Heavy Quark Production in Deep-Inelastic Scattering

Johannes Blümlein

in collaboration with J. Ablinger (JKU), I. Bierenbaum (U. Hamburg), A. De Freitas (DESY), A. Hasselhuhn (DESY), S. Klein (RWTH), C. Schneider (JKU), F. Wißbrock (DESY)

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References:

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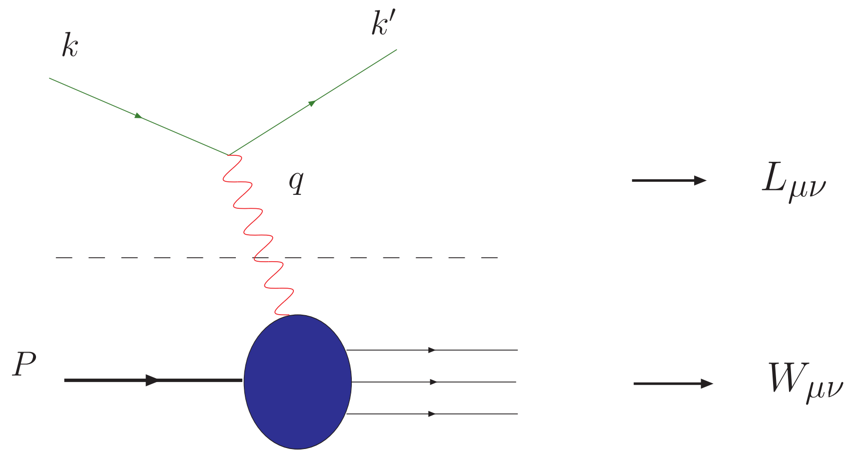
1. Introduction

Deep–Inelastic Scattering (DIS):



$$Q^2 := -q^2, \quad x := \frac{Q^2}{2pq} \quad \text{Bjorken–x}$$

$$\nu := \frac{Pq}{M} \; ,$$

$$\frac{d\sigma}{dQ^2 \; dx} \sim W_{\mu\nu} L^{\mu\nu}$$

The partonic picture of the proton at short distances:

[Feynman, 1969; Bjorken, Paschos, 1969.]

The operator picture of the proton at short distances - light cone expansion:

[Wilson, 1968; Bjorken 1969; Brandt & Preparata 1970; Frishman 1971.]

Notion of Twist (necessary): consistent renormaliztion [Gross and Treiman, 1971]

$\Longrightarrow$ Both concepts yield the same result at the level of Twist $\tau = 2$

- Some care is needed in the polarized case : $g_2(x, Q^2)$.

When is a parton ?

- The applicability of the parton picture rests on the comparison of two times.

[e.g. Drell, Yan, Levy, 1969]

- Consider the infinite momentum frame to describe deep-inelastic scattering, moving with momentum P
- The 2 characteristic times are :

$\tau_{\rm int}$ - the interaction time of the virtual gauge boson with the hadron.

$\tau_{\rm life}$ - the life-time of individual partons

- This is a non-covariant description, turning to a covariant one later.

$$\tau_{\rm int} \sim \frac{1}{q_0} = \frac{4Px}{Q^2(1-x)}$$
$$\tau_{\rm life} \sim \frac{1}{\sum_i (E_i - E)} = \frac{2P}{\sum_i (k_{\perp,i}^2 + m_i^2)/x_i - M^2}$$

- The ratio of these 2 times is covariant & independent of P.

When is a parton ?

- The single parton model is applicable iff:

$$\frac{\tau_{\rm life}}{\tau_{\rm int}} \gg 1 \ .$$

Light partons only: $m_i^2, M^2 \approx 0, \forall k_{\perp,i}^2 \equiv k_\perp^2, x_1 = x, x_2 = 1 - x$ as a model.

$$\frac{\tau_{\rm life}}{\tau_{\rm int}} = \frac{Q^2(1-x)^2}{2k_\perp^2} \gg 1$$

i.e.: $Q^2 \gg k_\perp^2, x$ neither close to 1 or 0

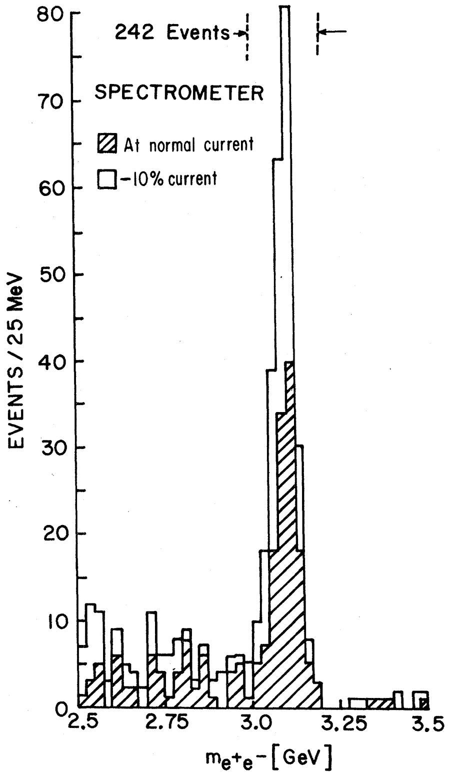
The latter condition follows from $|P_{\rm eff}| \gg 1$.

- There is no parton model at low Q^2. In this region also the light-cone expansion breaks down. This is the place, where vector meson dominance and other thoroughly non-perturbative phenomena live.

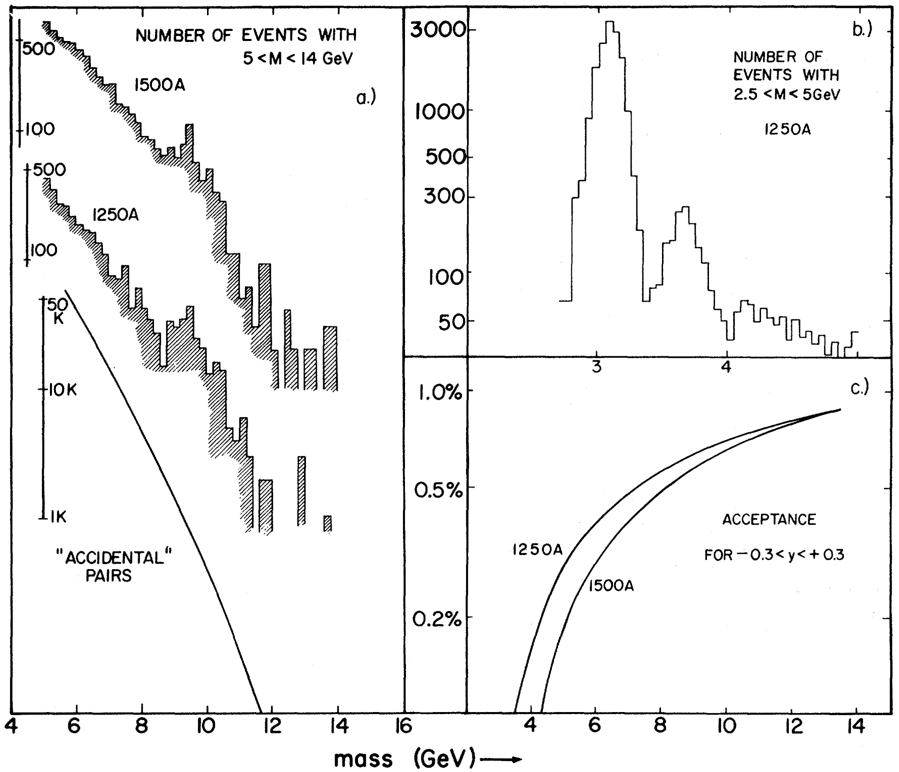
There are neither small- nor large-x parton densities.

- Light partons: $u, d, s, \overline{u}, \overline{d}, \overline{s}, g$

The Discovery of Heavy Quarks



$\mathcal{J}/\Psi$ [Aubert *et al.*, 1974.]



Υ [Herb *et al.*, 1977.]

- Masses of charm and bottom [PDG, 2006.]: $m_c \approx 1.3\,\text{GeV}$, $m_b \approx 4.2\,\text{GeV}$

When is a parton ?

- strange quarks, despite $m_s \sim 100$ MeV $\Lambda_{\rm QCD}$, is dealt with as massless.

Can heavy partons live in the nucleon ? [Brodsky, Hoyer, Peterson, Sakai, 1980]

- Intrinsic Charm
- Use old-vashioned perturbation theory.

$$Prob \sim \frac{1}{(M^2 - \sum_{i=1}^{5} m_\perp^2/x_i)^2}, \qquad m_{4,5} \to \infty$$

A more general question:

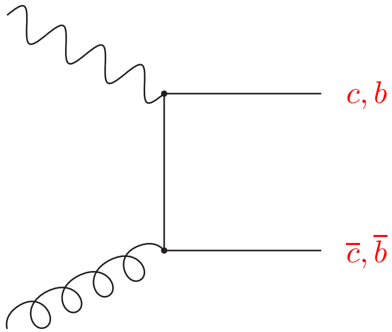
What is the heavy flavor content of DIS Structure Functions ?

