Renormalons

High-order behavior of perturbation theory in QFT

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Motivation

Renormalons:

Pattern of divergence in perturation series obtained in renormalizable QFT's related to their small & large momentum behavior.

- Most important for QCD
- I will discuss some basic aspects of renormalons relevant for practical work in particle physics.

For an excellent review see:

M. Beneke, Physics Reports, 317 (1999), p. 1-142
(hep-ph/9807443)

Motivation: Vacuum Polarization Function

$$(-i)\int d^4x\, e^{-iqx}\,\langle 0|T\,(j_\mu(x)j_\nu(0))|0\rangle = \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\,\Pi(Q^2)$$

$\Pi'_{\rm had}(q^2) \quad =$

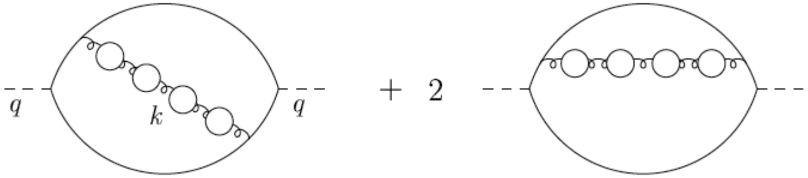
- hadronic contribution to $g-2$
- hadronic τ decay
- $\sigma(e^+e^- \to \text{hadrons})(q^2) \sim \text{Im}[\Pi(q^2)]$

Motivation: Vacuum Polarization Function

$$(-i)\int d^4x\, e^{-iqx}\,\langle 0|T\,(j_\mu(x)j_\nu(0))|0\rangle = \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\,\Pi(Q^2)$$

Task: Determine the massless quark bubble insertions to all orders to

$$D(Q^2) = 4\pi^2\,\frac{d\Pi(Q^2)}{dQ^2}\qquad (Q^2 = -q^2)$$



- gauge invariant
- . . . not dominating numerically

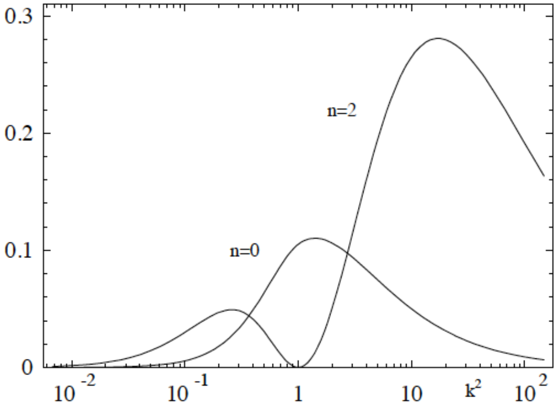
Motivation: Vacuum Polarization Function

$$D = \sum_{n=0}^{\infty} \alpha_s \int_0^{\infty} \frac{dk^2}{k^2}\, \underset{\uparrow}{F(k^2/Q^2)} \left[\frac{n_f}{6\pi}\alpha_s \ln\left(\frac{k^2 e^{-5/3}}{\mu^2}\right)\right]^n$$

"gluon momentum distribution"

$$F(\hat{k}^2 \ll 1) \;=\; \frac{3C_F}{2\pi}\,\hat{k}^4 + \mathcal{O}(\hat{k}^6 \ln \hat{k}^2)$$

$$F(\hat{k}^2 \gg 1) \;=\; \frac{C_F}{3\pi}\,\frac{1}{\hat{k}^2}\left(\ln \hat{k}^2 + \frac{5}{6}\right) + \mathcal{O}\left(\frac{\ln \hat{k}^2}{\hat{k}^4}\right)$$



We know (n=0):

- theory gives reliable answers only for $k^2 \sim Q^2$
- low and high momentum regions are suppressed

Motivation: Vacuum Polarization Function

$$D \sim \frac{C_F}{\pi}\sum_{n=0}^{\infty}\alpha_s^{n+1}\left[\frac{3}{4}A\Big(-\frac{n_f}{12\pi}\Big)^n n! + \frac{1}{3}A\Big(\frac{n_f}{6\pi}\Big)^n n!\,\Big(n+\frac{11}{6}\Big)\right]$$

"IR renormalon"
(small k)

"UV renormalon"
(large k)

For large n: The logarithmic behavior of the quark bubbles enhances the contributions coming from small and large momenta.

$$k_{\rm IR}^2 \sim \mu^2\, e^{5/3}\, e^{-n/2}$$

$$k_{\rm UV}^2 \sim \mu^2\, e^{5/3}\, e^{n}$$

- Does perturbation theory make sense?
- Why is perturbative QCD so successful?
- Can we . . . maybe . . . sum the series somehow?
- What can we learn from this behavior?

Math for Divergent Series

Konvergent Series:

series: $a_0 + a_1 + a_2 + \ldots = \sum_{n=0}^{\infty} a_n$

partial sum: $s_n = \sum_{i=0}^{n} a_n$

value of series: $S = \lim_{n\to\infty} s_n$

→ Convergence test:

$$\left|\frac{a_{n+1}}{a_n}\right| \rightarrow \rho \begin{cases} < 1\,, & \text{convergent} \\ = 1\,, & \text{?} \\ > 1\,, & \text{divergent} \end{cases}$$

→ Alternating series are convergent if

$$|a_{n+1}| \leq |a_n| \text{ and } \lim_{n\to\infty} a_n = 0.$$

Examples:

$$\sum_{i=0}^{\infty} \left(\tfrac{1}{2}\right)^i = 2$$

$$\sum_{i=0}^{\infty} x^i = \tfrac{1}{1-x}\,, \qquad (|x| < 1)$$

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{i} = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \ldots = \ln 2$$

Math for Divergent Series

Divergent Series:

A not that bad example:

$$\sum_{i=0}^{\infty} x^i = 1 + x + x^2 + x^3 + \ldots \qquad \text{(for } |x| > 1\text{)} \qquad \rightarrow \frac{1}{1-x} \text{ ?}$$

$$x = -1: \quad \sum_n a_n = 1 - 1 + 1 - 1 + \ldots \qquad \overset{?}{=} \frac{1}{2}$$

$$s_n = (1, 0, 1, 0, 1, \ldots)$$

$$x = -2: \quad \sum_n a_n = 1 - 2 + 4 - 8 + 16 - \ldots \qquad \overset{?}{=} \frac{1}{3}$$

$$s_n = (1, -1, 3, -5, 11, \ldots)$$

Math for Divergent Series

Divergent Series:

$\Rightarrow$ We need to extend our notion of a valid summation process !

$$\boxed{\text{Mathematics of divergent series}}$$

(Bernoulli, Euler
Abel, Boral, . . .)

Permanence Conditions:

1. Geneneralized summation processes have to be “reasonable”.
2. Every series convergent in the original sense with value S must also be summable with the same value S with a generalized summation process.
3. If a series is summable with respect to two processes its value must be the same with respect to both processes.

