Dispersive treatment of $\eta \rightarrow 3\pi$ and the determination of light quark masses

Gilberto Colangelo

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Outline

Introduction

Lattice determination of $m_u$, $m_d$ and $m_s$
  - FLAG
  - FLAG phase 2
  - Isospin limit
  - Isospin breaking

Quark mass ratios from CHPT

A new dispersive analysis of $\eta \rightarrow 3\pi$
  - Isospin breaking

Summary and Outlook
Quark masses

QCD Lagrangian:

\[ \mathcal{L}_{\text{QCD}} = -\frac{1}{4g^2} \text{Tr} G_{\mu\nu} G^{\mu\nu} + \sum_i \bar{q}_i (i\cancel{D} - m_q) q_i + \sum_j \bar{Q}_j (i\cancel{D} - m_Q) Q_j \]

- In the limit \( m_{q_i} \to 0 \) and \( m_{Q_j} \to \infty \):
  \[ M_{\text{hadrons}} \propto \Lambda \]

- Observe that \( m_{q_i} \ll \Lambda \) while \( m_{Q_j} \gg \Lambda \)
  \[ [\Lambda \sim M_N] \]

- Quarks do not propagate: quark masses are coupling constants! (not observables)
  they depend on the renormalization scale \( \mu \) (like \( \alpha_s \))
  for light quarks by convention: \( \mu = 2 \text{ GeV} \)

In the following

\[ m_q \equiv m_q(2 \text{ GeV}) \]
How to determine quark masses

- From their influence on the spectrum
  - \( m_Q \gg \Lambda \)
    \[ M_{\bar{Q}q_i} = m_Q + \mathcal{O}(\Lambda) \]
  - \( m_q \ll \Lambda \)
    \[ M_{\bar{q}_i q_j} = M_{0\ ij} + \mathcal{O}(m_{q_i}, m_{q_j}) \quad M_{0\ ij} = \mathcal{O}(\Lambda) \]

In both cases need to understand the \( \mathcal{O}(\Lambda) \) term
How to determine quark masses

- From their influence on the spectrum
  - \( m_Q \gg \Lambda \)
    \[ M_{\bar{Q}q_i} = m_Q + O(\Lambda) \]
  - \( m_q \ll \Lambda \)
    \[ M_{\bar{q}_i q_j} = M_{0\ ij} + O(m_{q_i}, m_{q_j}) \quad M_{0\ ij} = O(\Lambda) \]

  In both cases need to understand the \( O(\Lambda) \) term

- From their influence on any other observable

  Quark masses are coupling constants
  \[ \Rightarrow \text{exploit the sensitivity to them of any observable} \]
  \[ \text{[e.g. } \eta \text{ and } \tau \text{ decays]} \]
How to determine quark masses

Approaches

1. Chiral perturbation theory
   spectrum + low energy observables

2. Sum rules
   spectral functions

3. Lattice QCD
   spectrum
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Summary and Outlook
Lattice determinations of quark masses

First principle (and brute force) method:

- QCD Lagrangian as input
- calculate the spectrum of the low-lying states for different quark masses
- tune the values of the quark masses such that the QCD spectrum is reproduced
- the comparison is made for mass ratios — one mass or a dimensionful quantity (e.g. $F_\pi$) can be used to set the scale
Lattice determinations of quark masses

Systematic effects:

- finite volume
  exponentially suppressed effects, easy to correct for
- finite lattice spacing
  powerlike effects, extrapolate numerically
- unphysical quark masses
  extrapolate numerically with guidance from CHPT
  BMW and PACS-CS: simulations at the physical point!
- renormalization
  relation between bare (input parameter) and renormalized
  (physically relevant quantity) quark masses not easy
What/Who is FLAG (phase 1)?

**FLAG = FLAVIAnet Lattice Averaging Group**

**Members:**
Gilberto Colangelo (Bern)
Stephan Dürr (Jülich, BMW)
Andreas Jüttner (Southampton, RBC/UKQCD)
Laurent Lellouch (Marseille, BMW)
Heiri Leutwyler (Bern)
Vittorio Lubicz (Rome 3, ETM)
Silvia Necco (CERN, Alpha)
Chris Sachrajda (Southampton, RBC/UKQCD)
Silvano Simula (Rome 3, ETM)
Tassos Vladikas (Rome 2, Alpha and ETM)
Urs Wenger (Bern, ETM)
Hartmut Wittig (Mainz, Alpha)
What/Who is FLAG (phase 1)?

FLAG = FLAVIAnet Lattice Averaging Group

History and status:

- Beginning: FLAVIAnet meeting, Orsay, November 2007
- Start of the actual work: Bern, March 2008
- ...
- webpage made public in 2011: http://itpwiki.unibe.ch/flag
What exactly did FLAG-1 offer?

An answer to the questions

- what is the current lattice value for quantity $X$?
- what is a reliable estimate of the uncertainty?

in a way easily accessible to non-experts

Quantities considered in the first edition:

- light quark masses
- LEC
- decay constants (of pions and kaons)
- form factors (of pions and kaons)
- $B_K$
Color coding — FLAG-1 definition

- chiral extrapolation
  - ★ $M_{\pi,\text{min}} < 250$ MeV
  - ● $250$ MeV $\leq M_{\pi,\text{min}} \leq 400$ MeV
  - ■ $M_{\pi,\text{min}} > 400$ MeV
Color coding – FLAG-1 definition

- **chiral extrapolation**
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- **continuum extrapolation**
  - ★ 3 or more lattice spacings, at least 2 points below 0.1 fm
  - ● 2 or more lattice spacings, at least 1 point below 0.1 fm
  - ■ otherwise
Color coding — FLAG-1 definition

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  - ■ otherwise

- **finite volume effects**
  - ★ $(M_{\pi L})_{\text{min}} > 4$ or at least 3 volumes
  - ● $(M_{\pi L})_{\text{min}} > 3$ and at least 2 volumes
  - ■ otherwise
Color coding — FLAG-1 definition

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  - ■ otherwise

- renormalization (where applicable)
  - ★ non-perturbative
  - ● 2-loop perturbation theory (well behaved series)
  - ■ otherwise

[subject to change before each new edition]
Averages

Different lattice results were averaged *if*

- published
  - [lattice proceedings not enough]
- no red tags
- same $N_f$
  - [no average of $N_f = 2$ and $N_f = 3$ calculations]

Final FLAG number:

- average or single *no-red-tag* $N_f = 3$ number (if available)
- average or single *no-red-tag* $N_f = 2$ number (if available)

If *both* $N_f = 3$ and $N_f = 2$ numbers available:

  *agreement* $\Rightarrow$ more confidence in the final number
Similar initiative: Laiho, Lunghi and Van de Water

- began in 2009 to provide lattice-QCD inputs for the CKM unitarity-triangle analysis and other flavor-physics phenomenology

- main differences wrt FLAG-1:
  - only include $N_f = 2 + 1$ flavor results
  - no strict publication-only rule provided complete and reasonable systematic error budgets
  - heavy-quark quantities included from the start
  - unitarity triangle fits with lattice input
  - whenever a source of error is at all correlated between two lattice calculations (e.g. use the same gauge configurations, same theoretical tools, or experimental inputs), conservatively assume that the degree-of-correlation is 100%

- web page (www.latticeaverages.org) also popular

FLAG = Flavour Lattice Averaging Group
has now entered its phase 2 and has been extended in various directions

- quantities to be reviewed
  main extension: light quarks $\rightarrow$ + heavy quarks

- represented lattice collaborations:
  Alpha, BMW, ETMC, RBC/UKQCD $\rightarrow$ + CLS, Fermilab, HPQCD, JLQCD, MILC, PACS-CS, SWME

- represented world regions: Europe $\rightarrow$ + Japan and US

- number of people: 12 $\rightarrow$ 28
FLAG-2 organization

- **Advisory Board:**
  - S. Aoki, C. Bernard, C. Sachrajda

- **Editorial Board:**
  - GC, H. Leutwyler, T. Vladikas, U. Wenger

- **Working Groups**
  - **Quark masses**
    - L. Lellouch, T. Blum, V. Lubicz
  - **$V_{us}, V_{ud}$**
    - A. Jüttner, T. Kaneko, S. Simula
  - **LEC**
    - S. Dürr, H. Fukaya, S. Necco
  - **$B_K$**
    - H. Wittig, J. Laiho, S. Sharpe
  - **$\alpha_s$**
    - R. Sommer, R. Horsley, T. Onogi
  - **$f_B, B_B$**
    - A. El Khadra, Y. Aoki, M. Della Morte
  - **$B \to H\ell\nu$**
    - R. Van de Water, E. Lunghi, C. Pena, J. Shigemitsu
FLAG-2 plans and rules

- next review: work in progress (first half 2013)
- regularly update the webpage
- new published review: every 2nd year

Some internal FLAG rules

- members of the advisory board have a 4-year mandate
- \( AB = EU+J+US \)
- regular members can stay longer
- replacements must keep/improve the balance of FLAG
- WG members belong to 3 different lattice coll.
- a paper is not reviewed (color-coded) by an author
FLAG-2 status

- kick-off meeting: Les Houches, May 7-11 2012
- work on the update of the review is in progress
- WG for the new sections are working on defining their own quality criteria
- first draft (internal) of the new review: September 2012
- publication of the new review: early 2013
- it will cover all results published until December 31 2012
FLAG-2 status

Two important policy changes wrt FLAG-1

- use of one-loop renormalization will not necessarily mean a red tag

- error calculation for the averages: LLVdW procedure will be adopted as default (final error may be stretched if not convincingly conservative)
## Quark masses

<table>
<thead>
<tr>
<th>Collaboration</th>
<th>publ.</th>
<th>$m_{ud}$</th>
<th>$m_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PACS-CS 12</td>
<td>P</td>
<td>3.12(24)(8)</td>
<td>83.60(0.58)(2.23)</td>
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<tr>
<td>RBC/UKQCD 12</td>
<td>C</td>
<td>3.39(9)(4)(2)(7)</td>
<td>94.2(1.9)(1.0)(0.4)(2.1)</td>
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<tr>
<td>LVdW 11</td>
<td>C</td>
<td>3.31(7)(20)(17)</td>
<td>94.2(1.4)(3.2)(4.7)</td>
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<td>2.78(27)</td>
<td>86.7(2.3)</td>
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<tr>
<td>MILC 10A</td>
<td>C</td>
<td>3.19(4)(5)(16)</td>
<td>–</td>
</tr>
<tr>
<td>HPQCD 10</td>
<td>A</td>
<td>3.39(6)</td>
<td>92.2(1.3)</td>
</tr>
<tr>
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<td>3.46(47)(48)</td>
<td>95.5(1.1)(1.5)</td>
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<tr>
<td>RBC/UKQCD 10A</td>
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<td>Blum 10</td>
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<td>PACS-CS 09</td>
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<td>92.75(58)(95)</td>
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<td>3.40(7)</td>
<td>92.4(1.5)</td>
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<td>3.25(1)(7)(16)(0)</td>
<td>89.0(0.2)(1.6)(4.5)(0.1)</td>
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<td>PACS-CS 08</td>
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<td>2.527(47)</td>
<td>72.72(78)</td>
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<td>3.72(16)(33)(18)</td>
<td>107.3(4.4)(9.7)(4.9)</td>
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<tr>
<td>CP-PACS/JLQCD 07</td>
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<td>3.55(19)(56$^{+}_20$)</td>
<td>90.1(4.3)(16.7$^{-}_4.3$)</td>
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<tr>
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$$N_f = 2 + 1$$
## Quark masses

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<th>Collaboration</th>
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<th>renorm.</th>
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<th>$m_s$</th>
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<td>⬪</td>
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<td>⬪</td>
<td>⬪</td>
<td>−</td>
<td>3.52(10)(9)</td>
<td>97.0(2.6)(2.5)</td>
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<tr>
<td>ETM 10B</td>
<td>A</td>
<td>⬤</td>
<td>⬪</td>
<td>⬪</td>
<td>⬪</td>
<td>3.6(1)(2)</td>
<td>95(2)(6)</td>
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<td>JLQCD/TWQCD 08A</td>
<td>A</td>
<td>⬤</td>
<td>⬪</td>
<td>⬪</td>
<td>⬪</td>
<td>4.452(81)(38)(0)</td>
<td>−</td>
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<tr>
<td>RBC 07†</td>
<td>A</td>
<td>⬤</td>
<td>⬪</td>
<td>⬪</td>
<td>⬪</td>
<td>4.25(23)(26)</td>
<td>119.5(5.6)(7.4)</td>
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<tr>
<td>ETM 07</td>
<td>A</td>
<td>⬤</td>
<td>⬪</td>
<td>⬪</td>
<td>⬪</td>
<td>3.85(12)(40)</td>
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<td>⬤</td>
<td>⬪</td>
<td>⬪</td>
<td>⬪</td>
<td>4.08(23)(19)(23)</td>
<td>111(6)(4)(6)</td>
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<td>SPQcdR 05</td>
<td>A</td>
<td>⬤</td>
<td>⬪</td>
<td>⬪</td>
<td>⬪</td>
<td>4.3(4)(+1.1, -0.0)</td>
<td>101(8)(+25, -0)</td>
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<td>ALPHA 05</td>
<td>A</td>
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<td>⬪</td>
<td>⬪</td>
<td>⬪</td>
<td>−</td>
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<td>⬪</td>
<td>⬪</td>
<td>⬪</td>
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<td>119(5)(8)</td>
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<td>A</td>
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<td>⬪</td>
<td>⬪</td>
<td>⬪</td>
<td>3.223(+46, -69)</td>
<td>84.5(+12.0, -1.7)</td>
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<td>CP-PACS 01</td>
<td>A</td>
<td>⬤</td>
<td>⬪</td>
<td>⬪</td>
<td>⬪</td>
<td>3.45(10)(+11, -18)</td>
<td>89(2)(+2, -6)*</td>
</tr>
</tbody>
</table>