Introduction

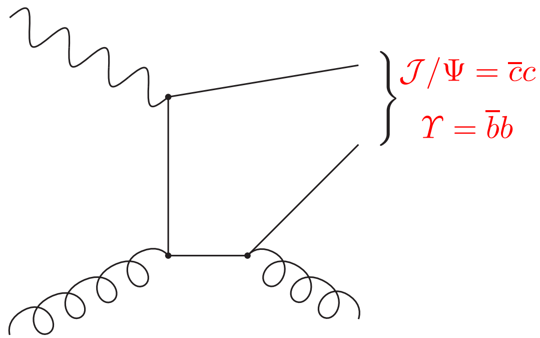
- Virtual Heavy Quark Loops: c,b,t.
- Heavy Quark final states in $F_i(x,Q^2), g_i(x,Q^2)$: $\theta(W^2-(2m_H+m_p)^2)$ and stronger.
- Both effects lead to different scaling violations in wide ranges of Q^2.
- Another Aspect: Inclusive Effects vs. Exclusive Reactions.
- Inclusive Effects: sum over all hadronic states in Fock space.
- Exclusive Reactions: tag heavy flavor in the final states.

These emerge as open heavy flavor or (in-)elastic heavy resonance production.

- open c(b) production:
  $D_u = \overline{u}c$, ...
  $B_u = \overline{u}b$, ...

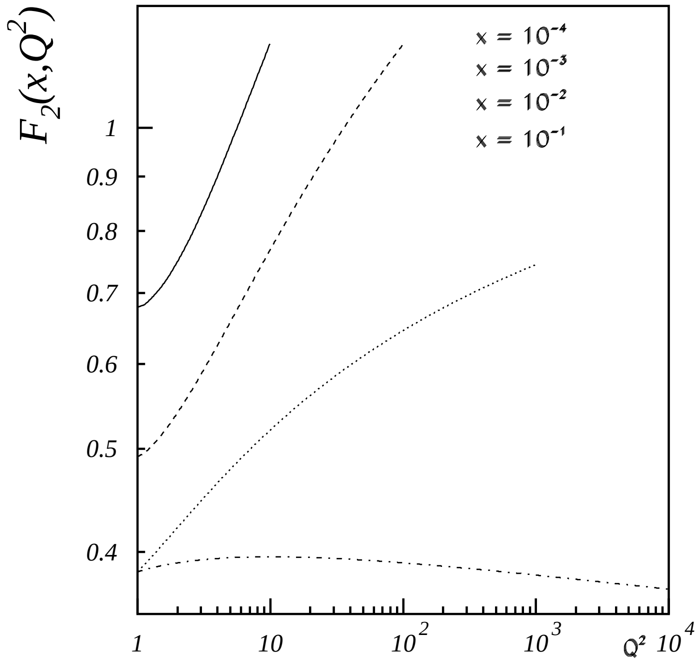


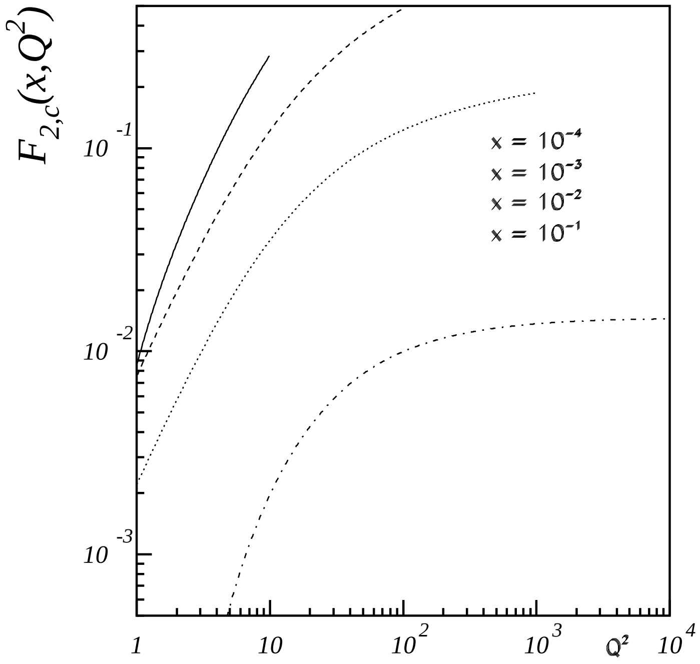
- heavy quark resonances:
  $\overline{c}c = \mathcal{J}/\Psi$
  $\overline{b}b = \Upsilon$.



Introduction

Let us compare the scaling violations in the massless and massive case.





LO charm contributions : PDFs from [Alekhin, Melnikov, Petriello, 2006.]

Introduction

- J/ψ and Υ production in the color singlet and octet contributions.

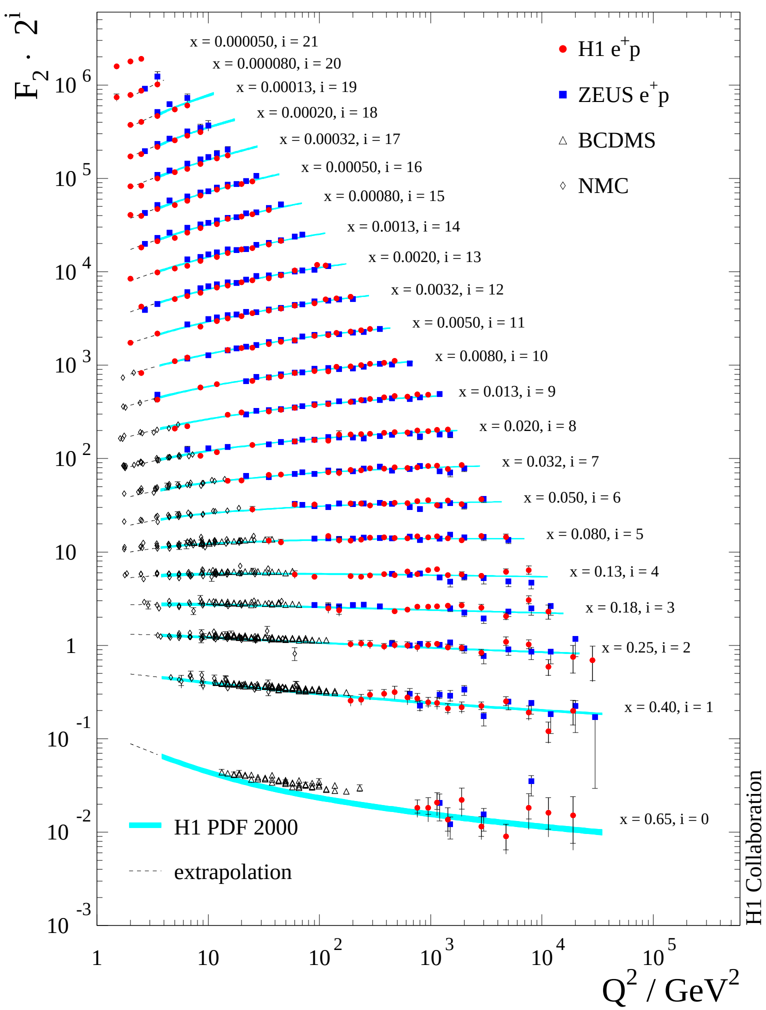
$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2} 64 g_s^4 e_Q^2 M_J^2 A^2 \frac{1}{12} \frac{s^2(s-M_J^2)^2 + t^2(t-M_J^2)^2 + u^2(u-M_J^2)^2}{(s-M_J^2)^2(t-M_J^2)^2(u-M_J^2)^2}$$

