a) + \chi_{D'}(a)$
- $\chi_{D \otimes D'}(a) = \chi_D(a)\, \chi_{D'}(a)$.

General properties: orthogonality theorem

Bilinear form on space of functions $G \to \mathbb{C}$:

$$(f|g) = \frac{1}{\mathrm{ord}\, G} \sum_{a \in G} f(a^{-1}) g(a)$$

Real subspace of functions with property $f(a^{-1}) = f^*(a)$
$\Rightarrow$ $(\cdot|\cdot)$ scalar product on this space.

Notation: $D^{(\alpha)}$ with $\dim D^{(\alpha)} = d_\alpha$ denotes all inequivalent irreps
Schur's Lemmata $\Rightarrow$

Orthogonality theorem

$$((D^{(\alpha)})_{ij} | (D^{(\beta)})_{kl}) = \frac{1}{d_\alpha} \delta_{\alpha\beta} \delta_{il} \delta_{jk}$$

General properties: characters

Orthogonality of characters

$$\left(\chi^{(\alpha)}|\chi^{(\beta)}\right) = \delta_{\alpha\beta} \quad \Leftrightarrow \quad \sum_{k=1}^{n_c} c_k \left(\chi_k^{(\alpha)}\right)^* \chi_k^{(\beta)} = \text{ord } G\, \delta_{\alpha\beta}$$

C_k .... class of G

n_c .... number of classes

c_k .... number of elements in C_k

$\chi_k^{(\alpha)}$ .... value of $\chi^{(\alpha)}$ on C_k

General properties

From orthogonality of characters it follows:

* number of inequivalent irreps $\leq n_c =$ number of classes *

However on can show that equality holds:

Theorem

number of inequivalent irreps = number of classes

Two more very important theorems:

Theorems on the dimensions of irreps

$$\sum_\alpha d_\alpha^2 = \text{ord}\, G$$

All d_α are divisors of ord G

General properties: character table

G	C_1	C_2	$\cdots$	C_{n_c}
$(\#\ C)$	(c_1)	(c_2)	$\cdots$	(c_{n_c})
$\mathrm{ord}\,(C)$	ν_1	ν_2	$\cdots$	ν_{n_c}
$D^{(1)}$	$\chi_1^{(1)}$	$\chi_2^{(1)}$	$\cdots$	$\chi_{n_c}^{(1)}$
$D^{(2)}$	$\chi_1^{(2)}$	$\chi_2^{(2)}$	$\cdots$	$\chi_{n_c}^{(2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$D^{(n_c)}$	$\chi_1^{(n_c)}$	$\chi_2^{(n_c)}$	$\cdots$	$\chi_{n_c}^{(n_c)}$

Lines $\Rightarrow$ ON system $\quad \left(\sqrt{\frac{c_1}{\mathrm{ord}\,G}}\,\chi_1^{(\alpha)}, \ldots, \sqrt{\frac{c_{n_c}}{\mathrm{ord}\,G}}\,\chi_{n_c}^{(\alpha)}\right)$

Columns $\Rightarrow$ ON system $\quad \sqrt{\frac{c_k}{\mathrm{ord}\,G}} \begin{pmatrix} \chi_k^{(1)} \\ \vdots \\ \chi_k^{(n_c)} \end{pmatrix} \quad (k = 1, \ldots, n_c)$

General properties: reduction of representations

Characters and character tables: means of finding irreducible components of a representation D

$$D = \bigoplus_\alpha m_\alpha D^{(\alpha)} \quad \Rightarrow \quad \chi_D = \sum_\alpha m_\alpha \chi^{(\alpha)}$$

Theorem

Let D be a representation of the group G $\Rightarrow$
The multiplicity m_α with which an irrep $D^{(\alpha)}$ occurs in D is given by

$$m_\alpha = (\chi^{(\alpha)}|\chi_D)$$

$*$ $(\chi_D|\chi_D) = \sum_\alpha m_\alpha^2 \Rightarrow D$ irreducible iff $(\chi_D|\chi_D) = 1$ $*$
Application to tensor products of irreps:

$$\chi^{(\alpha\otimes\beta)}(a) = \chi^{(\alpha)}(a) \times \chi^{(\beta)}(a) \quad \Rightarrow \quad m_\gamma = (\chi^{(\gamma)}|\chi^{(\alpha)} \times \chi^{(\beta)})$$