$N_f = 2$
# Quark masses

<table>
<thead>
<tr>
<th>Collaboration</th>
<th>$N_f$</th>
<th>$p_{ubl.}$</th>
<th>$a\rightarrow 0$</th>
<th>$FV$</th>
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<tr>
<td>PACS-CS 12</td>
<td>2+1</td>
<td>P</td>
<td>★</td>
<td>■</td>
<td>26.8(2.0)</td>
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<tr>
<td>LVdW 11</td>
<td>2+1</td>
<td>C</td>
<td>★</td>
<td>★</td>
<td>28.4(0.5)(1.3)</td>
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<tr>
<td>BMW 10A, 10B</td>
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<td>A</td>
<td>★</td>
<td>★</td>
<td>27.53(20)(8)</td>
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<tr>
<td>RBC/UKQCD 10A</td>
<td>2+1</td>
<td>A</td>
<td>★</td>
<td>★</td>
<td>26.8(0.8)(1.1)</td>
</tr>
<tr>
<td>Blum 10</td>
<td>2+1</td>
<td>A</td>
<td>★</td>
<td>★</td>
<td>28.31(0.29)(1.77)</td>
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<td>★</td>
<td>■</td>
<td>31.2(2.7)</td>
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<tr>
<td>MILC 09A</td>
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<td>C</td>
<td>★</td>
<td>★</td>
<td>27.41(5)(22)(0)(4)</td>
</tr>
<tr>
<td>MILC 09</td>
<td>2+1</td>
<td>A</td>
<td>★</td>
<td>★</td>
<td>27.2(1)(3)(0)(0)</td>
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<tr>
<td>PACS-CS 08</td>
<td>2+1</td>
<td>A</td>
<td>★</td>
<td>■</td>
<td>28.8(4)</td>
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<td>2+1</td>
<td>A</td>
<td>★</td>
<td>■</td>
<td>28.8(0.4)(1.6)</td>
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<td>MILC 04, HPQCD/</td>
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<td>A</td>
<td>★</td>
<td>●</td>
<td>27.4(1)(4)(0)(1)</td>
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<td>A</td>
<td>★</td>
<td>●</td>
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<td>ETM 10B</td>
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<td>A</td>
<td>★</td>
<td>●</td>
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<td>RBC 07</td>
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<td>●</td>
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<td>QCDSF/UKQCD 06</td>
<td>2</td>
<td>A</td>
<td>★</td>
<td>●</td>
<td>27.2(3.2)</td>
</tr>
</tbody>
</table>
Quark masses

\[ m_{ud} \]

**$N_f = 2 + 1$**

- PACS-CS 12
- RBC/UKQCD 12
- LVdW 11
- HPQCD 10
- PACS-CS 10
- MILC 10A
- BMW 10A
- RBC/UKQCD 10A
- Blum 10
- PACS-CS 09
- HPQCD 09
- MILC 09A
- MILC 09
- PACS-CS 08
- RBC/UKQCD 08
- CP-PACS/JLQCD 07
- HPQCD 05
- MILC 04, HPQCD/MILC/UKQCD 04

Our estimate for $N_f = 2 + 1$

**$N_f = 2$**

- ETM 10B
- JLQCD/TWQCD 08A
- RBC 07
- ETM 07
- QCDSF/UKQCD 06
- SPQcdR 05
- QCDSF/UKQCD 04
- JLQCD 02
- CP-PACS 01

Our estimate for $N_f = 2$
Quark masses

$m_s$ [MeV]
FLAG estimates of the light quark masses

\( N_f = 2 + 1: \)

\[
\begin{align*}
  m_s &= 94 \pm 3 \text{ MeV} \\
  m_{ud} &= 3.43 \pm 0.11 \text{ MeV} \\
  \frac{m_s}{m_{ud}} &= 27.4 \pm 0.4
\end{align*}
\]

\( N_f = 2: \)

\[
\begin{align*}
  m_s &= 95 \pm 2 \pm 6 \text{ MeV} \\
  m_{ud} &= 3.6 \pm 0.1 \pm 0.2 \text{ MeV} \\
  \frac{m_s}{m_{ud}} &= 27.3 \pm 0.5 \pm 0.7
\end{align*}
\]

PDG 2010: \( m_s = 101^{+29}_{-21} \) \( m_{ud} = 3.77^{+1.03}_{-0.77} \) \( \frac{m_s}{m_{ud}} = 26 \pm 4 \)
Isospin breaking on the lattice

- all lattice results described so far: isospin limit
  \[ m_u = m_d \quad \text{and} \quad \alpha_{em} = 0 \]

- need as input \( M_\pi \) and \( M_K \) in this limit
  at the current level of precision
  the difference \( M_P - M_P(\alpha_{em} = 0) \) matters!