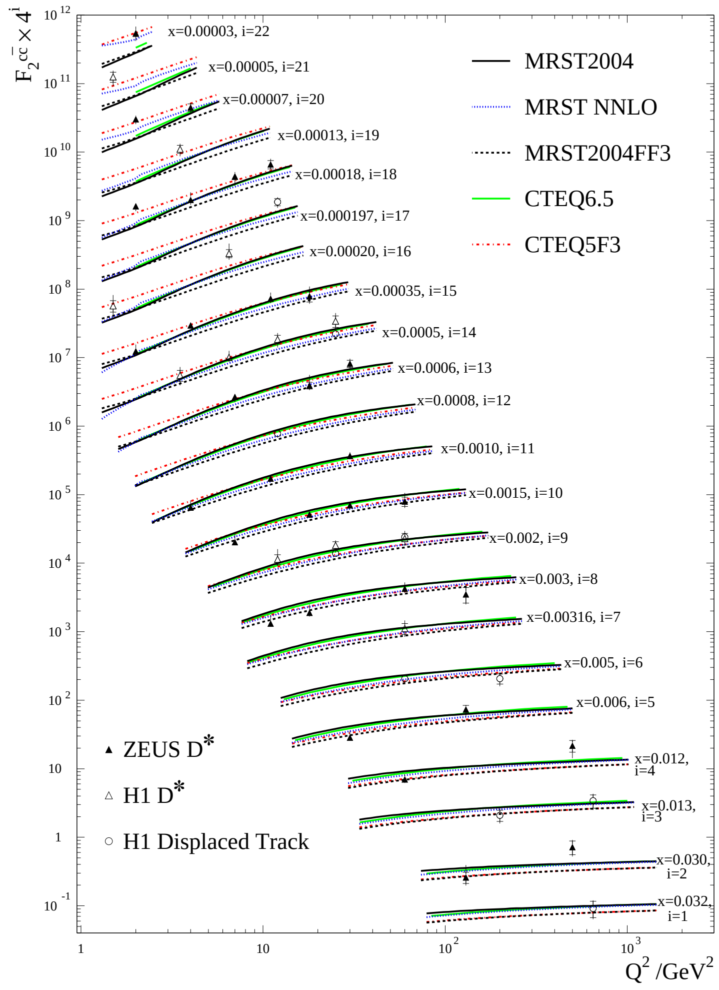
[Berger & Jones, 1980; Rückl & Baier, 1983; Körner, Cleymans, Kuroda & Gounaris, 1983; Krämer, Zunft & Steegborn, 1994; Krämer, 1995, Kniehl & Kramer, 1997; Cacciari, Greco & Krämer, 1997]

- Observation of charmonium in DIS [Aubert *et al.*, 1983.]
- These processes are rather sensitive to the gluon distribution $G(x, Q^2)$.
- Can one probe m_t due to virtual excitations?

[I do not discuss diffractive, pomeron-induced or photo & VMD production of open and resonant heavy flavor. [Jung, Schuler, et al. 1990/92]]

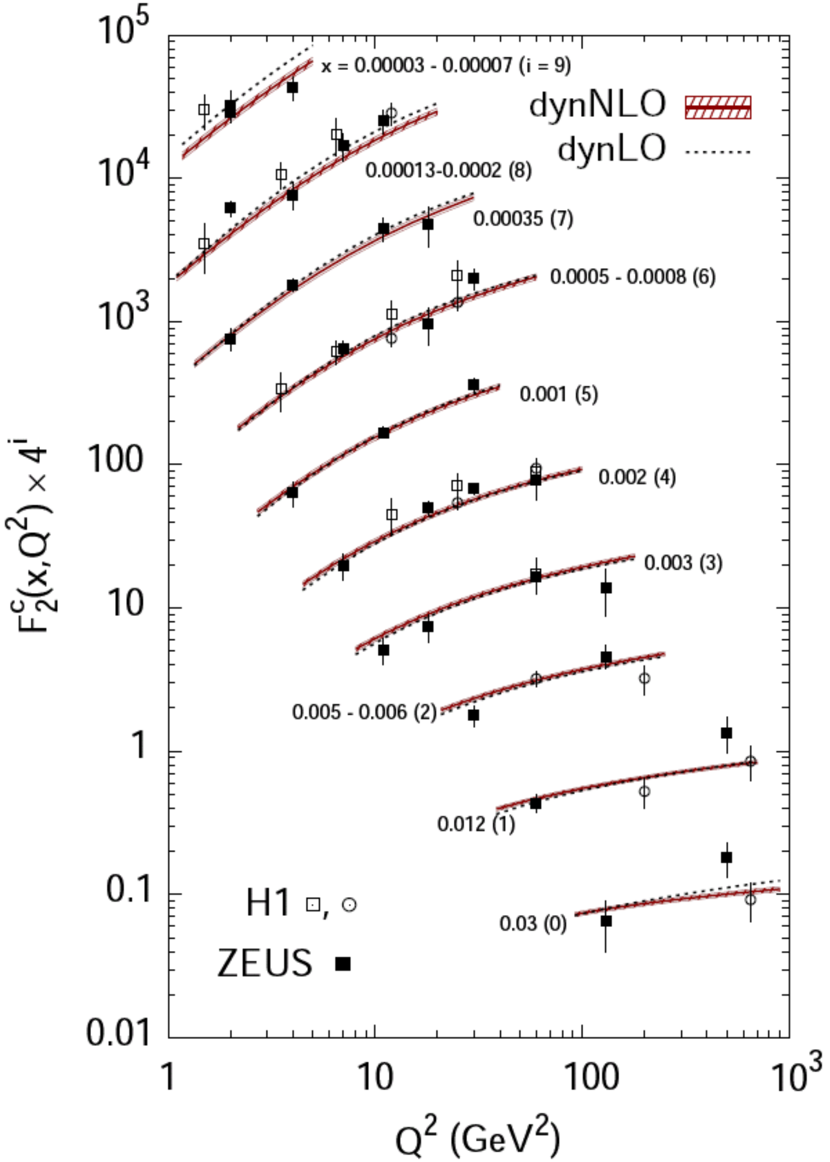
2. Light and Heavy Quark Contributions



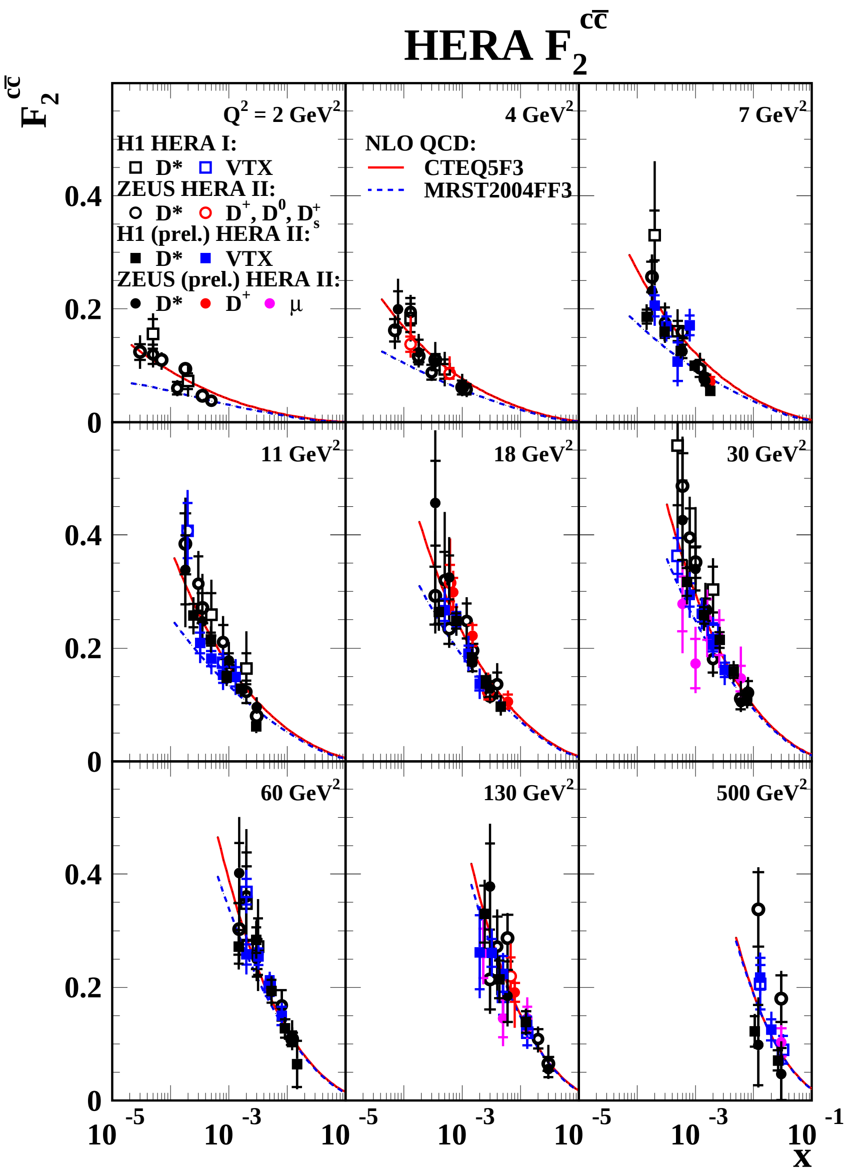


[Thompson, 2007]