Math for Divergent Series

Example:

The M-process:

A series is called M-summable if the following series is convergent:

$$s'_n = \frac{s_0 + s_1 + \ldots + s_n}{1 + 1 + \ldots + 1} = \frac{\sum_{i=0}^{n} s_i}{\sum_{i=0}^{n} 1} .$$

The value of the series is $S = \lim_{n\to\infty} s'_n$.

$$\sum_n a_n = 1 - 1 + 1 - 1 + \ldots \overset{?}{=} \tfrac{1}{1-(-1)} = \tfrac{1}{2}$$

$$s_n = (1, 0, 1, 0, 1, \ldots) \quad \to s'_n = \frac{1}{2} + \frac{1 + (-1)^n}{4(n+1)} \quad \to \frac{1}{2}$$

works!

$$\sum_n a_n = 1 - 2 + 4 - 8 + \ldots \overset{?}{=} \tfrac{1}{1-(-2)} = \tfrac{1}{3}$$

$$s_n = (1, -1, 3, -5, \ldots) \quad \to s'_n = \frac{1}{3} + \frac{2(1 + (-1)^n 2^{n+1})}{9(n+1)}$$

nope!

Math for Divergent Series

A better example:

The Borel-process:

A series is called Borel-summable if the following series is convergent, if

$$s'_n(t) \;=\; \frac{\sum_{i=0}^{n} s_i \frac{t^i}{i!}}{\sum_{i=0}^{n} \frac{t^i}{i!}}$$

converges a function $S(t)$ for $t > 0$ and the limit $S = \lim_{t\to\infty} S(t)$ exists. The value of the series is S.

Math for Divergent Series

The Borel-process: (for $\sum_{n=0}^{\infty} a_n x^n$)

The series is called Borel-summable if the sum

$$B(t) = \sum_{n=0}^{\infty} \frac{a_n t^n}{n!}$$

is convergent in the original sense for $t > 0$ and if the integral

$$S(x) = \frac{1}{x} \int_0^{\infty} dt\ e^{-t/x}\, B(t)$$

exists. The value of the series is $S(x)$.

$$\sum_n a_n = 1 - x + x^2 - x^3 + \ldots \overset{?}{=} \frac{1}{1-(-x)} = \frac{1}{1+x}$$

$$B(t) = e^{-t} \quad \rightarrow S(x) = \frac{1}{x} \int_0^{\infty} dt\, e^{-t(1+x)/x} = \frac{1}{1+x}$$

works!

Adler Function

$$D \sim \frac{C_F}{\pi}\sum_{n=0}^{\infty}\alpha_s^{n+1}\left[\frac{3}{4}A\Big(-\frac{n_f}{12\pi}\Big)^n n! + \frac{1}{3}A\Big(\frac{n_f}{6\pi}\Big)^n n!\,\Big(n+\frac{11}{6}\Big)\right]$$

Assumption:

The perturbative series is "asymptotic", i.e.

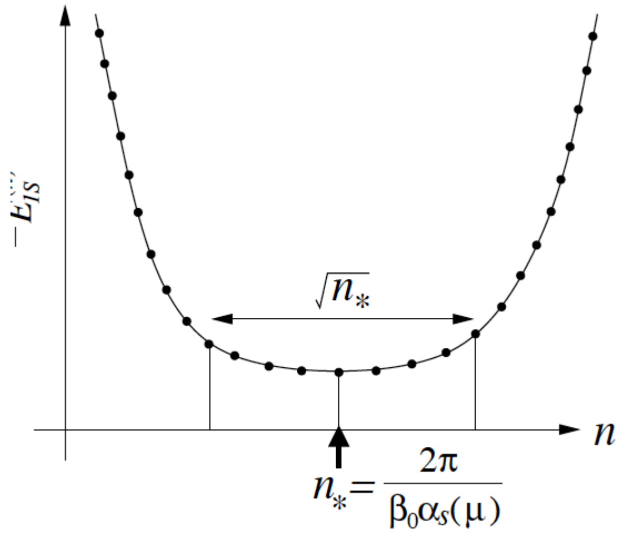
$$\left|R(\alpha_s) - \sum_{i=0}^{n} r_i\alpha_s^i\right|/\alpha_s^n \;\rightarrow\; 0$$

for any n and $\alpha_s \rightarrow 0$ (or $Q^2 \rightarrow \infty$).

- non-trival ("Nothing large is being missed in pert. theory.")
- cannot be proved (. . . maybe with the lattice in some time)
- but reasonable (phenomenology works quite well)

Adler Function

$$D \sim \frac{C_F}{\pi}\sum_{n=0}^{\infty}\alpha_s^{n+1}\left[\frac{3}{4}A\left(-\frac{n_f}{12\pi}\right)^n n! + \frac{1}{3}A\left(\frac{n_f}{6\pi}\right)^n n!\,\left(n+\frac{11}{6}\right)\right]$$

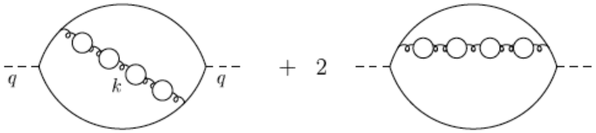


- meaningful only to order $n_* \pm \sqrt{n_*}$
- for finite α_s an irreducible error remains
- What is the parametric uncertainty?

Adler Function

$$D \sim \frac{C_F}{\pi} \sum_{n=0}^{\infty} \alpha_s^{n+1} \left[\frac{3}{4} A\left(-\frac{n_f}{12\pi}\right)^n n! + \frac{1}{3} A\left(\frac{n_f}{6\pi}\right)^n n! \left(n + \frac{11}{6}\right)\right]$$

→ We noted earlier that the massless quark bubbles are not really dominating numerically.



"Naive Non-Abelianization": $n_f \rightarrow -\frac{3}{2}\left(11 - \frac{2}{3} n_f\right) = -\frac{3}{2}\beta_0$

("large-β_0 approximation")

- gauge-invariant
- cannot be proven
- appears to work very well numerically

Adler Function

$$D \sim \frac{C_F}{\pi}\sum_{n=0}^{\infty}\alpha_s^{n+1}\left[\frac{3}{4}A\Big(-\frac{n_f}{12\pi}\Big)^n n! + \frac{1}{3}A\Big(\frac{n_f}{6\pi}\Big)^n n!\,\Big(n+\frac{11}{6}\Big)\right]$$

"IR renormalon"
(same sign)

"UV renormalon"
(sign alternating)