Remark: \( M_{\pi^+}(\alpha_{em} = 0) \approx M_{\pi^0}(\alpha_{em} = 0) \approx M_{\pi^0} \)
\[ \Rightarrow \text{need to know } (M_{K^+} - M_{K^0})_{em} \]

- alternatively:
  simulate QCD+QED on the lattice and compare to the real-world spectrum
Electromagnetic effects in lattice calculations

Lattice calculations with QCD+QED

- Duncan, Eichten, Thacker (96)
  QCD quenched approximation, estimate for meson mass em splittings; systematic error not estimated
  \((M_{K^+} - M_{K^0})_{em} = 1.9\ \text{MeV}\)

- Blum et al. (RBC 07)
  QCD with 2 dynamical quarks, but QED quenched
  \((M_{K^+} - M_{K^0})_{em} = 1.443(55)\ \text{MeV}\)

- Blum et al. (10)
  QCD with 3 dynamical quarks (RBC/UKQCD configurations), but QED quenched
  \((M_{K^+} - M_{K^0})_{em} = 1.87(10)\ \text{MeV}\)
Mixed approach

1. take as an input the phenomenological estimates of 
   \((M_{K^+} - M_{K^0})_{em}\)
2. use the kaon masses to determine \(m_u/m_d\)

MILC adopts the value

\[(M_{K^+} - M_{K^0})_{em} = 2.8(6) \text{ MeV}\]

and obtains

\[
\frac{m_u}{m_d} = 0.432(1)(9)(0)(39)
\]

Bijnens-Prades 97, Donoghue-Perez 97
Quark masses from lattice + $Q$ from $\eta \to 3\pi$

Alternative approach to determine $m_u$, $m_d$ and $m_s$:

- determine $m_{ud}$ and $m_s$ on the lattice in the isospin limit
- determine $Q$ from $\eta \to 3\pi$
- combine the two pieces of information

Example: BMW 10

$$m_s = 95.5(1.1)(1.5) \text{ MeV} \quad m_{ud} = 3.469(47)(48) \text{ MeV}$$

Add $Q$ from $\eta \to 3\pi$:

$$m_u = 2.15(03)(10) \text{ MeV} \quad m_d = 4.79(07)(12) \text{ MeV}$$
Quark masses from lattice \( +Q \) from \( \eta \rightarrow 3\pi \)

![Graph showing quark masses from lattice calculations.](attachment:image.png)

- **Legend:**
  - $m_s/m_d$ values from various sources:
    - MILC 09
    - PACS-CS 08
    - RBC/UKQCD 08
    - PDG 08
    - RBC 07
    - Bijnens & Ghorbani 07
    - Namekawa & Kikukawa 06
    - MILC 04
    - Nelson, Fleming & Kilcup 03
    - Gao, Yan & Li 97
    - Kaiser 97
    - Leutwyler 96
    - Schechter et al. 93
    - Donoghue, Holstein & Wyler 92
    - Gerard 90
    - Cline 89
    - Gasser and Leutwyler 82
    - Langacker & Pagels 79
    - Weinberg 77
    - Gasser & Leutwyler 75

- **Notes:**
  - $\chi$PT fails at higher values of $m_s/m_d$.
  - $\chi$PT must be reordered.

- **Citations:**
  - Cline 89
  - Gasser & Leutwyler 82
  - Langacker & Pagels 79
  - Weinberg 77
  - Gasser & Leutwyler 75
  - Leutwyler PoS, CD09
## Lattice determinations of $m_u$ and $m_d$

<table>
<thead>
<tr>
<th>Collaboration</th>
<th>publ.</th>
<th>$m_u$</th>
<th>$m_d$</th>
</tr>
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<tbody>
<tr>
<td>PACS-CS 12</td>
<td>A</td>
<td>2.57(26)(7)</td>
<td>3.68(29)(10)</td>
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<tr>
<td>LVdW 11</td>
<td>C</td>
<td>1.90(8)(21)(10)</td>
<td>4.73(9)(27)(24)</td>
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<td>HPQCD 10</td>
<td>A</td>
<td>2.01(14)</td>
<td>4.77(15)</td>
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<tr>
<td>BMW 10A, 10B</td>
<td>A</td>
<td>2.15(03)(10)</td>
<td>4.79(07)(12)</td>
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<tr>
<td>Blum et al. 10</td>
<td>P</td>
<td>2.24(10)(34)</td>
<td>4.65(15)(32)</td>
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<tr>
<td>MILC 09A</td>
<td>C</td>
<td>1.96(0)(6)(10)(12)</td>
<td>4.53(1)(8)(23)(12)</td>
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<tr>
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<td>1.9(0)(1)(1)(1)</td>
<td>4.6(0)(2)(2)(1)</td>
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<tr>
<td>MILC 04, HPQCD/MILC/UKQCD 04</td>
<td>A</td>
<td>1.7(0)(1)(2)(2)</td>
<td>3.9(0)(1)(4)(2)</td>
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<tr>
<td>RM123 11</td>
<td>A</td>
<td>2.43(20)(12)</td>
<td>4.78(20)(12)</td>
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<tr>
<td>Dürr 11</td>
<td>A</td>
<td>2.18(6)(11)</td>
<td>4.87(14)(16)</td>
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<tr>
<td>RBC 07</td>
<td>A</td>
<td>3.02(27)(19)</td>
<td>5.49(20)(34)</td>
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### FLAG-1 summary of the quark masses

<table>
<thead>
<tr>
<th>$N_f$</th>
<th>$m_u$</th>
<th>$m_d$</th>
<th>$m_s$</th>
<th>$m_{ud}$</th>
</tr>
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<tbody>
<tr>
<td>2+1</td>
<td>2.19(15)</td>
<td>4.67(20)</td>
<td>94(3)</td>
<td>3.43(11)</td>
</tr>
<tr>
<td>2</td>
<td>–</td>
<td>–</td>
<td>95(6)</td>
<td>3.6(2)</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$N_f$</th>
<th>$m_u/m_d$</th>
<th>$m_s/m_{ud}$</th>
<th>$R$</th>
<th>$Q$</th>
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<tbody>
<tr>
<td>2+1</td>
<td>0.47(4)</td>
<td>27.4(4)</td>
<td>36.6(3.8)</td>
<td>22.8(1.2)</td>
</tr>
<tr>
<td>2</td>
<td>–</td>
<td>27.3(9)</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

All masses in MeV
Outline

Introduction

Lattice determination of $m_u$, $m_d$ and $m_s$
  FLAG
  FLAG phase 2
  Isospin limit
  Isospin breaking

Quark mass ratios from CHPT

A new dispersive analysis of $\eta \to 3\pi$
  Isospin breaking

Summary and Outlook
Expansion around the chiral limit

\[ \mathcal{H}_{\text{QCD}} = \mathcal{H}_{\text{QCD}}^0 + \mathcal{H}_m \quad \mathcal{H}_m := \bar{q} \mathcal{M} q \quad \mathcal{M} = \begin{pmatrix} m_u & m_d & m_s \\ \end{pmatrix} \]

\[ \bar{q} = (\bar{u}, \bar{d}, \bar{s}) \]

the mass term \( \mathcal{H}_m \) will be treated as a small perturbation
Expansion around the chiral limit

\[ \mathcal{H}_{\text{QCD}} = \mathcal{H}_{\text{QCD}}^0 + \mathcal{H}_m \quad \mathcal{H}_m := \bar{q} \mathcal{M} q \quad \mathcal{M} = \begin{pmatrix} m_u & m_d \\ m_d & m_s \end{pmatrix} \]