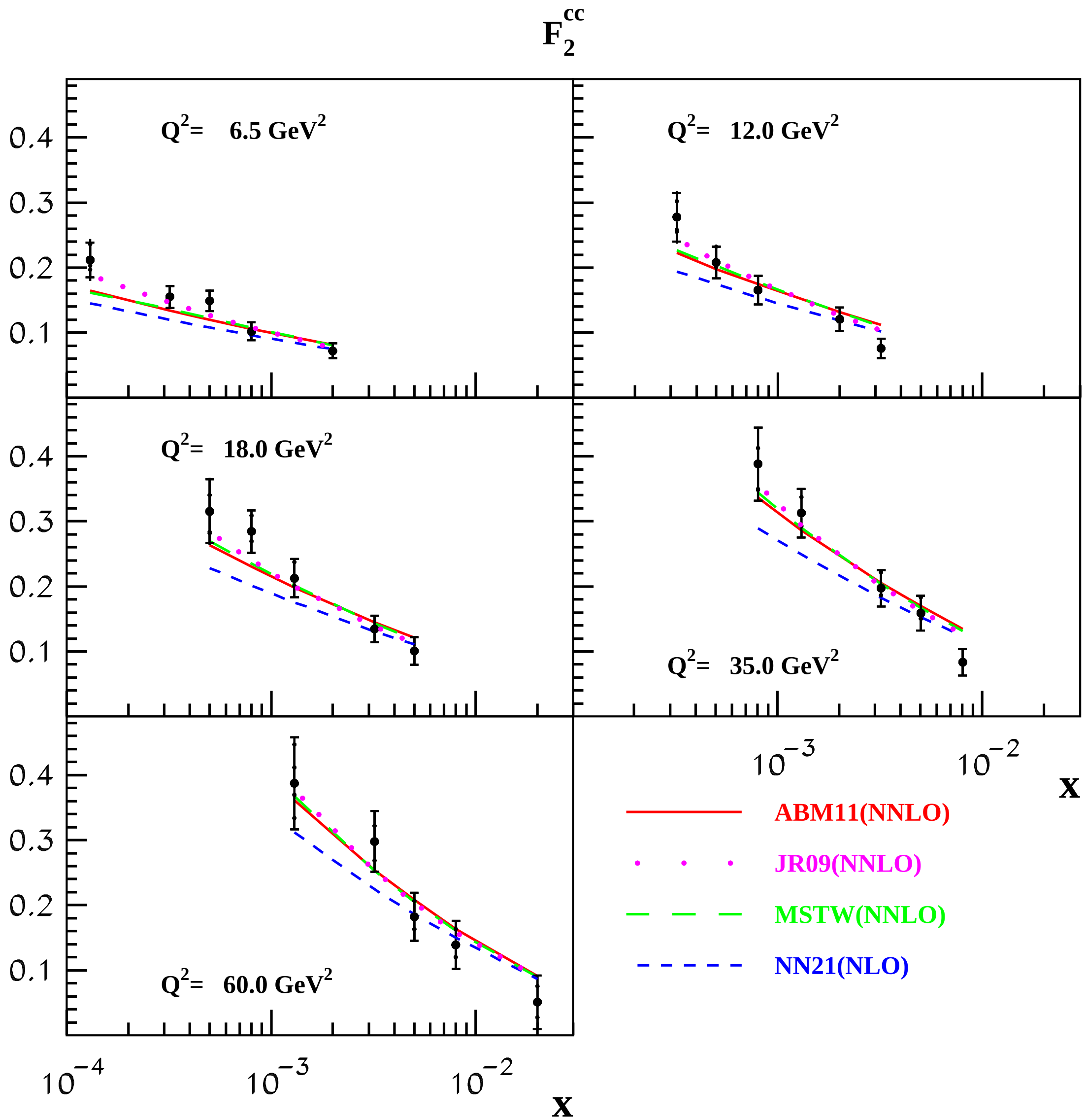
High statistics for F_2 and $F_2^{c\bar{c}}$ $\Longrightarrow$ Accuracy will increase in the future.



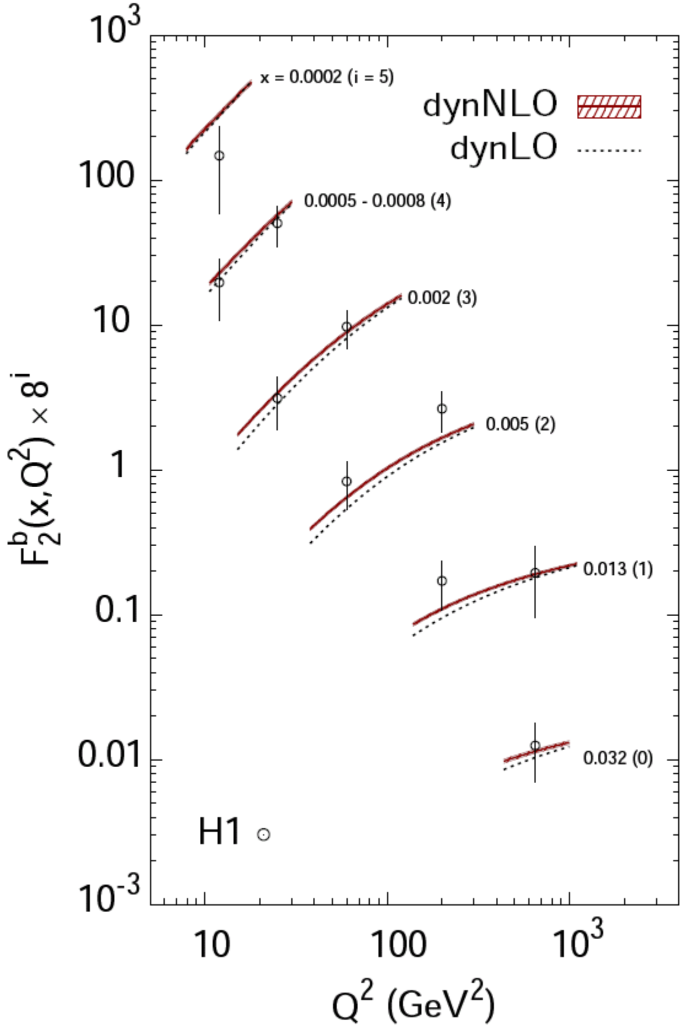
[Glück, Jimenez–Delgado, Reya, 2007.]



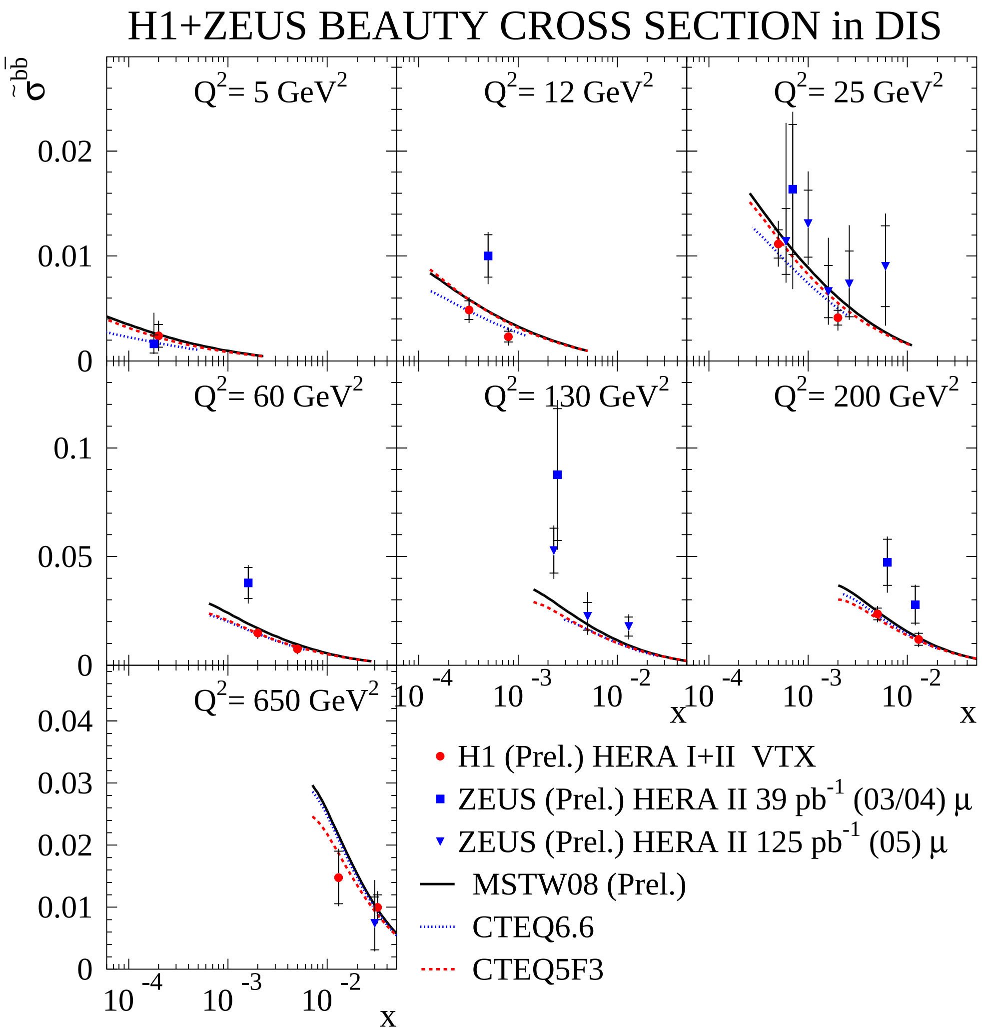
[Krüger (H1 and Z. Coll.), 2008.]