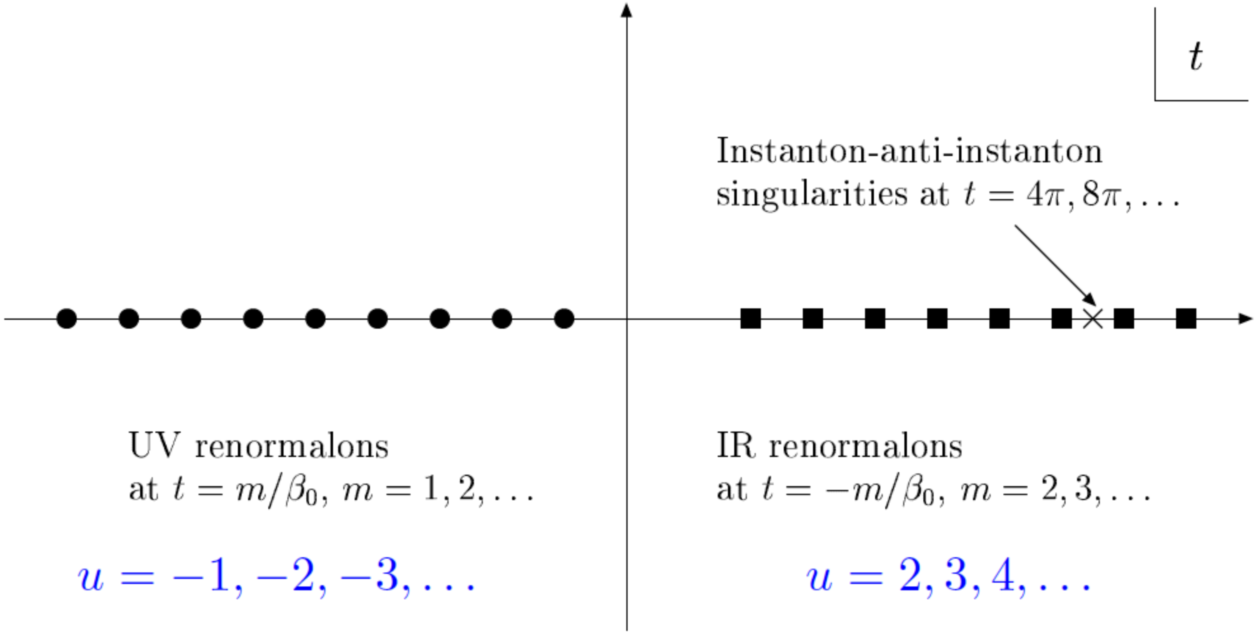
<u>Borel transform:</u> $(u = t\frac{\beta_0}{4\pi})$

$$B_D(u) \sim \frac{A_2}{u-2} + \ldots$$ (IR renormalons)

$$+ A_{-1}\left[\frac{1}{(u+1)^2} + \frac{5}{6}\frac{1}{u+1}\right] + \ldots$$ (UV renormalons)

Adler Function

$$D \sim \frac{C_F}{\pi}\sum_{n=0}^{\infty}\alpha_s^{n+1}\left[\frac{3}{4}A\left(-\frac{n_f}{12\pi}\right)^n n! + \frac{1}{3}A\left(\frac{n_f}{6\pi}\right)^n n!\left(n+\frac{11}{6}\right)\right]$$



$u = -1, -2, -3, \ldots$

$u = 2, 3, 4, \ldots$

Adler Function

$$D \sim \frac{C_F}{\pi} \sum_{n=0}^{\infty} \alpha_s^{n+1} \left[\frac{3}{4} A\left(-\frac{n_f}{12\pi}\right)^n n! + \frac{1}{3} A\left(\frac{n_f}{6\pi}\right)^n n! \left(n + \frac{11}{6}\right)\right]$$

→ What do the poles mean? Let's sum the series !

$$D = \int_0^{\infty} du \, e^{-\frac{4\pi u}{\beta_0 \alpha_s}} \, B_D(u)$$

- UV renormalons can be Borel-summed !

  in practice: divergence not very serious

- IR renormalons are not Borel-summable ! → Ambiguity!

  in practice: serious, we have something to do about it

Adler Function

$$D \sim \frac{C_F}{\pi} \sum_{n=0}^{\infty} \alpha_s^{n+1} \left[\frac{3}{4} A\Big(-\frac{n_f}{12\pi}\Big)^n n! + \frac{1}{3} A\Big(\frac{n_f}{6\pi}\Big)^n n! \left(n + \frac{11}{6}\right)\right]$$

→Ambiguity from a pole at $u = k$:

$$\Delta \left[\int_0^\infty du \; e^{-\frac{4\pi u}{\beta_0 \alpha_s}} \frac{1}{u-k} \right] \sim e^{-(4\pi/\beta_0\alpha_s)\,k} \sim \left(\frac{\Lambda^2_{\rm QCD}}{Q^2}\right)^k$$

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda^2_{\rm QCD})}$$

“Power-Ambiguity”

- smaller for $Q^2 \to \infty$ (asymptotic freedom)
- pole closest to origin most important
- signals non-perturbative effects, power corrections
  → Operator-Product-Expansion

Adler Function

$$D \sim \frac{C_F}{\pi}\sum_{n=0}^{\infty}\alpha_s^{n+1}\left[\frac{3}{4}A\Big(-\frac{n_f}{12\pi}\Big)^n n! + \frac{1}{3}A\Big(\frac{n_f}{6\pi}\Big)^n n!\,\Big(n+\frac{11}{6}\Big)\right]$$

$$\boxed{\frac{1}{u-2} \;\leftrightarrow\; \langle 0|G^A_{\mu\nu}G^{\mu\nu}_A|0\rangle \;\sim\; \Lambda^4_{\mathrm{QCD}}}$$

Operator-Product-Expansion:

$$\begin{aligned} D(Q) \;&=\; C_0^{\mathrm{pert.th.}}(Q^2/\mu^2) \qquad \leftarrow \text{pert. series} \\ &\quad + \frac{1}{Q^4}\left[C_{GG}(Q^2/\mu^2)\,\langle 0|G^A_{\mu\nu}G^{A,\mu\nu}|0\rangle(\mu) + \ldots\right] \;+\; \mathcal{O}(1/Q^6), \end{aligned}$$

pert. Wilson coeff.

“Gluon-Condensate” $= (0.5 \pm 0.1\ \text{GeV})^4$

Adler Function

Operator-Product-Expansion:

$$D(Q) = C_0^{\text{pert.th.}}(Q^2/\mu^2) \qquad \leftarrow \text{pert. series}$$

$$+ \frac{1}{Q^4}\left[C_{GG}(Q^2/\mu^2)\,\langle 0|G^A_{\mu\nu}G^{A,\mu\nu}|0\rangle(\mu) + \ldots\right] + \mathcal{O}(1/Q^6),$$

pert. Wilson coeff.

"Gluon-Condensate" $= (0.5 \pm 0.1\ \text{GeV})^4$

- One can fit for the power correction from experiment
- One might determine it from lattice QCD
- Power corrections "absorb" the ambituities.
- As long as the series is convergent: condensates can be determined more and more accurately
- Intrisic limit (ideal world): up to power corrections not included

Divergent Series in QCD

…. arise in the context of

- power corrections in the OPE using the MS (dim reg) scheme
- an infrared sensitive choice of renormalization conditions

OPE

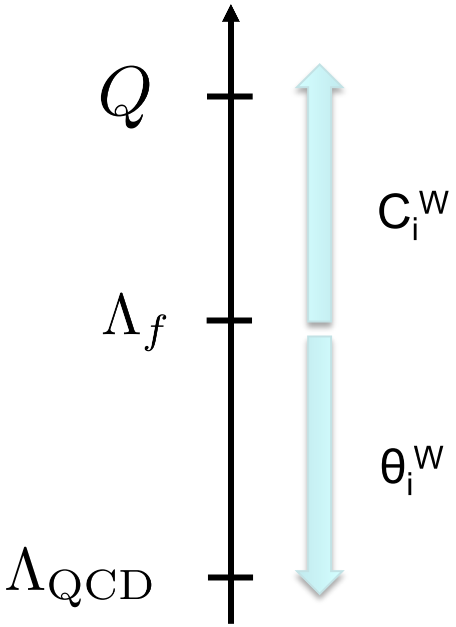
→ Standard QCD tool for perturbative computations.

Separation of process-dependent perturbative from universal nonperturbative contributions.