Expansion around \( \mathcal{H}_{\text{QCD}}^0 \equiv \text{expansion in powers of } m_q \)

General quark mass expansion for a particle \( P \):

\[ M^2 = M_0^2 + (m_u + m_d) \langle P | \bar{q} q | P \rangle + O(m_q^2) \]
Expansion around the chiral limit

\[ \mathcal{H}_{\text{QCD}} = \mathcal{H}_{\text{QCD}}^0 + \mathcal{H}_m \quad \mathcal{H}_m := \bar{q} \mathcal{M} q \quad \mathcal{M} = \begin{pmatrix} m_u \\ m_d \\ m_s \end{pmatrix} \]

Expansion around \( \mathcal{H}_{\text{QCD}}^0 \equiv \) expansion in powers of \( m_q \)

For a Goldstone bosons \( M_0^2 = 0 \):

\[ M_\pi^2 = - (m_u + m_d) \frac{1}{F_\pi^2} \langle 0 | \bar{q} q | 0 \rangle + O(m_q^2) \]

where we have used a Ward identity: \[ \langle \pi | \bar{q} q | \pi \rangle = - \frac{1}{F_\pi^2} \langle 0 | \bar{q} q | 0 \rangle =: B_0 \]

Gell-Mann, Oakes and Renner (68)
Quark masses

Consider the whole pseudoscalar octet:

\[ M_{\pi}^2 = B_0 (m_u + m_d) \quad M_{K^+}^2 = B_0 (m_u + m_s) \quad M_{K^0}^2 = B_0 (m_d + m_s) \]
**Quark masses**

Consider the whole pseudoscalar octet:

\[
M_{\pi}^2 = B_0 (m_u + m_d) \quad M_{K^+}^2 = B_0 (m_u + m_s) \quad M_{K^0}^2 = B_0 (m_d + m_s)
\]

Quark mass ratios:

\[
\frac{m_u}{m_d} \lesssim \frac{M_{\pi^+}^2 - M_{K^0}^2 + M_{K^+}^2}{M_{\pi^+}^2 + M_{K^0}^2 - M_{K^+}^2} \lesssim 0.67
\]

\[
\frac{m_s}{m_d} \lesssim \frac{M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \lesssim 20
\]
Quark masses

Consider the whole pseudoscalar octet:

\[ M_{\pi}^2 = B_0(m_u + m_d) \quad M_{K^+}^2 = B_0(m_u + m_s) \quad M_{K^0}^2 = B_0(m_d + m_s) \]

Quark mass ratios:

\[
\frac{m_u}{m_d} \approx \frac{M_{\pi^+}^2 - M_{K^0}^2 + M_{K^+}^2}{M_{\pi^+}^2 + M_{K^0}^2 - M_{K^+}^2} \approx 0.67
\]

\[
\frac{m_s}{m_d} \approx \frac{M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \approx 20
\]

\[ \hat{m} \equiv (m_u + m_d)/2 \approx 5.4 \text{ MeV} \]

SU(6) relation, Leutwyler (75)

\[ m_u \approx 4 \text{ MeV} \quad m_d \approx 6 \text{ MeV} \quad m_s \approx 135 \text{ MeV} \]

Gasser and Leutwyler (75)
Electromagnetic corrections to the masses

According to Dashen’s theorem

\[
\begin{align*}
M_{\pi^0}^2 &= B_0(m_u + m_d) \\
M_{\pi^+}^2 &= B_0(m_u + m_d) + \Delta_{\text{em}} \\
M_{K^0}^2 &= B_0(m_d + m_s) \\
M_{K^+}^2 &= B_0(m_u + m_s) + \Delta_{\text{em}}
\end{align*}
\]

Extracting the quark mass ratios gives

\[
\begin{align*}
\frac{m_u}{m_d} &= \frac{M_{K^+}^2 - M_{K^0}^2 + 2M_{\pi^0}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} = 0.56 \\
\frac{m_s}{m_d} &= \frac{M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} = 20.1
\end{align*}
\]

Weinberg (77) estimated \( m_s \) from the splitting in baryon octet

\[
m_u = 4.2 \text{ MeV} \quad m_d = 7.5 \text{ MeV} \quad m_s = 150 \text{ MeV}
\]
Higher order chiral corrections

Mass formulae to second order

\[
\frac{M_K^2}{M_\pi^2} = \frac{m_s + \hat{m}}{2\hat{m}} \left[ 1 + \Delta_M + \mathcal{O}(m^2) \right]
\]

\[
\frac{M_{K^0}^2 - M_{K^+}^2}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}} \left[ 1 + \Delta_M + \mathcal{O}(m^2) \right]
\]

\[
\Delta_M = \frac{8(M_K^2 - M_\pi^2)}{F_\pi^2} (2L_8 - L_5) + \chi\text{-logs}
\]

The same \(\mathcal{O}(m)\) correction appears in both ratios

\Rightarrow\text{ this double ratio is free from } \mathcal{O}(m) \text{ corrections}

\[
Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} = \frac{M_K^2}{M_\pi^2} \frac{M_{K^0}^2 - M_{K^+}^2}{M_K^2} \left[ 1 + \mathcal{O}(m^2) \right]
\]
Higher order chiral corrections

Mass formulae to second order

\[
\frac{M_K^2}{M_\pi^2} = \frac{m_s + \hat{m}}{2\hat{m}} \left[ 1 + \Delta_M + O(m^2) \right]
\]

\[
\frac{M_{K0}^2 - M_{K+}^2}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}} \left[ 1 + \Delta_M + O(m^2) \right]
\]

\[
\Delta_M = \frac{8(M_K^2 - M_\pi^2)}{F_\pi^2} (2L_8 - L_5) + \chi\text{-logs}
\]

The same \(O(m)\) correction appears in both ratios
\[\Rightarrow\] this double ratio is free from \(O(m)\) and em corrections

\[
Q_D^2 \equiv \frac{(M_{K0}^2 + M_{K+}^2 - M_{\pi^+}^2 + M_{\pi^0}^2)(M_{K0}^2 + M_{K+}^2 - M_{\pi^+}^2 - M_{\pi^0}^2)}{4M_{\pi^0}^2(M_{K0}^2 - M_{K+}^2 + M_{\pi^+}^2 - M_{\pi^0}^2)} = 24.2
\]
Leutwyler’s ellipse

Information on $Q$ amounts to an elliptic constraint in the plane of $\frac{m_s}{m_d}$ and $\frac{m_u}{m_d}$

$$\left( \frac{m_s}{m_d} \right)^2 \frac{1}{Q^2} + \left( \frac{m_u}{m_d} \right)^2 = 1$$