[data from: H1 coll. arXiv:1106.1028.]



[Glück, Jimenez–Delgado, Reya, 2007.]



[Krüger (H1 and Z. Coll.), 2008.]

Intrinsic Charm

• Charm quarks as massive partons in the IMF in the infinite mass limit.
[Why not b and t too ?]

$$P(x_4, x_5) = N_5 \frac{x_4^n x_5^n}{(x_4 + x_5)^n} \theta(W^2 - (2m_H + m_p)^2)$$

$$P(x) = \frac{N_5}{2} x^2 \left[\frac{1}{3}(1-x)(1+10x+x^2) + 2x(1+x)\ln(x)\right] \theta(W^2 - (2m_H + m_p)^2),$$

$$n = 2, N_5 = 36$$

$n = 2$ corresponds to the usual Fock space [There may be other phenomenological choices.]; N_5 is chosen to yield a $\sim 1\%$ overall effect.

• NLO calculation : [Hoffmann and Moore, 1983]

• Phenomenological data analyzes: [Smith, Harris, R. Vogt, 1996, Gunion and R. Vogt, 1997]

Effect: $\leq 1\%$ compatible with 0 within errors.

Slow Rescaling

- Neutral or Charged currents hit massive fermion lines m_i, m_f inside a massive target M. [Georgi & Politzer 1976; Barnett 1976; K. Ellis, Parisi, Petronzio, 1976; Barbieri, J. Ellis, Gaillard, G. Ross, 1976; Brock, 1980]

$$\xi = \frac{Q^2 + m_f^2 - m_i^2 + \sqrt{(Q^2 + m_f^2 - m_i^2)^2 + 4m_i^2 Q^2}}{2\left(\nu + \sqrt{\nu^2 + M^2 Q^2}\right)}$$

$$\xi(m_i = 0, M = 0) = x\left[1 + \frac{m_f^2}{Q^2}\right]$$

$$\xi(m_i = m_f, M = 0) = \frac{x}{2}\left[1 + \sqrt{1 + \frac{4m_f^2}{Q^2}}\right]$$

- Purely kinematic approach. Very early, too naive to substitute the description of heavy flavor.
- Still some numeric success for $s(d) \to \bar{c}$ inclusive charged current transitions.
- Heavy Quark parton densities cannot be coined in this way.

The hadronic tensor cannot be calculated perturbatively. Using symmetry considerations, it can be decomposed into several scalar structure functions. For DIS via single photon exchange, it is given by:

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle$$

$$\text{unpol.}\left\{ \; = \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2}g_{\mu\nu}\right) F_2(x,Q^2)\right.$$

$$\text{pol.}\left\{ \; -\frac{M}{2Pq}\varepsilon_{\mu\nu\alpha\beta}q^\alpha\left[s^\beta g_1(x,Q^2) + \left(s^\beta - \frac{sq}{Pq}p^\beta\right) g_2(x,Q^2)\right]\right. .$$

For light quarks: In the Bjorken limit, $\{Q^2,\ \nu\} \rightarrow \infty$, x fixed, at twist $\tau = 2$–level:

$$\underbrace{F_i(x,Q^2)}_{\text{structure functions}} = \sum_j \underbrace{C_i^j\left(x, \frac{Q^2}{\mu^2}\right)}_{\text{Wilson coefficients, perturbative}} \otimes \underbrace{f_j(x,\mu^2)}_{\text{parton densities, non-perturbative}},$$

This representation applies both to light and heavy quarks.

Evolution of Light Quark Distributions

- The scaling violations are described by the splitting functions $P_{ij}(x, a_s)$.
- They describe the probability to find parton i radiated from parton j and carrying its momentum fraction x.
- They are related to the anomalous dimensions via a Mellin–Transform:

$$\mathbf{M}[f](N) := \int_0^1 dz z^{N-1} f(z) \; , \; \gamma_{ij}(N, a_s) := -\mathbf{M}[P_{ij}](N, a_s) \; .$$

- The splitting functions govern the scale–evolution of the parton densities.

$$\frac{d}{d \ln Q^2} \begin{pmatrix} \Sigma(N, Q^2) \\ G(N, Q^2) \end{pmatrix} = - \begin{pmatrix} \gamma_{qq} & \gamma_{qg} \\ \gamma_{gq} & \gamma_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma(N, Q^2) \\ G(N, Q^2) \end{pmatrix} \; ,$$

$$\frac{d}{d \ln Q^2} q_{NS}(N, Q^2) = -\gamma_{qq,NS} \otimes q_{NS} \; .$$

- The singlet light flavor density is defined by

$$\Sigma(n_f, \mu^2) = \sum_{i=1}^{n_f} (f_i(n_f, \mu^2) + \bar{f}_i(n_f, \mu^2)) \; .$$

- The anomalous dimensions are presently known at NNLO [Moch, Vermaseren, Vogt, 2004.])

Heavy Quark Contributions emerge in the Wilson Coefficients only.

$\Longrightarrow$ This description is called : Fixed Flavor Scheme, $N_{\rm light} = 3$.

Any fixed twist calculation of DIS heavy flavor contributions uses this frame.

The range of validity of this approach is discussed later.

For some applications one may construct associated variable flavor number schemes.

3. Leading and Next-to-Leading Order Contributions

Leading Order : $F_{2,L}(x,Q^2)$ [Witten, 1976; Babcock, Sivers, 1978; Shifman, Vainshtein, Zakharov, 1978; Leveille, Weiler, 1979; Glück, Reya, 1979; Glück, Hoffmann, Reya, 1982.]

$$
\begin{aligned}
F_{2,L}(x,Q^2) &= e_Q^2 a_s(Q^2) \int_{ax}^{1} \frac{dy}{y} C^{(1)}_{F_2(F_L)}\left(\frac{x}{y}, m_Q^2, Q^2\right) G(y,Q^2) \\
C^{(1)}_{F_2}\left(z, m_Q^2, Q^2\right) &= 8T_R \left\{ \beta \left[-\frac{1}{2} + 4z(1-z) + \frac{m_Q^2}{Q^2} z(2z-1) \right] \right. \\
&+ \left. \left[-\frac{1}{2} + z - z^2 + 2\frac{m_Q^2}{Q^2} z(3z-1) - 4\frac{m_Q^4}{Q^4} z^2 \right] \ln\left(\frac{1-\beta}{1+\beta}\right) \right\} \\
C^{(1)}_{F_L}\left(z, m_Q^2, Q^2\right) &= 16T_R \left[z(1-z)\beta - \frac{m_Q^2}{Q^2} z^2 \ln\left|\frac{1+\beta}{1-\beta}\right| \right] \\
\beta &= \sqrt{1 - \frac{4m_Q^2 z}{1-z}}, \qquad a = 1 + \frac{4m_Q^2}{Q^2}
\end{aligned}
$$

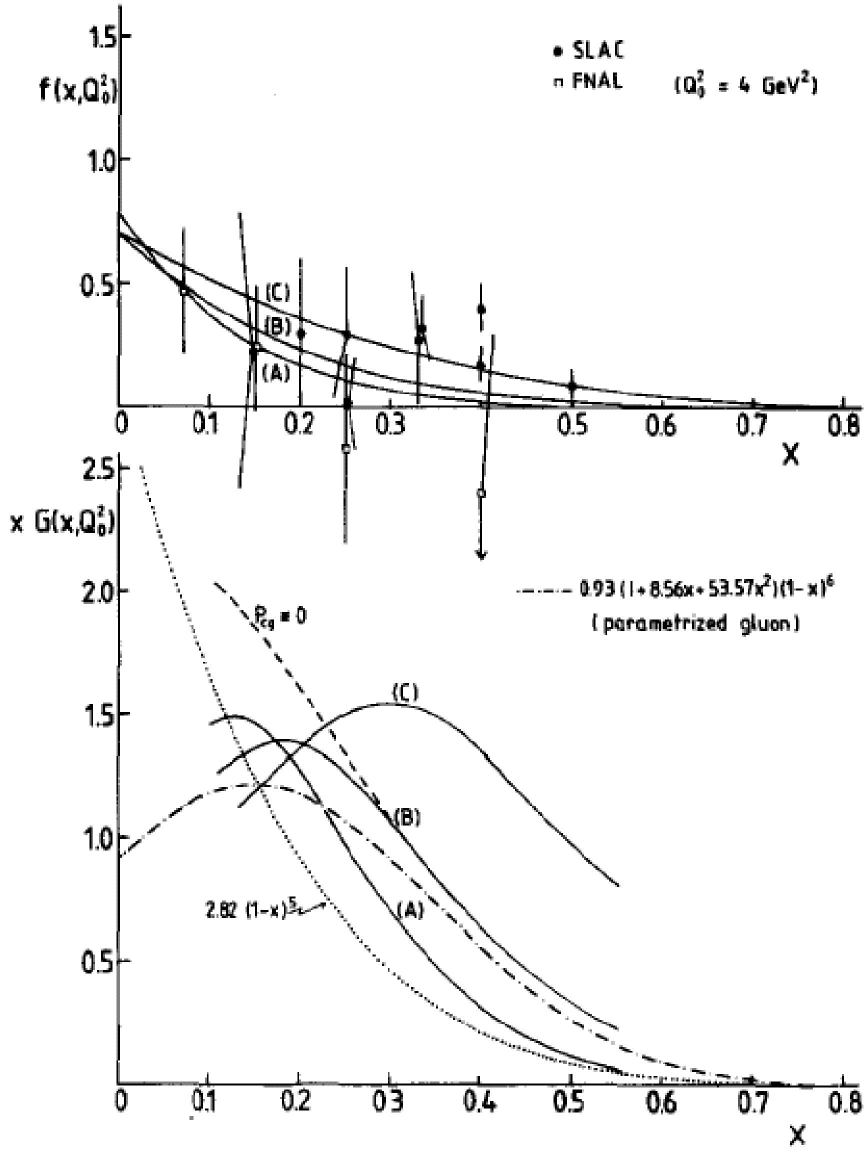
- $\exists$ non-power and power terms m_Q^2/Q^2
- Power corrections always remain process dependent @ twist-2
- Never ever tailor on power corrections.