Consider: dimensionless observable σ for some hard scattering process at the scale Q with $Q \gg \Lambda_{\rm QCD}$

Wilson's OPE: $\Lambda_{\rm QCD} < \Lambda_f < Q$ (cutoff regulator)

$$\sigma = C_0^W(Q,\Lambda^f)\theta_0^W(\Lambda^f) + C_1^W(Q,\Lambda^f)\,\frac{\theta_1^W(\Lambda^f)}{Q^p} + \ldots .$$



- Infrared-insensitive & physical (no IR renormalons)
- Inpractical for computations

OPE

→ Standard QCD tool for perturbative computations.

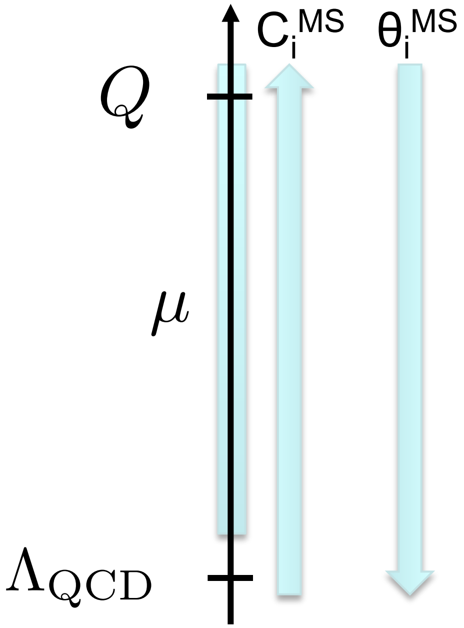
Separation of process-dependent perturbative from universal nonperturbative contributions.

Consider: dimensionless observable σ for some hard scattering process at the scale Q with $Q \gg \Lambda_{\mathrm{QCD}}$

OPE in $\overline{\text{MS}}$: (dimensional regularization)

$$\sigma = \bar{C}_0(Q,\mu)\bar{\theta}_0(\mu) + \bar{C}_1(Q,\mu)\frac{\bar{\theta}_1(\mu)}{Q^p} + \dots$$

- Infrared-sensitive & unphysical
- Practical for computations



OPE and Renormalons

Mueller
Beneke
Luke, Manohar

- Pattern in the coefficients of perturbation theory that leads to worse convergence behavior.

$$\alpha_s(k) = \alpha_s(\mu)\sum_{i=0}^{\infty}\left(\frac{\alpha_s(\mu)\beta_0}{4\pi}\right)^i \ln^i\left(\frac{\mu^2}{k^2}\right)$$

$$\bar{C}_0(\mu,Q) \sim \bar{C}_0^{\rm phys}(\mu,Q) + \underbrace{\frac{1}{Q^p}\int_0^\mu dk\, k^{p-1}\alpha_s(k)}_{\displaystyle \frac{\mu^p}{Q^p}\frac{2\pi}{\beta_0}\sum_{i=0}^{\infty}\left(\frac{\alpha_s(\mu)\beta_0}{2p\pi}\right)^{i+1} i!}$$

$$\frac{\mu^p}{Q^p}\frac{2\pi}{\beta_0}\sum_{i=0}^{\infty}\left(\frac{\alpha_s(\mu)\beta_0}{2p\pi}\right)^{i+1} i! \;\longrightarrow\; B(u) \sim -\frac{\mu^p}{Q^p}\frac{2\pi}{\beta_0}\frac{1}{u-\frac{p}{2}}$$

$\Downarrow$

numerical cancellation

$\Uparrow$

$$\frac{\bar{\theta}_1(\mu)}{Q^p} \sim \frac{\bar{\theta}_1^{\rm phys}(\mu)}{Q^p} + \underbrace{\frac{1}{Q^p}\int_\mu^\infty dk\, k^{p-1}\alpha_s(k)}_{\displaystyle -\frac{\mu^p}{Q^p}\frac{2\pi}{\beta_0}\sum_{i=0}^{\infty}\left(\frac{\alpha_s(\mu)\beta_0}{2p\pi}\right)^{i+1} i!}$$

$$-\frac{\mu^p}{Q^p}\frac{2\pi}{\beta_0}\sum_{i=0}^{\infty}\left(\frac{\alpha_s(\mu)\beta_0}{2p\pi}\right)^{i+1} i! \;\longrightarrow\; B(u) \sim \frac{\mu^p}{Q^p}\frac{2\pi}{\beta_0}\frac{1}{u-\frac{p}{2}}$$

OPE in an R-Scheme

→ Define Wilson coefficients with infrared subtractions

Reintroduce power dependence on the cutoff scale.

OPE in R-schemes:

Subtracts bad renormalon contributions

$$C_0(Q,R,\mu) \quad = \bar{C}_0(Q,\mu) - \delta C_0(Q,R,\mu)$$

$$\delta C_0(Q,R,\mu) = \boxed{\left(\frac{R}{Q}\right)^p} \sum_{n=1}^{\infty} d_n\left(\frac{\mu}{R}\right)\left[\frac{\alpha_s(\mu)}{(4\pi)}\right]^n \qquad d_n(\mu/R) = \sum_{k=0} d_{nk} \ln^k(\mu/R)$$

$$\bar{\theta}_1(\mu) = \theta_1(R,\mu) - [Q^p \delta C_0(Q,R,\mu)]\bar{\theta}_0(\mu)$$

$$\boxed{\sigma = C_0(Q,R,\mu)\bar{\theta}_0(\mu) + \bar{C}_1(Q,\mu)\frac{\theta_1(R,\mu)}{Q^p} + \ldots}$$

Avoids numerical renormalon cancelation beween different terms in the OPE.

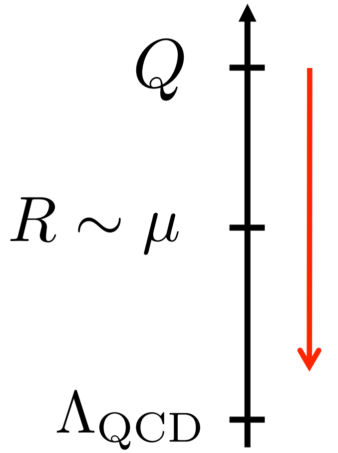
OPE in an R-Scheme

$$\sigma = C_0(Q,R,\mu)\bar{\theta}_0(\mu) + \frac{\theta_1(R,\mu)}{Q^p} + \ldots$$

Scale setting:

$$C_0(Q,R,\mu):\quad \ln(\tfrac{\mu}{Q})\,,\ \ln(\tfrac{\mu}{R}) \quad \longrightarrow \quad R\sim\mu\sim Q$$

$$\bar{\theta}_1(\mu)\sim\theta_1(R,\mu)\sim\Lambda^p_{\rm QCD}:\quad \ln(\tfrac{\mu}{\Lambda_{\rm QCD}})\,,\ \ln(\tfrac{\mu}{R}) \quad \longrightarrow \quad R\sim\mu\gtrsim\Lambda_{\rm QCD}$$



→ R and μ have to be evolved simultaneously.

Here we take $R=\mu$.