Leutwyler
Information on $Q$ amounts to an elliptic constraint in the plane of $\frac{m_s}{m_d}$ and $\frac{m_u}{m_d}$

$$\left( \frac{m_s}{m_d} \right)^2 \frac{1}{Q^2} + \left( \frac{m_u}{m_d} \right)^2 = 1$$

Leutwyler's ellipse

Weinberg (77)
Estimate of $Q$: violation of Dashen’s theorem

\[
\left( M_{K^+}^2 - M_{K^0}^2 \right)_{\text{em}} = \left( M_{\pi^+}^2 - M_{\pi^0}^2 \right)_{\text{em}} \Rightarrow \left( M_{K^+} - M_{K^0} \right)_{\text{em}} = 1.3 \text{ MeV}
\]
Estimate of $Q$: violation of Dashen’s theorem

\[
\left( M_{K^+}^2 - M_{K^0}^2 \right)_{em} = \left( M_{\pi^+}^2 - M_{\pi^0}^2 \right)_{em} \Rightarrow \left( M_{K^+} - M_{K^0} \right)_{em} = 1.3 \text{ MeV}
\]

Higher order corrections change the numerical value:

\[
(M_{K^+} - M_{K^0})_{em} = \begin{cases} 
1.9 \text{ MeV} & \text{Duncan et al. (96)} \quad Q = 22.8 \\
2.3 \text{ MeV} & \text{Bijnens-Prades (97)} \quad Q = 22 \\
2.6 \text{ MeV} & \text{Donoghue-Perez (97)} \quad Q = 21.5 \\
3.2 \text{ MeV} & \text{Anant-Moussallam (04)} \quad Q = 20.7 
\end{cases}
\]

Most recent evaluation: Kastner-Neufeld (08): $Q = 20.7 \pm 1.2$
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  Isospin breaking

Summary and Outlook
Q from the decay $\eta \rightarrow 3\pi$

Decay amplitude at leading order

$$A(\eta \rightarrow \pi^0 \pi^+ \pi^-) = -\frac{\sqrt{3}}{4} \frac{m_u - m_d}{m_s - \hat{m}} s - \frac{4M^2_\pi}{3} F^2_\pi$$
**Q from the decay $\eta \rightarrow 3\pi$**

Decay amplitude

$$A(\eta \rightarrow \pi^0\pi^+\pi^-) = -\frac{1}{Q^2} \frac{M_K^2(M_K^2 - M^2_\pi)}{3\sqrt{3}M^2_\pi F^2_\pi} M(s, t, u)$$

The decay width can be written as

$$\Gamma(\eta \rightarrow \pi^0\pi^+\pi^-) = \Gamma_0 \left(\frac{Q_D}{Q}\right)^4 = (295 \pm 20) \text{ eV} \quad \text{PDG (08)}$$

- isospin-breaking sensitive process
- em contributions suppressed (Sutherland's theorem)
  - mainly sensitive to $m_u - m_d$
- strong decay width $\Gamma_0$ difficult to estimate
**Q from the decay $\eta \rightarrow 3\pi$**

Decay amplitude

$$A(\eta \rightarrow \pi^0\pi^+\pi^-) = -\frac{1}{Q^2} \frac{M_K^2(M_K^2 - M_\pi^2)}{3\sqrt{3}M_\pi^2F_\pi^2} M(s, t, u)$$

The decay width can be written as

$$\Gamma(\eta \rightarrow \pi^0\pi^+\pi^-) = \Gamma_0 \left(\frac{Q_D}{Q}\right)^4 = (295 \pm 20)\text{ eV} \quad \text{PDG (08)}$$

$$\Gamma_0 = \begin{cases} (167 \pm 50)\text{ eV} & \text{Gasser-Leutwyler (85)} \quad Q = 21.1 \pm 1.6 \\ (219 \pm 22)\text{ eV} & \text{Anisovich-Leutwyler (96)} \quad Q = 22.6 \pm 0.7 \\ (209 \pm 20)\text{ eV} & \text{Kambor et al (96)} \quad Q = 22.3 \pm 0.6 \end{cases}$$

Gasser Leutwyler (85) based on one-loop CHPT

The other two evaluations based on dispersion relations

See also: analysis of KLOE data on $\eta \rightarrow 3\pi$  

$$Q = 22.8 \pm 0.4$$
Q from the decay $\eta \rightarrow 3\pi$

Decay amplitude

$$A(\eta \rightarrow \pi^0\pi^+\pi^-) = -\frac{1}{Q^2} \frac{M_K^2(M_K^2 - M^2_{\pi})}{3\sqrt{3}M^2_{\pi}F^2_{\pi}} M(s, t, u)$$

The decay width can be written as

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$$\Gamma_0 = \begin{cases} 
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(209 \pm 20)\text{ eV} & \text{Kambor et al (96)} \quad Q = 22.3 \pm 0.6 
\end{cases}$$

Gasser Leutwyler (85) based on one-loop CHPT
The other two evaluations based on dispersion relations

See also: full two-loop calculation of $\eta \rightarrow 3\pi$ Bijnens-Ghorbani (07)

$$Q = 23.2$$
A new analysis is in progress

- recent measurements of the Dalitz plot
  ⇒ test the calculation of the strong dynamics of the decay

- dispersive analysis based on $\pi\pi$ scattering phases
  recent improvements must be taken into account

- recent progress in dealing with isospin breaking (NREFT)
  can be applied also here
Dispersion relation for $\eta \to 3\pi$

Based on the representation

$$M(s, t, u) = M_0(s) - \frac{2}{3} M_2(s) + [(s - u) M_1(t) + M_2(t) + (t \leftrightarrow u)]$$

valid if the discontinuities of $D$ waves and higher are neglected

Dispersion relation for the $M_I$'s

$$M_I(s) = \Omega_I(s) \left\{ P_I(s) + \frac{s^n}{\pi} \int_{4M^2_\pi}^{\infty} ds' \frac{\sin \delta_I(s')}{|\Omega_I(s)|^{s^n}(s' - s)} \right\}$$

where

$$\Omega(s) = \exp \left[ \frac{s}{\pi} \int_{4M^2_\pi}^{\infty} ds' \frac{\delta_I(s')}{s'(s' - s)} \right]$$
Dispersion relation for $\eta \rightarrow 3\pi$