Symmetric and alternating groups

Symmetric group S_n: Group of all permutations of n objects

$$p = \begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}, \quad \text{ord}\, S_n = n!$$

Cycle of length r: $(n_1 \to n_2 \to n_3 \to \cdots n_r \to n_1) \equiv (n_1 n_2 n_3 \cdots n_r)$
All numbers $n_1, \ldots, n_r$ are different

Theorem

Every permutation is a unique product of cycles which have no common elements

Example: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 5 & 1 & 2 \end{pmatrix} = (145)(3)(26)$

Remarks:
Cycles which have no common element commute
A cycle which consists of only one element is identical with the unit element of S_n

Symmetric and alternating groups

Even and odd permutations:

Every permutation of S_n is associated with an $n \times n$ permutation matrix $M(p)$

Even and odd permutations

Sign of a permutation: $\mathrm{sgn}(p) = \det M(p)$

A permutation p is called even (odd) if $\mathrm{sgn}(p) = +1$ (-1)

Alternating group A_n: Group of all *even* permutations of n objects

Theorem

A_n is a normal subgroup of S_n with $n!/2$ elements

$$S_n \cong A_n \rtimes \mathbb{Z}_2$$

Proof: Every element $p \in S_n \backslash A_n$ can be written as $p = st$ with $s \in A_n$ and a fixed transposition $t \in S_n$, for instance, $t = (12)$.

Symmetric and alternating groups

Classes of S_n and A_n

The classes of S_n:

Consist of the permutations with the same cycle structure

The classes of A_n:

Obtained from those of S_n in the following way:

- All classes of S_n with even permutations are also classes of A_n,
- except those which consist exclusively of cycles of unequal odd length.
- Each of the latter classes of S_n refines in A_n into two classes of equal size.

Examples:

S_4: $(a)(b)(c)(d), (a)(b)(cd), (a)(bcd), (abcd), (ab)(cd) \Rightarrow n_c = 5$

A_4: $(a)(b)(c)(d)$, $(a)(bcd) \rightarrow 2$ classes, $(ab)(cd) \Rightarrow n_c = 4$

Symmetric and alternating groups

One-dimensional irreps of S_n

S_n has exactly two 1-dimensional irreps:
$p \mapsto 1$ and $p \mapsto \text{sgn}(p)$

Discussion of S_4 and A_4

Dimensions if irreps of S_4: $1^2 + 1^2 + d_3^2 + d_4^2 + d_5^2 = 24 \Rightarrow$
$d_3 = 2,\ d_4 = d_5 = 3$

Dimensions if irreps of A_4: $1^2 + d_2^2 + d_3^2 + d_4^2 = 12 \Rightarrow$
$d_2 = d_3 = 1,\ d_4 = 3$

Structure of A_4 and S_4:

Klein's four-group: $k_1 k_2 = k_2 k_1 = k_3$ plus permutations of indices

$$K = \{e,\ (12)(34),\ (14)(23),\ (13)(24)\} \equiv \{e,\ k_1,\ k_2,\ k_3\}$$

$*$ K is a normal subgroup of A_4 and S_4 $*$

K and $s \equiv (123)$ generate A_4:

$$k_1^2 = k_2^2 = k_3^2 = e, \quad s^3 = e, \quad s k_1 s^{-1} = k_2, \quad s k_2 s^{-1} = k_3$$

K, s and $t \equiv (12)$ generate S_4:

$$t^2 = e, \quad t k_1 t^{-1} = k_1, \quad t k_2 t^{-1} = k_3, \quad t s t^{-1} = s^2$$

Symmetric and alternating groups

Theorem

Every element of $p \in S_4$ can be uniquely decomposed into $p = kq$ with $k \in K$ and q being a permutation of the numbers 1,2,3.