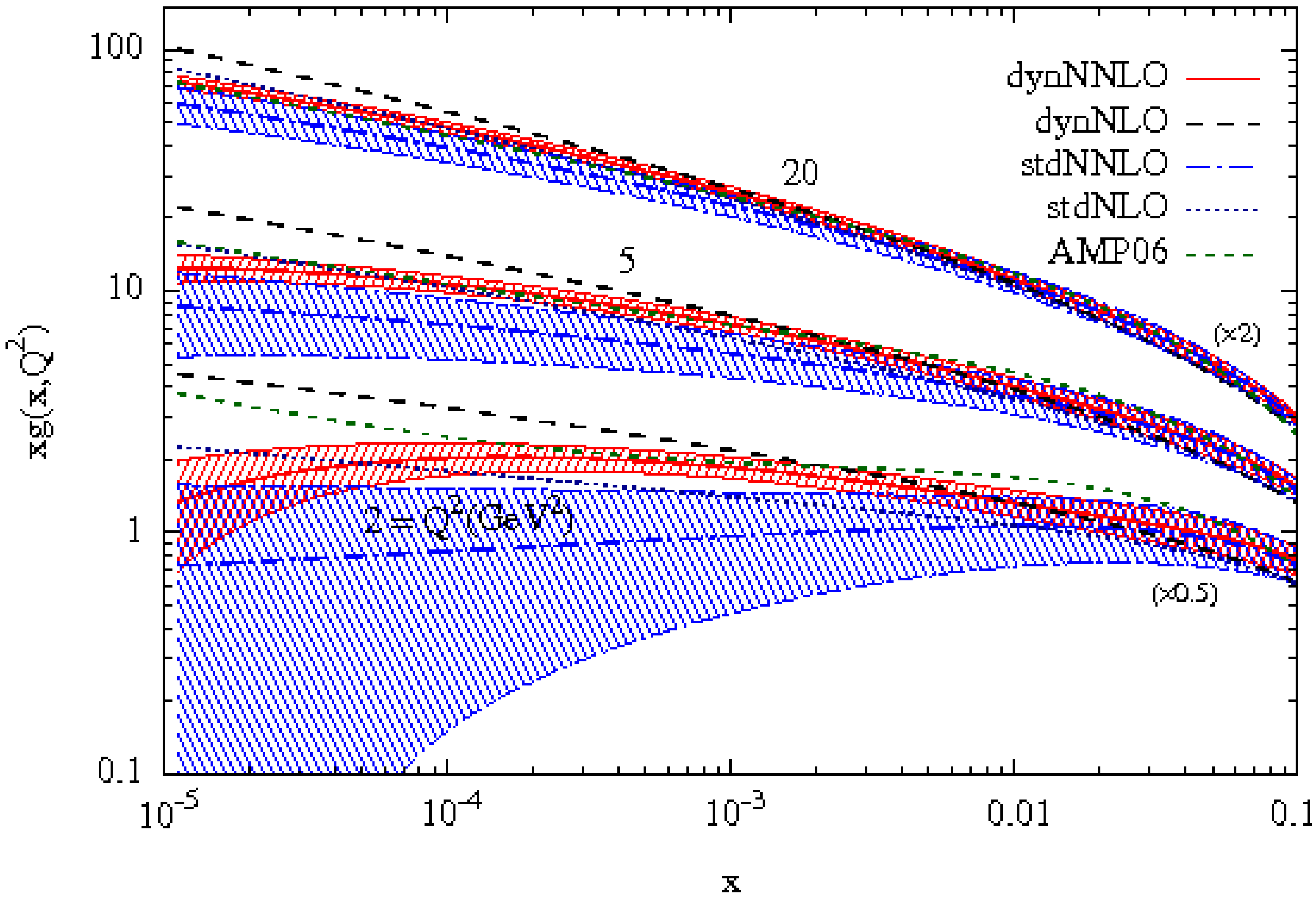
Unfolding the Gluon Distribution

- Gluon carries roughly 50% of the proton momentum.
- Heavy quark production excellent way to extract Gluon density via measurement of
  - scaling–violations of F_2
  - $F_L^{Q\overline{Q}}$
- First extraction of the gluon density including heavy quark effects by [Glück, Hoffmann, Reya, 1982.]:

- Unfold the gluon density via

$$G(x, Q^2) = P_{qg}^{-1} \otimes \left[\frac{f(x, Q^2)}{x} - \frac{2}{3} P_{cg} \otimes G(x, Q^2) \right] .$$





[Jimenez–Delgado, Reya, 2008.]; other fits: MSTW, CTEQ, AMP, BB, NN-collab

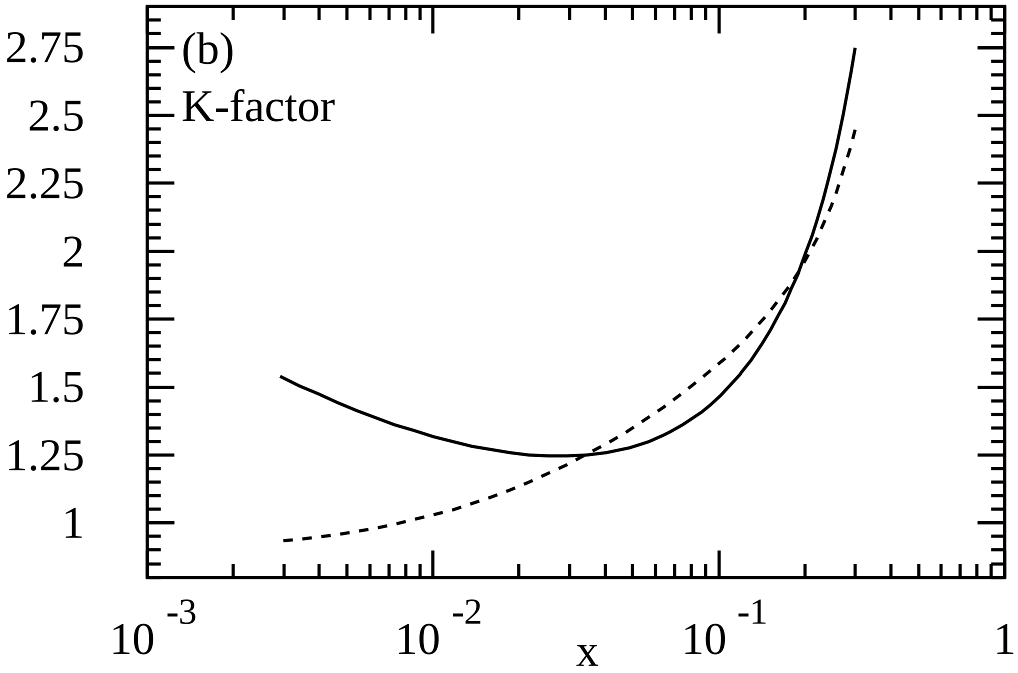
Threshold Resummation

- Soft resummation may be applied in the threshold region at fixed small rapidity of the produced heavy quark pair. [Laenen & Moch, 1998]

$$
\begin{aligned}
\frac{d^3\sigma(x,M^2,\theta,y,Q^2)}{dM^2 d\cos\theta dy} &= \frac{1}{S'^2}\int dz \int \frac{dx'}{x'} \Phi_{g/P}(x',\mu^2)\delta(y-\ln(1/x')/2)\delta(z-4m^2x/Q^2/(x'-x)) \\
&\times \quad \omega(z,\theta,(Q^2/\mu^2),(M^2/\mu^2),a_s)
\end{aligned}
$$

$$
\begin{aligned}
\omega^{(k)}(s',\theta,M) &= K^{(k)}\sigma^{\rm Born}(s',\theta,M) \\
K^{(1)} &= 4a_s\Bigg[2C_A\left[\frac{\ln(1-z)}{1-z}\right]_+ + \left[\frac{1}{1-z}\right]_+ \Bigg\{C_A\Bigg(-2\ln\left(\frac{1}{1-x}\right)+{\rm Re}L_\beta \\
&+\ln\left(\frac{t_1u_1}{m^4}\right)-\ln\left(\frac{M^2}{m^2}\right)\Bigg)-2C_F({\rm Re}L_\beta+1)\Bigg\}\Bigg] \\
K^{(2)} &= 16a_s^2\Bigg[2C_A^2\left[\frac{\ln^3(1-z)}{1-z}\right]_+ \left[\frac{\ln^2(1-z)}{1-z}\right]_+ \Bigg\{3C_A^2\Bigg(\ln\left(\frac{t_1u_1}{m^4}\right)+{\rm Re}L_\beta \\
&-6\ln\left(\frac{1}{1-x}\right)-2\ln\left(\frac{M^2}{m^2}\right)\Bigg)-2C_A((11C_A-2N_f)/12+3C_F({\rm Re}L_\beta+1))\Bigg\}\Bigg]
\end{aligned}
$$