Based on the representation

$$M(s, t, u) = M_0(s) - \frac{2}{3} M_2(s) + \left[(s - u) M_1(t) + M_2(t) + (t \leftrightarrow u)\right]$$

valid if the discontinuities of $D$ waves and higher are neglected

Dispersion relation for the $M_I$'s

$$M_I(s) = \Omega_I(s) \left\{ P_I(s) + \frac{s^n}{\pi} \int_{4M^2_\pi}^{\infty} ds' \frac{\sin\delta_I(s') \hat{M}_I(s')}{|\Omega_I(s)| s'^n(s' - s)} \right\}$$

where

$$\Omega(s) = \exp \left[ \frac{s}{\pi} \int_{4M^2_\pi}^{\infty} ds' \frac{\delta_I(s')}{s'(s' - s)} \right]$$

given $\delta_I(s)$, the solution depends on subtraction constants only
Subtraction constants

Extended the number of parameters w.r.t. Anisovich and Leutwyler (96):

\[ P_0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3 \]
\[ P_1(s) = \alpha_1 + \beta_1 s + \gamma_1 s^2 \]
\[ P_2(s) = \alpha_2 + \beta_2 s + \gamma_2 s^2 + \delta_2 s^3 \]

Solution linear in the subtraction constants: \text{Anisovich, Leutwyler, unpublished}

\[ M(s, t, u) = \alpha_0 M^{\alpha_0}(s, t, u) + \beta_0 M^{\beta_0}(s, t, u) + \ldots \]

makes fitting of data very easy.
Taylor coefficients

Subtraction constants $\alpha_I$, $\beta_I$, $\gamma_I$, ... can be replaced by Taylor coefficients: the relation between the two sets is linear

\[ M_0(s) = a_0 + b_0 s + c_0 s^2 + d_0 s^3 + \ldots \]
\[ M_1(s) = a_1 + b_1 s + c_1 s^2 + \ldots \]
\[ M_2(s) = a_2 + b_2 s + c_2 s^2 + d_2 s^3 + \ldots \]

Not all Taylor coefficients are physically relevant:
\[ \exists \text{ 5-parameter family of polynomials } \delta M_I(s) \text{ that added to } M_I(s) \text{ do not change } M(s, t, u) \text{ (reparametrization invariance)} \]
Taylor coefficients

Subtraction constants $\alpha_i, \beta_i, \gamma_i, \ldots$ can be replaced by Taylor coefficients: the relation between the two sets is \textit{linear}

$$
M_0(s) = a_0 + b_0 s + c_0 s^2 + d_0 s^3 + \ldots \\
M_1(s) = a_1 + b_1 s + c_1 s^2 + \ldots \\
M_2(s) = a_2 + b_2 s + c_2 s^2 + d_2 s^3 + \ldots
$$

▶ use reparametrization invariance to arbitrarily fix 5 coefficients: \text{tree-level ChPT or } \delta_2 = 0
Taylor coefficients

Subtraction constants $\alpha_I, \beta_I, \gamma_I, \ldots$ can be replaced by Taylor coefficients: the relation between the two sets is \textit{linear}

\[
M_0(s) = a_0 + b_0 s + c_0 s^2 + d_0 s^3 + \ldots
\]
\[
M_1(s) = a_1 + b_1 s + c_1 s^2 + \ldots
\]
\[
M_2(s) = a_2 + b_2 s + c_2 s^2 + d_2 s^3 + \ldots
\]

- use reparametrization invariance to arbitrarily fix 5 coefficients: tree-level ChPT or $\delta_2 = 0$
- fix the remaining ones with one-loop ChPT
Taylor coefficients

Subtraction constants \( \alpha_I, \beta_I, \gamma_I, \ldots \) can be replaced by Taylor coefficients: the relation between the two sets is linear

\[
M_0(s) = a_0 + b_0 s + c_0 s^2 + d_0 s^3 + \ldots \\
M_1(s) = a_1 + b_1 s + c_1 s^2 + \ldots \\
M_2(s) = a_2 + b_2 s + c_2 s^2 + d_2 s^3 + \ldots
\]

- use reparametrization invariance to arbitrarily fix 5 coefficients: tree-level ChPT or \( \delta_2 = 0 \)
- fix the remaining ones with one-loop ChPT
- either set \( d_0 = c_1 = 0 \) \( \Rightarrow \) dispersive, one loop
  or fix \( d_0, c_1 \) by fitting data \( \Rightarrow \) dispersive, fit to KLOE
Taylor coefficients

Subtraction constants $\alpha_I, \beta_I, \gamma_I, \ldots$ can be replaced by Taylor coefficients: the relation between the two sets is *linear*

\[
\begin{align*}
M_0(s) &= a_0 + b_0 s + c_0 s^2 + d_0 s^3 + \ldots \\
M_1(s) &= a_1 + b_1 s + c_1 s^2 + \ldots \\
M_2(s) &= a_2 + b_2 s + c_2 s^2 + d_2 s^3 + \ldots 
\end{align*}
\]

- use reparametrization invariance to arbitrarily fix 5 coefficients: tree-level ChPT or $\delta_2 = 0$
- fix the remaining ones with one-loop ChPT
- either set $d_0 = c_1 = 0 \Rightarrow \text{dispersive, one loop}$
- or fix $d_0, c_1$ by fitting data $\Rightarrow \text{dispersive, fit to KLOE}$
- Dalitz-plot data are insensitive to the normalization: ChPT fixes the normalization and allows the extraction of $Q$
Dalitz distribution for $\eta \rightarrow \pi^+ \pi^- \pi^0$
Dalitz distribution for $\eta \rightarrow \pi^+ \pi^- \pi^0$
Dalitz distribution for $\eta \rightarrow \pi^+ \pi^- \pi^0$
Dalitz distribution for $\eta \rightarrow 3\pi^0$

![Graph showing the Dalitz distribution for $\eta \rightarrow 3\pi^0$]
Dalitz distribution for $\eta \rightarrow 3\pi^0$

$\Gamma(Z)$ vs. $Z$

- One-loop $\chi$PT
- MAMI-C
- Dispersive, one loop

Preliminary
Dalitz distribution for $\eta \rightarrow 3\pi^0$

![Graph showing the Dalitz distribution for $\eta \rightarrow 3\pi^0$](image-url)

- Preliminary
- One-loop $\chi$PT
- MAMI-C
- Dispersive, one loop
- Dispersive, fit to KLOE
Comparison of $\alpha$

- $\chi$PT $O(p^4)$ [ GL '85, Bijnens&Gasser '02 ]
- $\chi$PT $O(p^6)$ [ Bijnens&Ghorbani '07 ]
- Kambor et al. [ Kambor et al. '96 ]
- Kampf et al. [ Kampf et al. '11 ]
- NREFT [ Schneider et al. '11 ]