$$A_4 \cong K \rtimes \mathbb{Z}_3 \quad \text{and} \quad S_4 \cong K \rtimes S_3$$

Note: $\{e\} \lhd K \lhd A_4 \lhd S_4$
Every kernel of a non-faithful irrep is a normal subgroup $\Rightarrow$
In non-faithful irrep of A_4 and S_4 always $K \mapsto \mathbb{1}$

Side remark: $*$ Simple groups have only faithful non-trivial irreps $*$

Irreps of A_4:

One-dimensional irreps:

$\mathbf{1}^{(p)}$: $k_i \mapsto 1,\ s \mapsto \omega^p$ $(p = 0, 1, 2)$ with $\omega = e^{2\pi i/3}$

Three-dimensional irrep: K represented as diagonal matrices

$$\mathbf{3}: \quad k_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} =: A, \quad s \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} =: E$$

$\Rightarrow$

$k_2 = s k_1 s^{-1} \mapsto \mathrm{diag}\,(-1, -1, 1),\ k_3 = s k_2 s^{-1} \mapsto \mathrm{diag}\,(-1, 1, -1)$

Irreps of S_4:

$$\mathbf{1}: \quad p \mapsto 1$$

$$\mathbf{1}': \quad p \mapsto \mathrm{sign}(p)$$

$$\mathbf{3}: \quad k_1 \mapsto A, \quad s \mapsto E, \quad t \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} =: R_t$$

$$\mathbf{3}': \quad k_1 \mapsto A, \quad s \mapsto E, \quad t \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{2}: \quad k_i \mapsto 1, \quad s \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Remarks: $\mathbf{2}$ is irrep of $S_3 \cong S_4/K$, $\mathbf{3}' = \mathbf{1}' \otimes \mathbf{3}$

Character table of A_4:

$T \cong A_4$	$C_1(e)$	$C_2(s)$	$C_3(s^2)$	$C_4(k_1)$
$(\# \ C)$	(1)	(4)	(4)	(3)
$\mathrm{ord}\,(C)$	1	3	3	2
$\mathbf{1}^{(0)}$	1	1	1	1
$\mathbf{1}^{(1)}$	1	ω	ω^2	1
$\mathbf{1}^{(2)}$	1	ω^2	ω	1
$\mathbf{3}$	3	0	0	-1

Character table of S_4: $r := s^{-1} k_1 s t = (1423)$

$O \cong S_4$ $(\# \ C)$ $\mathrm{ord}\,(C)$	$C_1(e)$ (1) 1	$C_2(t)$ (6) 2	$C_3(k_1)$ (3) 2	$C_4(s)$ (8) 3	$C_5(r)$ (6) 4
$\mathbf{1}$	1	1	1	1	1
$\mathbf{1}'$	1	-1	1	1	-1
$\mathbf{2}$	2	0	2	-1	0
$\mathbf{3}$	3	-1	-1	0	1
$\mathbf{3}'$	3	1	-1	0	-1

Symmetric and alternating groups

Remark:
Of the non-trivial symmetric and alternating groups, only
S_3, A_4, S_4, A_5
can be considered as finite subgroups of $SO(3)$,
i.e. possess a faithful representation by 3×3 rotation matrices.

- $S_3 \cong S_\triangle =$ symmetry group of unilateral triangle
- $A_4 \cong T =$ symmetry group of tetrahedron
- $S_4 \cong O =$ symmetry group of octahedron
- $A_5 \cong I =$ symmetry group of icosahedron

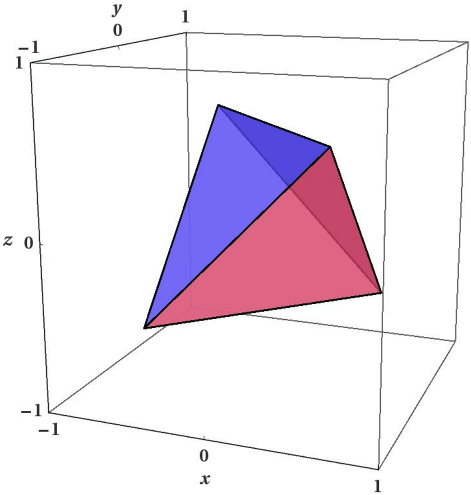
Symmetric and alternating groups

Classes of rotation groups: $R_2\, R(\alpha, \vec{n})\, R_2^{-1} = R(\alpha, R_2 \vec{n}) \Rightarrow$ rotations through the same angle about equivalent axes are equivalent