Threshold Resummation



The x-dependence of the ratios $F^{\rm charm}_{2\,(NLO)}/F^{\rm charm}_{2\,(LO)}$ (solid line) and $F^{\rm charm}_{2\,(NNLO)}/F^{\rm charm}_{2\,(NLO)}$ (dashed line) with $F^{\rm charm}_{2\,(NNLO)}$ in the improved NLL approximation (exact NLO result plus NLL approximate NNLO result with the damping factor $1/\sqrt{1+\eta}$).

Leading and Next-to-Leading Order Contributions

Next-to-Leading Order : $F_{2,L}(x,Q^2)$ [Laenen, Riemersma, Smith, van Neerven, 1993, 1995]

asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven, 1996; Bierenbaum, J.B., Klein, 2007]

New variables :

$$\xi = \frac{Q^2}{m^2}, \quad \eta = \frac{s}{4m^2} - 1 \geq 0 \, .$$

$$z = \frac{Q^2}{Q^2+s} = \frac{\xi/4}{1+\eta+\xi/4}, \quad z \, \epsilon \left[x, \frac{Q^2}{Q^2+4m^2}\right]$$

$$L(\eta) = \ln\left[\frac{(1+\eta)^{1/2}+\eta^{1/2}}{(1+\eta)^{1/2}-\eta^{1/2}}\right]$$

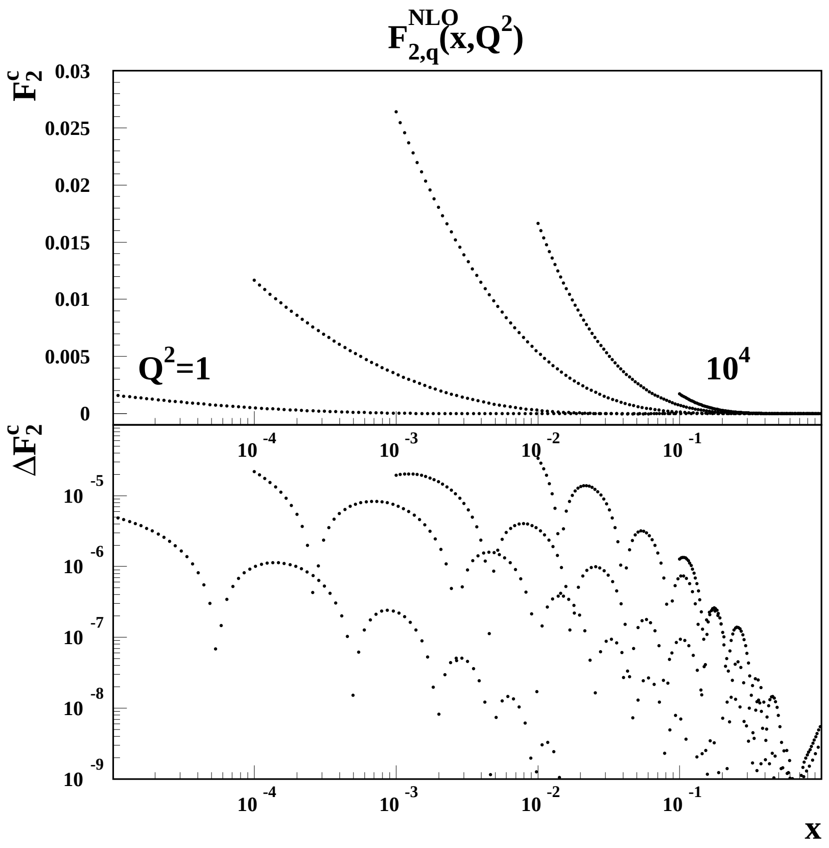
$$F_k^{\rm NLO}(x,Q^2,m^2) = \frac{Q^2}{\pi m^2}\alpha_s^2(\mu^2)\int_x^{z_{\max}}\frac{dz}{z}\left\{e_Q^2 f_g\left(\frac{x}{z},\mu^2\right)\left[c_{k,g}^{(1)}(\xi,\eta)+\overline{c}_{k,g}^{(1)}(\xi,\eta)\ln\left(\frac{\mu^2}{m^2}\right)\right]\right.$$

$$+\sum_{i=q,\overline{q}}^{3}\left\{e_Q^2 f_i\left(\frac{x}{z},\mu^2\right)\left[c_{k,i}^{(1)}(\xi,\eta)+\overline{c}_{k,i}^{(1)}(\xi,\eta)\ln\left(\frac{\mu^2}{m^2}\right)\right]\right.$$

$$\left.+e_i^2 f_i\left(\frac{x}{z},\mu^2\right)\left[d_{k,i}^{(1)}(\xi,\eta)+\overline{d}_{k,i}^{(1)}(\xi,\eta)\right]\right., \qquad \overline{d}_{L,q}^{(1)}(\xi,\eta)=0.$$

• Semi-analytic expressions for $c_{k,i}^{(1)}(\xi,\eta), \overline{c}_{k,i}^{(1)}(\xi,\eta), c_{d,i}^{(1)}(\xi,\eta), \overline{d}_{k,i}^{(1)}(\xi,\eta)$; not all integrals could be done analytically.

• Fast semi-analytic representation in Mellin space: [Alekhin & J.B., 2003] This includes power corrections.



The Non-Power Contributions

- massless RGE and light–cone expansion in Bjorken–limit $\{Q^2,\ \nu\} \ \to\ \infty,\ x$ fixed:

$$\lim_{\xi^2\to 0}\Big[J(\xi), J(0)\Big] \propto \sum_{i,N,\tau} c^N_{i,\tau}(\xi^2,\mu^2)\xi_{\mu_1}...\xi_{\mu_N} O^{\mu_1\cdots\mu_m}_{i,\tau}(0,\mu^2)\ .$$

- Operators: flavor non-singlet (≤ 3), pure–singlet and gluon; consider leading twist.
- RGE for collinear singularities: mass factorization of the structure functions into Wilson coefficients and parton densities:

$$F_i(x,Q^2) = \sum_j \underbrace{C_i^j\left(x,\frac{Q^2}{\mu^2}\right)}_{\text{perturbative}} \otimes \underbrace{f_j(x,\mu^2)}_{\text{non-perturbative}}$$

- Light-flavor Wilson coefficients: process dependent ($O(a_s^3)$: [Moch, Vermaseren, Vogt, 2005.])

$$C^{\rm fl}_{(2,L);i}\left(\frac{Q^2}{\mu^2}\right) = \delta_{i,q} + \sum_{l=1}^{\infty} a_s^l C^{{\rm fl},(l)}_{(2,L),i}, \qquad i = q, g$$

- Heavy quark contributions given by heavy quark Wilson coefficients, $H^{\rm S,NS}_{(2,L),i}\left(\frac{Q^2}{\mu^2},\frac{m^2}{Q^2}\right)$.

- In the limit $Q^2 \gg m_h^2$ $[Q^2 \approx 10\ m^2$ for $F_2, g_1]$:
massive RGE, derivative $m^2\partial/\partial m^2$ acts on Wilson coefficients only: all terms but power corrections calculable through partonic operator matrix elements, $\langle i|A_l|j\rangle$, which are process independent objects!

$$H_{(2,L),i}^{\mathrm{S,NS}}\left(\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \underbrace{A_{k,i}^{\mathrm{S,NS}}\left(\frac{m^2}{\mu^2}\right)}_{\text{massive OMEs}} \otimes \underbrace{C_{(2,L),k}^{\mathrm{S,NS}}\left(\frac{Q^2}{\mu^2}\right)}_{\text{light–parton–Wilson coefficients}}.$$

- holds for polarized and unpolarized case. OMEs obey expansion

$$A_{k,i}^{\mathrm{S,NS}}\left(\frac{m^2}{\mu^2}\right) = \langle i|O_k^{\mathrm{S,NS}}|i\rangle = \delta_{k,i} + \sum_{l=1}^{\infty} a_s^l A_{k,i}^{\mathrm{S,NS},(l)}\left(\frac{m^2}{\mu^2}\right), \qquad i = q, g$$

[Buza, Matiounine, Migneron, Smith, van Neerven, 1996; Buza, Matiounine, Smith, van Neerven, 1997.]