OPE in an R-Scheme

$$\ln C_0(Q,R,\mu) \equiv \sum_{n=1}^{\infty} \left\{ a_n\left(\tfrac{\mu}{Q}\right) - \tfrac{R^p}{Q^p} a_n\left(\tfrac{\mu}{R}\right) \right\} \frac{\alpha_s^n(\mu)}{(4\pi)^n}$$

R-evolution:

$$R\frac{d}{dR} \ln C_0(Q,R,R) = \underset{\downarrow}{\bar\gamma[\alpha_s(R)]} - \left(\frac{R}{Q}\right)^p \underset{\downarrow}{\gamma[\alpha_s(R)]}$$

$\overline{\text{MS}}$ anomalous dimension

R-evolution equation

renormalon-free !

$$\gamma[\alpha_s] = \sum_{n=0}^{\infty} \gamma_n \left(\tfrac{\alpha_s(R)}{4\pi}\right)^{n+1}$$

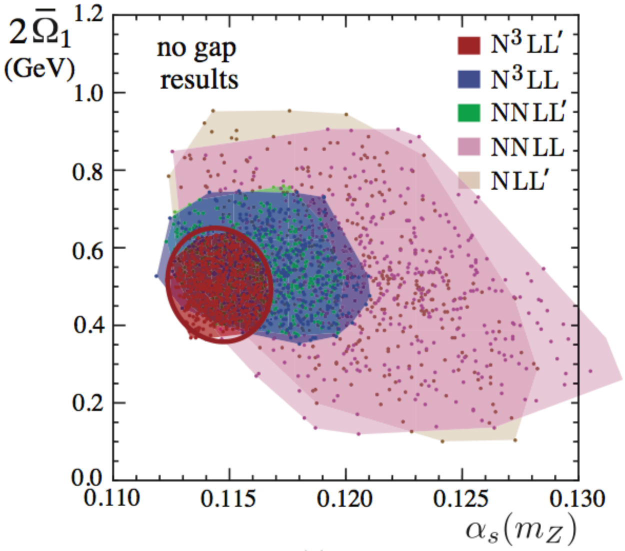
$$\gamma_{n-1} \approx p\left[a_n - \frac{2(n-1)}{p} a_{n-1}\beta_0\right]$$

- $\bar\gamma[\alpha_s(R)] = 0$: improved pert. convergence for small $R \sim \mu$

Strong Coupling from Thrust

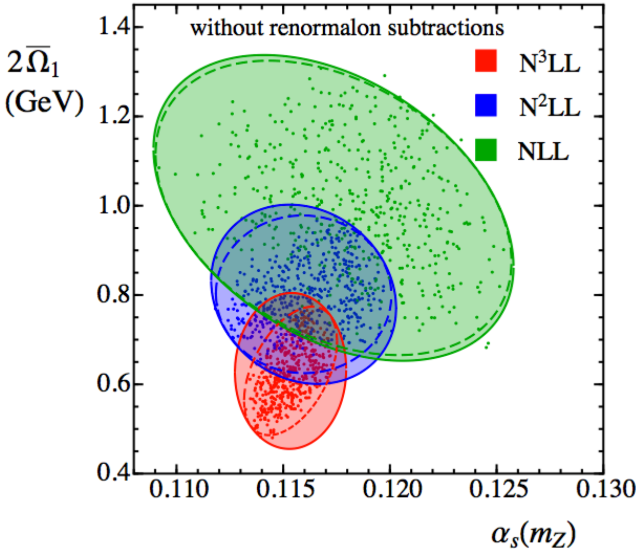
Thrust distribution:

Abbate, Ficking, AH, Mateu, Stewart
Phys. Rev.D83 (2011) 074021



Thrust moments:

Abbate, Ficking, AH, Mateu, Stewart
arXiv:1204.5746



OPE in $\overline{\text{MS}}$: no renormalon subtraction

Strong Coupling from Thrust

Thrust distribution:

Abbate, Ficking, AH, Mateu, Stewart
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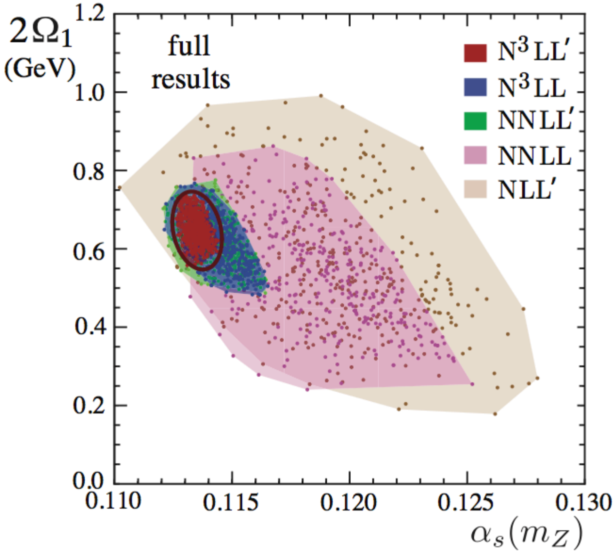
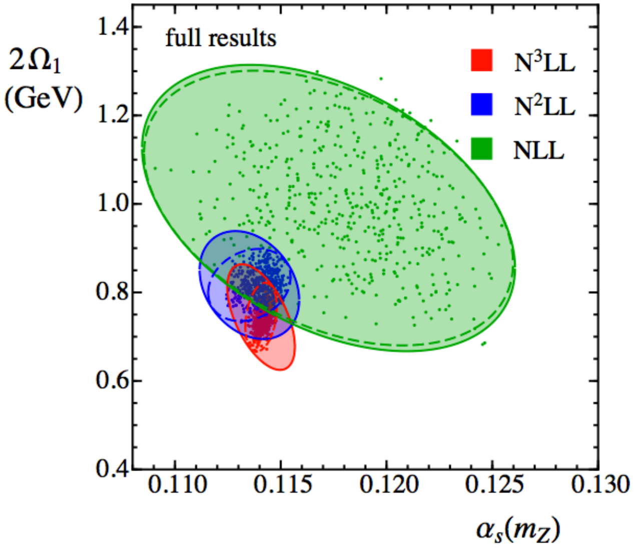


$$\alpha_s(m_Z) = 0.1135 \pm 0.0002(ex) \pm 0.0010(th)$$

Thrust moments:

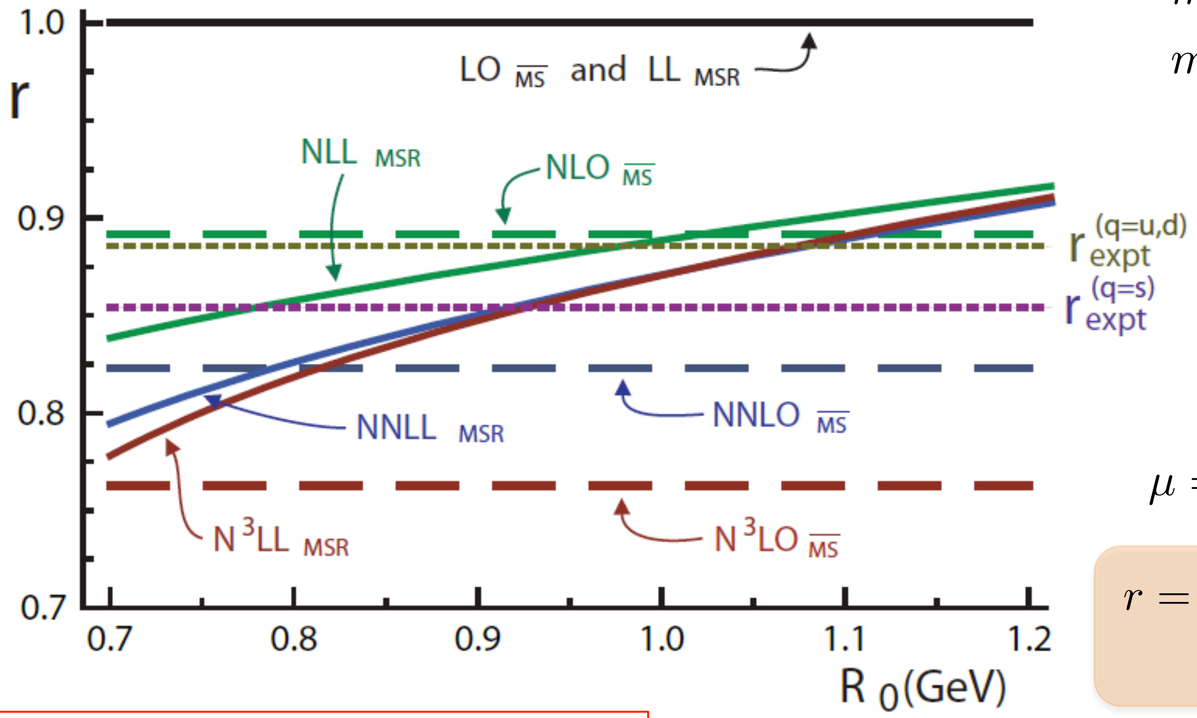
Abbate, Ficking, AH, Mateu, Stewart
arXiv:1204.5746



$$\alpha_s(m_Z) = 0.1141 \pm 0.0004(ex) \pm 0.016(th)$$

Meson Mass Differences (p.th.)

$$r_{p.th.} = \frac{C_G(m_b, R_1, R_1)U_R(m_b, R_1, R_0)}{C_G(m_c, R_1, R_1)U_R(m_c, R_1, R_0)}$$



$$m_b = 4.7 \text{ GeV}$$

$$m_c = 1.6 \text{ GeV}$$

$$\mu = R_1 = \sqrt{m_b m_c}$$

$$r = 0.860 \pm (0.065)_{\Sigma_\rho}$$
$$\pm(0.008)_{\text{pert.}}$$

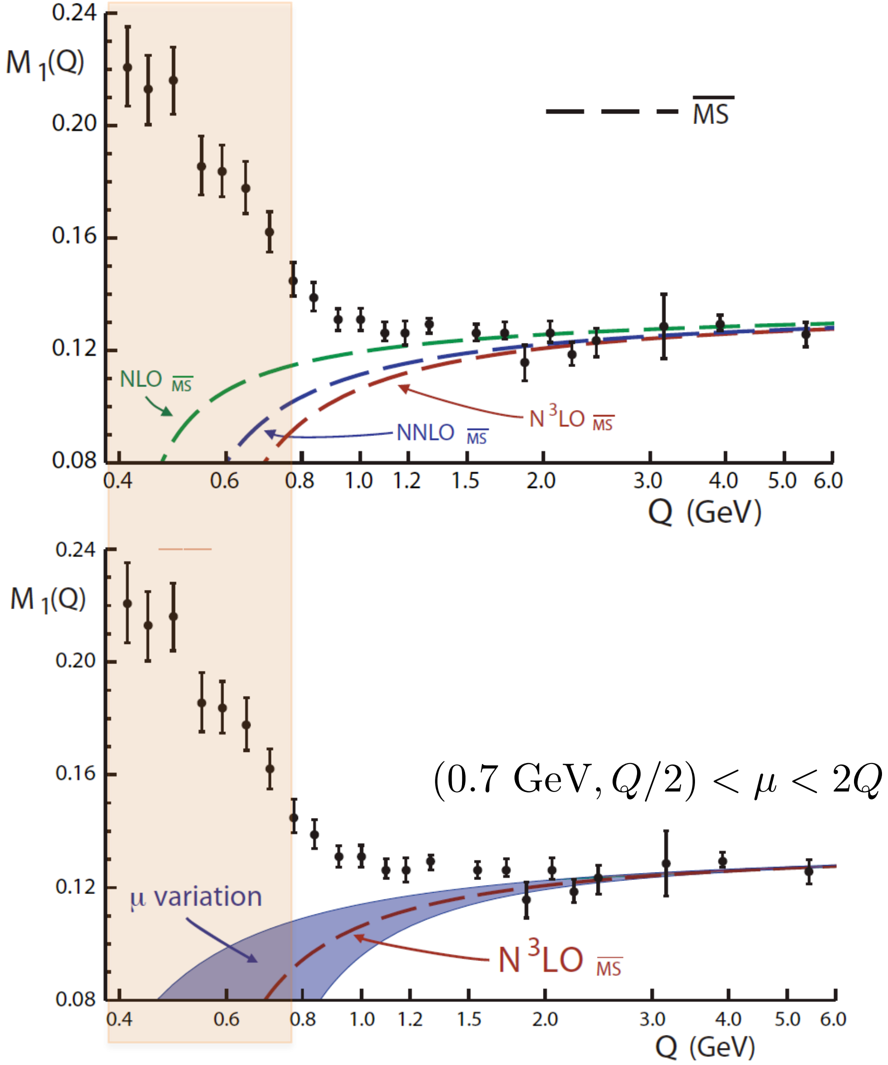
MSR: $\Delta r_{p.th.} = 0.008$

$$\frac{1}{2}\sqrt{m_b m_c} < R_1 < 2\sqrt{m_b m_c}$$

- perturbative expansion in good shape
- MSR power corrections small and stable
- R_0 dependence estimates OPE uncertainty

Ellis Jaffe Sum Rule (p.th.)

$$M_{1,\text{p.th.}}^{p,\overline{MS}}(Q) = \left[\,\bar{C}_B(Q,\mu)\,\theta_B + \bar{C}_0(Q,\mu)\,\frac{\hat{a}_0}{9}\,\right]$$