Classes of the tetrahedral group:

- The class of the identity element
  $C_1 = \{e\}$,
- the class of rotations through 120° about the four three-fold axes
  $C_2 = \{a_1,\, a_2,\, a_3,\, a_4\}$,
- the class of rotations through 240° about the four three-fold axes
  $C_3 = \{a_1^2,\, a_2^2,\, a_3^2,\, a_4^2\}$,
- the class of rotations about the three two-fold axes
  $C_4 = \{b_1,\, b_2,\, b_3\}$.

Symmetric and alternating groups



Symmetric and alternating groups

Generators of 3-dimensional irrep of A_4:

$$R(2\pi/3, \vec{n}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \equiv E \quad \text{with} \quad \vec{n} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$R(\pi, \vec{e}_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \equiv A$$

Vertices of the tetrahedron:

Set of four points which is invariant under E and A

$$\frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

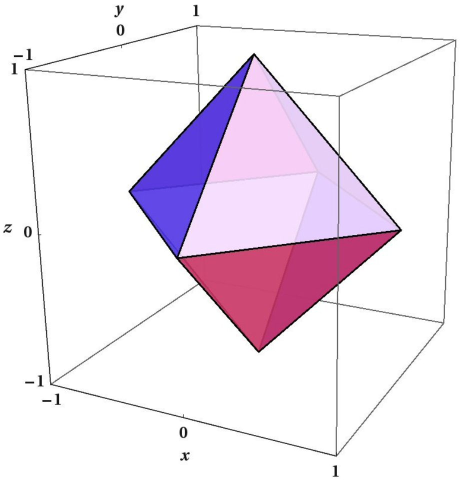
Symmetric and alternating groups

Symmetry axes of the symmetry group of the octahedron, O:

- Type 1: Three axes connecting two opposite vertices $\Rightarrow$ six rotations through $\pm 90^\circ$, three rotations through 180°
- Type 2: Six axes passing through the centers of two opposite edges $\Rightarrow$ six rotations through 180°
- Type 3: Four axes passing through the centers of two opposite faces $\Rightarrow$ eight rotations through 120°

Classes of the octahedron:

axis type	0	2	1	3	1
$(\# C)$	(1)	(6)	(3)	(8)	(6)
$\mathrm{ord}\, C$	1	2	2	3	4
element of S_4	e	t	k_1	s	r


y
1
0
−1
1
z
0
−1
−1
0
x
1

Generators of **3** of S_4:

$$s = (123) \mapsto E, \quad k_1 = (12)(34) \mapsto A, \quad t = (12) \mapsto R_t$$

$$R_t = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Set of vertices of the octahedron, invariant under E, A, R_t:
$\pm\vec{e}_x$, $\pm\vec{e}_y$, $\pm\vec{e}_z$

The finite subgroups of $SU(3)$

H.F. Blichfeldt (1916)[1]:

Classification of the finite subgroups of SU(3) into five types:

(A) Abelian groups.

(B) Finite subgroups of $SU(3)$ with faithful 2-dimensional representations.

(C) The groups $C(n,a,b)$ generated by the matrices

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad F(n,a,b) = \mathrm{diag}(\eta^a, \eta^b, \eta^{-a-b}),$$

where $\eta = \exp(2\pi i/n)$.

[1] G.A. Miller, H.F. Blichfeldt and L.E. Dickson: Theory and applications of finite groups, New York (1916)

(D) The groups $D(n, a, b; d, r, s)$ generated by E, $F(n, a, b)$ and

$$\widetilde{G}(d, r, s) = \begin{pmatrix} \delta^r & 0 & 0 \\ 0 & 0 & \delta^s \\ 0 & -\delta^{-r-s} & 0 \end{pmatrix},$$

where $\delta = \exp(2\pi i/d)$.