- Crystal Barrel@LEAR (1998) [ Abele et al. '98 ]
- Crystal Ball@BNL (2001) [ Tippens et al. '01 ]
- SND (2001) [ Achasov et al. '01 ]
- WASA@CELSIUS (2007) [ Bashkanov et al. '07 ]
- WASA@COSY (2008) [ Adolph et al. '09 ]
- Crystal Ball@MAMI-B (2009) [ Unverzagt et al. '09 ]
- Crystal Ball@MAMI-C (2009) [ Prakhov et al. '09 ]
- KLOE (2010) [ Ambrosino et al. '10 ]
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- Dispersive, one loop
- Dispersive, fit to KLOE

Preliminary
## Comparison of $Q$

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<th>$Q$</th>
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<td>no Dashen violation</td>
<td>[Weinberg '77]</td>
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<tr>
<td>with Dashen violation</td>
<td>[Anant et al. '04, Kastner &amp; Neufeld '08]</td>
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</tbody>
</table>
Comparison of $Q$

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- no Dashen violation [Weinberg '77]
- with Dashen violation [Anant et al. '04, Kastner & Neufeld '08]
- dispersive, one loop
- dispersive, fit to KLOE

preliminary

$Q$ [20, 21, 22, 23, 24]
Dispersive analysis by Kampf et al.

The Adler zero has not been imposed as constraint
Intermediate summary

- dispersive representation with purely theoretical input fails to correctly describe the momentum dependence both in the charged and neutral channel

- extend the framework by two more parameters (higher chiral order) \( \Rightarrow \) good description of Dalitz plot data

- fit of momentum dependence in the charged channel leads to a correct prediction for \( \alpha \)

- the value of \( Q \) is also affected by the fit to data
A different determination of the Taylor coefficients

\[
M_0(s) = a_0 + b_0 s + c_0 s^2 + d_0 s^3 + \ldots \\
M_1(s) = a_1 + b_1 s + c_1 s^2 + \ldots \\
M_2(s) = a_2 + b_2 s + c_2 s^2 + d_2 s^3 + \ldots 
\]

- reparametrization invariance \(\Rightarrow\) fix 5 Taylor coefficients
A different determination of the Taylor coefficients

\[ M_0(s) = a_0 + b_0 s + c_0 s^2 + d_0 s^3 + \ldots \]
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- reparametrization invariance \( \Rightarrow \) fix 5 Taylor coefficients
- fix the others by requiring:
  1. one-loop ChPT value of \( c_0, b_1, b_2 \) and \( c_2 \) holds to 30%
  2. one-loop prediction for Adler zero and the amplitude derivative at the zero hold to 10%
A different determination of the Taylor coefficients

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  1. one-loop ChPT value of \( c_0, b_1, b_2 \) and \( c_2 \) holds to 30%
  2. one-loop prediction for Adler zero and the amplitude derivative at the zero hold to 10%
- fit the data in addition:
  - PDG value for \( \alpha \)
  - KLOE in charged channel
  - MAMi in the neutral channel
  - WASA in the charged channel
Importance of data vs theory

![Graph showing the comparison between data and theory for the process $\eta \rightarrow 3\pi$. The graph includes lines and curves representing different experimental data sets (MAMI, KLOE, WASA) and theoretical models. The axes are labeled $d_0$ on the horizontal axis and $c_1$ on the vertical axis. Lines and curves are color-coded to distinguish between acceptable chiral expansion models, PDG values, and experimental data distributions.]
Importance of data vs theory

Isospin breaking not accounted for

- Yellow: acceptable chiral expansion
- Cyan: PDG value of $\alpha_{3\pi^0}$
- Blue: Z-distribution of MAMI
- Black: Dalitz plot of MAMI
- Red: Dalitz plot of KLOE
- Green: Dalitz plot of WASA
Importance of data vs theory

Value of $Q$ obtained from rate of neutral decay

Preliminary, isospin breaking not accounted for

- **acceptance chiral expansion**
- PDG value of $\alpha$ $3\pi^0$
- Z-distribution of MAMI $3\pi^0$
- Dalitz plot of MAMI $3\pi^0$
- Dalitz plot of KLOE $\pi^+\pi^-\pi^0$
- Dalitz plot of WASA $\pi^+\pi^-\pi^0$

![Graph showing the value of Q obtained from rate of neutral decay with different data sets and their significance.](image-url)
Importance of data vs theory

Isospin breaking not accounted for

- Acceptable chiral expansion
- PDG value of $\alpha_{3\pi^0}$
- Z-distribution of MAMI
- Dalitz plot of MAMI
- Dalitz plot of KLOE
- Dalitz plot of WASA

Authors:
- Kambor et al. 1996
- Bijnens and Ghorbani 2007
- Kampf et al. 2011
- Lanz 2011
Isospin breaking

Dispersive calculation performed in the isospin limit:

\[ M_\pi = M_{\pi^+} \quad e = 0 \]

- we correct for \( M_{\pi^0} \neq M_{\pi^+} \) by “stretching” \( s, t, u \Rightarrow \) boundaries of isospin-symmetric phase space = boundaries of physical phase space

- physical thresholds inside the phase space can also be mimicked “by hand”

- analysis of Ditsche, Kubis, Meissner (09) used as guidance and check. Same for Gullström, Kupsc and Rusetsky (09)

- \( e \neq 0 \) effects partly corrected for in the data analysis for the rest we rely on one-loop ChPT – formulae given by Ditsche, Kubis, Meissner (09)

- ⇒ to be completed
Isospin breaking

Dispersive calculation performed in the isospin limit:

\[ M_\pi = M_{\pi^+} \quad e = 0 \]

- NREFT approach (Schneider, Kubis, Ditsche (11)): systematic method to take into account isospin breaking
- matching between dispersive representation and NREFT in the isospin limit \( \Rightarrow \) determine NREFT isospin-conserving parameters
- switch on isospin breaking and fit the data
- for the future
Outline

Introduction

Lattice determination of $m_u$, $m_d$ and $m_s$
- FLAG
- FLAG phase 2
- Isospin limit
- Isospin breaking

Quark mass ratios from CHPT

A new dispersive analysis of $\eta \to 3\pi$
- Isospin breaking

Summary and Outlook
Summary

- Quark masses are fundamental and yet unexplained parameters of the standard model.
- I have reviewed the determination based on lattice and chiral perturbation theory.
- I have discussed the extraction of the quark mass ratio $Q$ from $\eta \rightarrow 3\pi$ decays based on dispersion relations.
- A combination of high precision lattice determinations in the isospin limit and $Q$ from $\eta \rightarrow 3\pi$ is at present the best method to determine $m_u$ and $m_d$. 