$$(0.7\ \mathrm{GeV}, Q/2) < \mu < 2Q$$

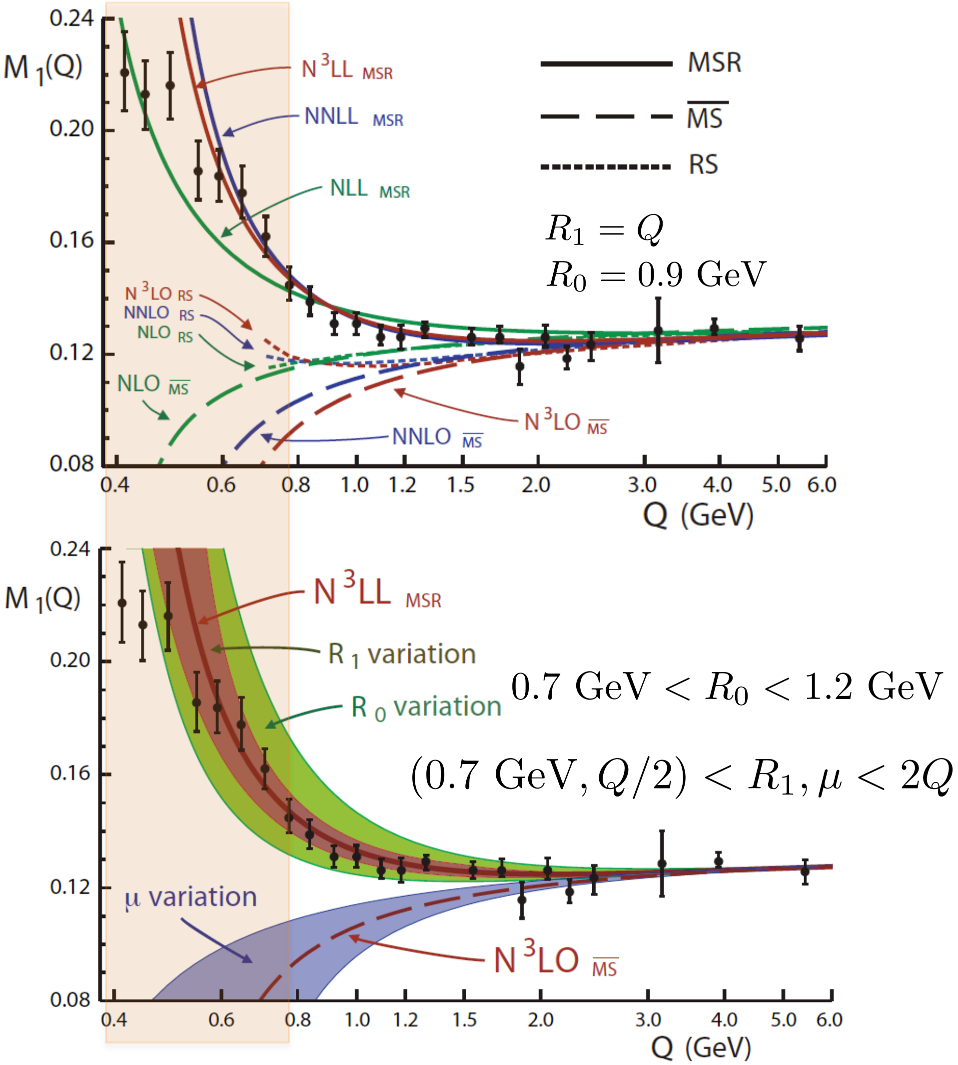
Ellis Jaffe Sum Rule (p.th.)

$$M^{p,\overline{MS}}_{1,\text{p.th.}}(Q) = \left[\,\bar{C}_B(Q,\mu)\,\theta_B + \bar{C}_0(Q,\mu)\,\frac{\hat{a}_0}{9}\,\right]$$

$$M^{p,\text{MSR}}_{1,\text{p.th.}}(Q) = \left[\,\bar{C}_B(Q,R_0)\,\theta_B + \bar{C}_0(Q,R_0)\,\frac{\hat{a}_0}{9}\,\right]$$

MSR scheme:

- perturbative series excellent
- power corrections small
- R_0 dependence estimates OPE uncertainty



$R_1 = Q$

$R_0 = 0.9$ GeV

$0.7\ \text{GeV} < R_0 < 1.2\ \text{GeV}$

$(0.7\ \text{GeV}, Q/2) < R_1, \mu < 2Q$

Heavy Quark Masses

- Important QCD input parameters for SM predictions
- Confinement $\Longrightarrow$ quark masses not physical observables

$$\mathcal{L}_{\rm QCD} = \tfrac{1}{4} G_{\mu\nu}^2 + \sum_i \bar{\psi}_i \left(\not{D} - \boxed{M_i} \right) \psi$$

  - ▷ defined as formal parameters in QCD action
  - ▷ (renormalization) scheme dependent
  - ▷ to be well defined: $m_q^{\rm schemeA} = m_q^{\rm schemeB}(1 + \alpha_s + \alpha_s^2 + \ldots)$
  - ▷ heavy quark masses: $m_Q > \Lambda_{\rm QCD}$
  - ▷ some schemes more appropriate than others

Heavy Quark Masses

- Important QCD input parameters for SM predictions
- Confinement $\Longrightarrow$ quark masses not physical observables

$$\mathcal{L}_{\rm QCD} \;=\; \tfrac{1}{4}\,G_{\mu\nu}^2 \;+\; \textstyle\sum_i \bar{\psi}_i\,(\,\not{D} - \enclose{circle}{M_i})\,\psi$$

  - ▷ defined as formal parameters in QCD action

- **pole mass scheme:**
  - ▷ 2-point-function: $\not{p} - m_Q + \Sigma(m_Q^2) \equiv \not{p} - m_Q^{\rm pole}$
  - ▷ scheme-invariant, gauge-invariant
- **$\overline{\text{MS}}$ scheme:**
  - ▷ only $1/\epsilon$ divergences absorbed in $\overline{\text{MS}}$ mass
  - ▷ “running” $\overline{m}_q(\mu)$, gauge invariant

Heavy Quark Masses

→ Let us consider $\Delta = m_Q^{\text{pole}} - \overline{m}_Q(\mu)$:

$$B_\Delta(u) \sim -C_F \frac{2}{b_0} e^{-C/2} \mu \frac{1}{(u - \frac{1}{2})} + \cdots$$

(additional poles at $u = 1, \frac{3}{2}, 2, \ldots$)

→ Example: bottom quark

$$\overline{m}_b(\overline{m}_b) = 4.2 \text{ GeV}\,, \quad \alpha_s(4.7\,\text{GeV}) = 0.216$$

$$m_b^{\text{pole}} = (4.2 + 0.39 + 0.20 + 0.16 + \ldots) \text{ GeV}$$

Heavy Quark Masses

→ Let us consider $\Delta = m_Q^{\rm pole} - \overline{m}_Q(\mu)$:

$$B_\Delta(u) \sim -C_F \frac{2}{b_0} e^{-C/2} \mu \frac{1}{(u - \frac{1}{2})} + \cdots$$

(additional poles at $u = 1, \frac{3}{2}, 2, \ldots$)

$\frac{1}{u-1/2}$ ↔ Ambiguity of $\mathcal{O}(\Lambda_{\rm QCD})$, “Pole mass renormalon”

- Is there a corresponding non-perturbative matrix element?
  → NO!
- Is the ambiguity relevant? → that depends . . .