(E) Six exceptional finite subgroups of $SU(3)$:

- $\Sigma(60) \cong A_5$, $\Sigma(168) \cong PSL(2, 7)$
- $\Sigma(36 \times 3)$, $\Sigma(72 \times 3)$, $\Sigma(216 \times 3)$ and $\Sigma(360 \times 3)$,

as well as the direct products $\Sigma(60) \times \mathbb{Z}_3$ and $\Sigma(168) \times \mathbb{Z}_3$.

The finite subgroups of $SU(3)$: (A) Abelian groups

Simple (but powerful) theorem: P.O. Ludl (2011)

Abelian finite subgroups of SU(3)

Every finite Abelian subgroup $\mathcal{A}$ of SU(3) is isomorphic to

$$\mathbb{Z}_m \times \mathbb{Z}_n,$$

where

$$m = \max_{a \in \mathcal{A}} \mathrm{ord}(a)$$

and n is a divisor of m.

$\Rightarrow$ Possible structures of Abelian finite subgroups of SU(3) are strongly restricted!

Examples:

- Rotations about one axis (cyclic groups $\mathbb{Z}_m$)
- Klein's four group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Finite subgroups of $SO(3)$

1. The uniaxial groups: $\mathbb{Z}_n$
2. The dihedral groups D_n ($\mathrm{ord}\, D_n = 2n$)
3. The rotation groups of the Platonic solids: T, O, I
   symmetry group of cube $\cong O$
   symmetry group of dedecahedron $\cong I$

The finite subgroups of $SU(3)$: (B) Groups with two-dimensional faithful representations

Every finite subgroup of $SU(2)$ can be conceived as a finite subgroup of $SU(3)$:

$$A \in SU(2) \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in SU(3)$$

Even true for the finite subgroups of $U(2)$:

$$A \in U(2) \Rightarrow \begin{pmatrix} \det A^* & 0 \\ 0 & A \end{pmatrix} \in SU(3)$$

Examples:

- Dihedral groups D_n (finite subgroups of $SO(3)$).
- Double covers of the finite 3-dimensional rotation groups ($\widetilde{T}$, $\widetilde{O}$, $\widetilde{I}$, $\widetilde{D_n}$).

The finite subgroups of $SU(3)$: groups of type (C)

Generated by

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad F(n,a,b) = \mathrm{diag}(\eta^a, \eta^b, \eta^{-a-b}),$$

where $\eta = \exp(2\pi i/n)$.

Structure: $F(n,a,b)$ diagonal $\Rightarrow EF(n,a,b)E^{-1}$ also diagonal.

$\Rightarrow$ Subgroup $N(n,a,b)$ of diagonal matrices is a normal subgroup.

$$\Rightarrow C(n,a,b) \cong N(n,a,b) \rtimes \mathbb{Z}_3.$$

We also know that $N(n,a,b)$ is an Abelian finite subgroup of $SU(3)$, thus

$$C(n,a,b) \cong (\mathbb{Z}_m \times \mathbb{Z}_p) \rtimes \mathbb{Z}_3.$$

The finite subgroups of $SU(3)$: groups of type (C)

$$C(n,a,b) \cong (\mathbb{Z}_m \times \mathbb{Z}_p) \rtimes \mathbb{Z}_3.$$

Note: $n, a, b \Rightarrow m, p$ in a complicated way, p divisor of m

Special cases:

- $p = 1 \Rightarrow$ Groups of the type $T_m \cong \mathbb{Z}_m \rtimes \mathbb{Z}_3$ where m is a product of powers of primes of the form $6k + 1$.
- $p = m \Rightarrow$ Groups of the type $(\mathbb{Z}_m \times \mathbb{Z}_m) \rtimes \mathbb{Z}_3 \cong \Delta(3m^2)$.