Top Mass at the ILC

Linear Collider

threshold: $\sigma(e^+e^- \to t\bar{t})$ at $\sqrt{s} \approx 350$ GeV

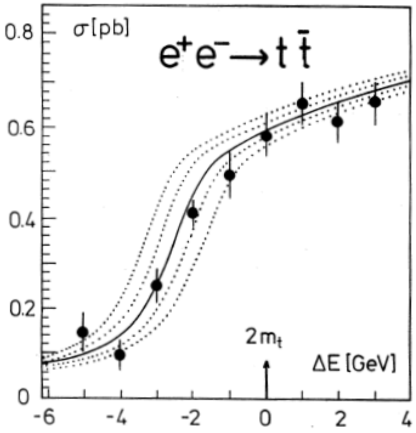
$\to \; V(r) = -\frac{C_F \alpha_s}{r} - \frac{g^2_{\rm tth}}{r} e^{-m_h r}$

$\mathcal{O}(5 - 10\,\%)$ correction

$\to \; \delta g_{\rm tth}/g_{\rm tth} = 20 - 50\,\%$

$\delta M_{1S} \le 50$ MeV, $\delta\Gamma_t \sim 30$ MeV, $\delta\alpha_s \sim 0.001$

$(\mathcal{L} = 300\,\text{fb}^{-1}$, $m_h \lesssim 120$ GeV)

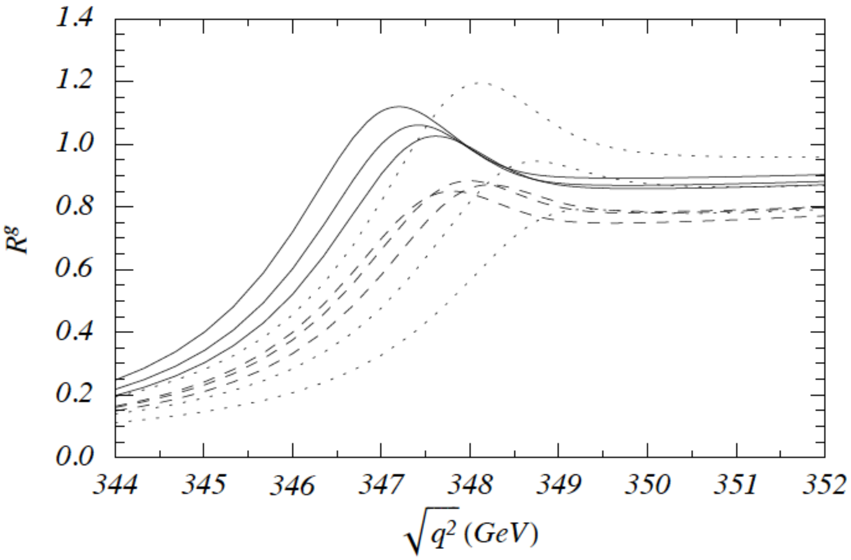


Martinez, Miquel

- perfect knowledge of luminsity spectrum assumed
- Yukawa coupling only for light Higgs
- depends on knowledge other parameters, QCD theory

Top Mass at the ILC

→ Theoretical Prediction in the pole mass scheme:



$m_t^{\rm pole} = 175$ GeV

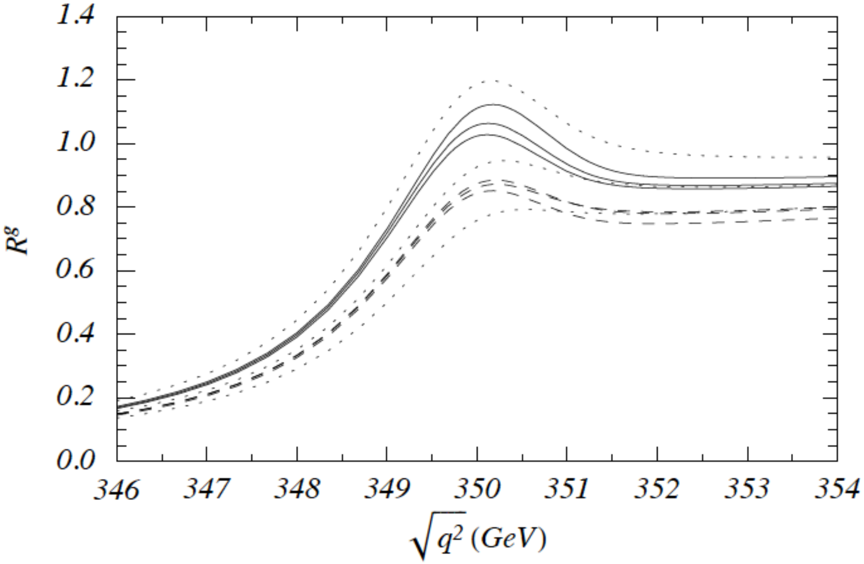
LO, NLO, NNL0

→ Pole mass obtained from experiments:

- order-dependent
- renormalization group independent
- uncertainties: $\delta m_t^{\rm pole} \sim 200 - 300$ MeV

Heavy Quark Masses

→ Theoretical Prediction in the 1S mass scheme:



$\overline{M}_{1S} = 175$ GeV

LO, NLO, NNL0

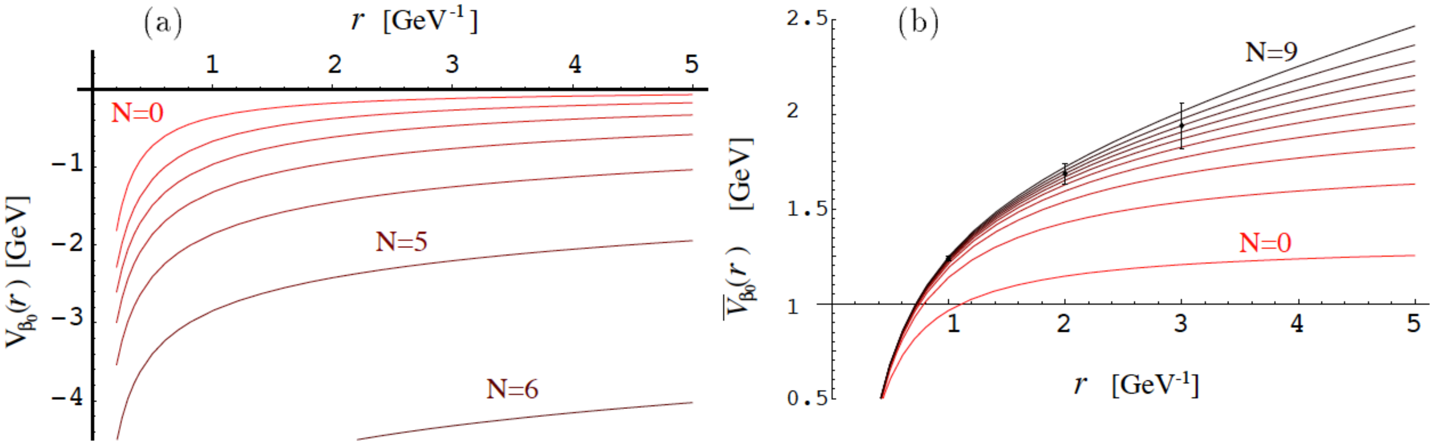
→ 1S mass obtained from experiments:

- very good convergence small corrections at low orders
- used for experimental simulations
- 1S mass is very close to a direkt physical observable

Top Mass at the ILC

→ Let us replace the pole mass by the $\overline{\text{MS}}$ mass:

$$E_{\text{stat}} = 2m_Q^{\text{pole}} + V(r) = 2\overline{m}_Q + [2\Delta + V(r)]\ , \quad V(r) = \frac{-C_F\alpha_s}{r}$$



Conclusions

- The main aim of this talk was to make you aware of patterns of perturbation theory at higher order that need considerations way beyond methods related to how to compute them.

- Renormalons are divergence patterns of perturbative coefficients in gauge theories (with massless gauge boson) that can spoil the convergence of perturbative series. This is particularly important in QCD where the coupling is relatively large, even for high energies.

- Understanding the concept of renormalons can provide guidelines how to deal with these divergence patterns, and how to improve the behavior of perturbation theory.