Examples:

- Well-known groups such as $A_4 \cong T \cong \Delta(12)$, $\Delta(27)$, T_7, T_{13}.
- Smallest group of type (C) which is neither of the form T_n nor of the form $\Delta(3n^2)$:

$$C(9,1,1) \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3.$$

Note: dimensions of irreps can only be 1 and 3 (Grimus, Ludl (2011)).

The finite subgroups of $SU(3)$: groups of type (D)

The group $D(n, a, b; d, r, s)$ is generated by the generators of $C(n, a, b)$ and

$$\widetilde{G}(d, r, s) = \begin{pmatrix} \delta^r & 0 & 0 \\ 0 & 0 & \delta^s \\ 0 & -\delta^{-r-s} & 0 \end{pmatrix},$$

where $\delta = \exp(2\pi i/d)$.

By means of a unitary transformation one obtains a different set of generators (Grimus, Ludl (2011), review):

- Three diagonal matrices,
- and the two S_3-generators

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

$$\Rightarrow D(n,a,b;d,r,s) \cong N(n,a,b;d,r,s) \rtimes S_3$$
$$\Rightarrow D(n,a,b;d,r,s) \cong (\mathbb{Z}_m \times \mathbb{Z}_{m'}) \rtimes S_3.$$

Special cases:

- $m = m' \Rightarrow$ Groups of the type $(\mathbb{Z}_m \times \mathbb{Z}_m) \rtimes S_3 \cong \Delta(6m^2)$.

Examples:

- Well-known groups such as $S_4 \cong \Delta(24)$, $\Delta(54)$.
- Smallest group of type (D) which is neither a direct product nor of the form $\Delta(6n^2)$:

$$D(9,1,1;2,1,1) \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes S_3.$$

Note: dimensions of irreps can only be 1, 2, 3 and 6 (Grimus, Ludl (2011)).

Summary: "number theorems" of finite groups

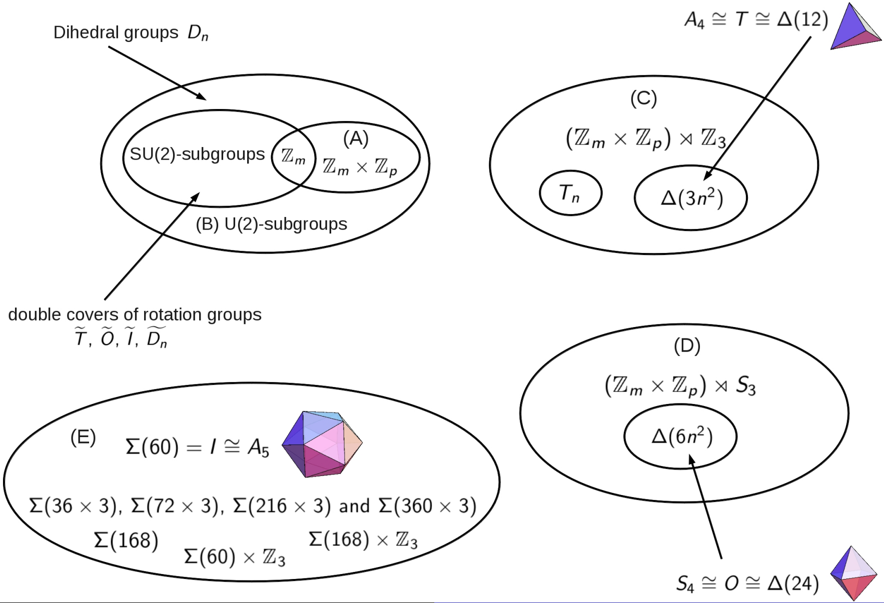
Divisors of ord G:

- order of a subgroup
- order of an element
- number of elements in a class
- dimension of an irrep

number if inequivalent irreps = number of classes

$$\text{Irreps } D^{(\alpha)} \text{ with } \dim D^{(\alpha)} = d_\alpha \quad \Rightarrow \quad \sum_\alpha d_\alpha^2 = \text{ord } G$$

Summary: The finite subgroups of SU(3)



Thank you for your attention!