- Heavy OMEs are the transition functions to define a VFNS starting from a fixed flavor number scheme(FFNS).

[Buza, Matiounine, Smith, van Neerven, 1998; Chuvakin, Smith, van Neerven, 1998.]

One–Loop Example

Consider $F_2^{Q\overline{Q}}(x, Q^2)$:

$$H_{2,g}^{(1)}\left(z, \frac{m^2}{Q^2}\right) = 8T_R a_s \left\{ v \left[-\frac{1}{2} + 4z(1-z) + \frac{m^2}{Q^2} z(2z-1) \right] + \left[-\frac{1}{2} + z - z^2 + 2\frac{m^2}{Q^2} z(3z-1) - 4\frac{m^4}{Q^4} z^2 \right] \ln\left(\frac{1-v}{1+v}\right) \right\} ,$$

$$\lim_{Q^2 \gg m^2} H_{2,g}^{(1)}\left(z, \frac{m^2}{Q^2}\right) = 4T_R a_s \left\{ [z^2 + (1-z)^2] \ln\left(\frac{Q^2}{m^2} \frac{1-z}{z}\right) + 8z(1-z) - 1 \right\} .$$

$\overline{\rm MS}$ result for $m^2 = 0$:

$$C_{2,g}^{(1)}\left(z, \frac{Q^2}{\mu^2}\right) = 4T_R a_s \left\{ [z^2 + (1-z)^2] \ln\left(\frac{Q^2}{\mu^2} \frac{1-z}{z}\right) + 8z(1-z) - 1 \right\} .$$

Massive operator matrix element:

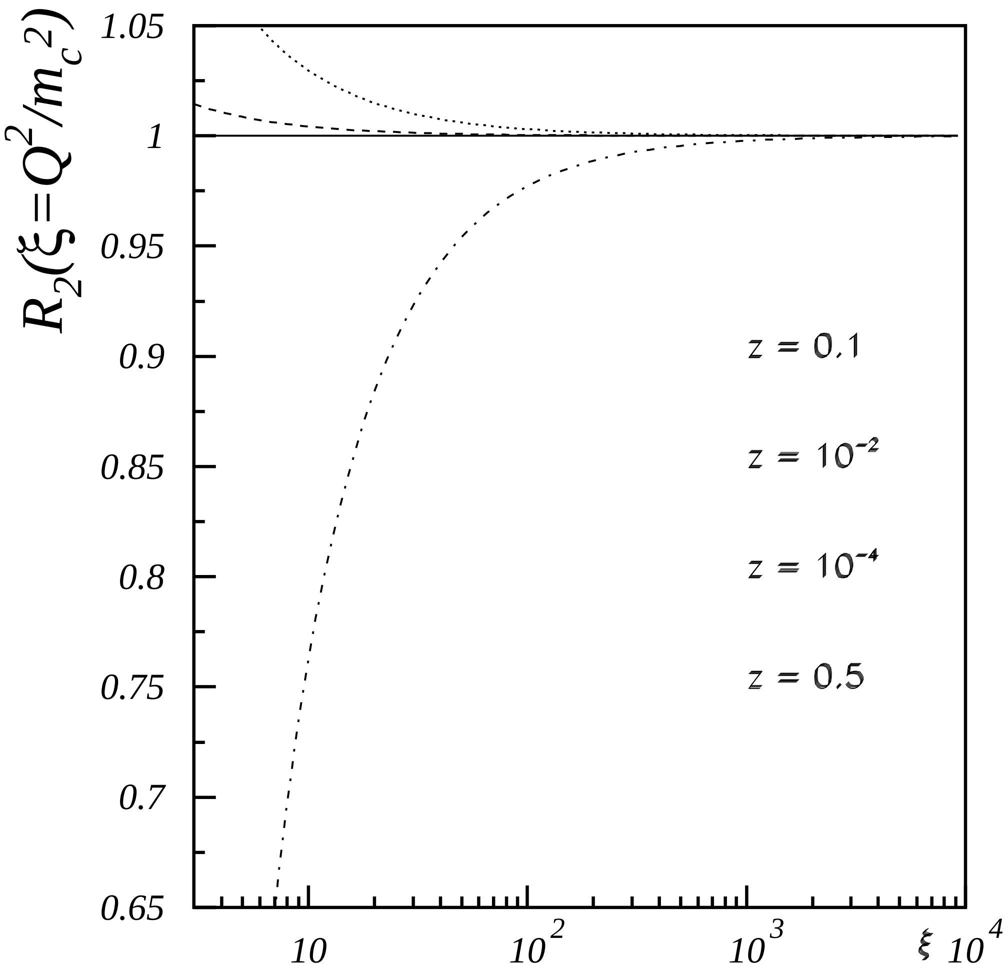
$$A_{Qg}^{(1)}\left(z, \frac{m^2}{\mu^2}\right) = -4T_R a_s [z^2 + (1-z)^2] \ln\left(\frac{m^2}{\mu^2}\right) + a_{Qg}^{(1)} , \quad a_{Qg}^{(1)} = 0 .$$

$$\Longrightarrow \lim_{Q^2 \gg m^2} H_{2,g}^{(1)}\left(z, \frac{m^2}{Q^2}\right) = C_{2,g}^{(1)}\left(z, \frac{Q^2}{\mu^2}\right) + A_{Qg}^{(1)}\left(z, \frac{m^2}{\mu^2}\right) . \quad A_{Qg}^{(1)}(z, 1) = 0$$

- Comparison for LO:

$$R_2\left(\xi \equiv \frac{Q^2}{m^2}\right) \equiv \frac{H_{2,g}^{(1)}}{H_{2,g,(asym)}^{(1)}} \,.$$

- Comparison to exact order $O(a_s^2)$ result: asymptotic formulae valid for $Q^2 \geq 20$ $(\text{GeV}/c)^2$ in case of $F_2^{c\bar{c}}(x,Q^2)$ and $Q^2 \geq 1000$ $(\text{GeV}/c)^2$ for $F_L^{c\bar{c}}(x,Q^2)$
- Drawbacks:
  - Power corrections $(m^2/Q^2)^k$ can not be calculated using this method.
  - Two heavy quark masses are still too complicated $\Longrightarrow$ 2 scale problem to be treated analytically.
  - Only inclusive quantities can be calculated $\Longrightarrow$ structure functions.

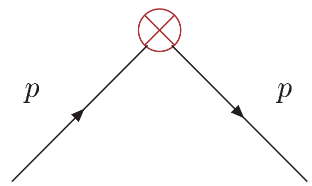


Computing Massive OMEs

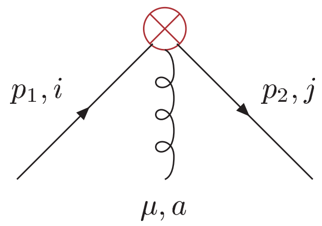
- Operator insertions in light–cone expansion

E.g. singlet heavy quark operator:

$$O_Q^{\mu_1 \ldots . \mu_N}(z) = \frac{1}{2} i^{N-1} S[\overline{q}(z)\gamma^{\mu_1} D^{\mu_2} \ldots . D^{\mu_N} q(z)] - \mathsf{TraceTerms}\ .$$



$$\not\Delta \gamma_\pm (\Delta \cdot p)^{N-1}\ ,$$



$$g t^a_{ji} \Delta^\mu \not\Delta \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2}\ ,$$

$$\gamma_+ = 1\ , \quad \gamma_- = \gamma_5\ .$$

Δ: light–like momentum, $\Delta^2 = 0$.

$\Longrightarrow$ additional vertices with 2 and more gluons at higher orders.

Renormalization

$$\hat{\hat{A}}_{ij} = \delta_{ij} + \sum_{k=0}^{\infty} \hat{a}_s^k \hat{\hat{A}}_{ij}^{(k)}$$

- Mass renormalization (on–mass shell scheme)
- Charge renormalization

→ use $\overline{\mathrm{MS}}$ scheme ($D = 4 + \varepsilon$) and
decoupling formalism [Ovrut, Schnitzer, 1981; Bernreuther, Wetzel, 1982.].

- Renormalization of ultraviolet singularities
  $\Longrightarrow$ are absorbed into Z–factors given in terms of anomalous dimensions γ_{ij}.
- Factorization of collinear singularities
  $\Longrightarrow$ are factored into Γ–factors Γ_{NS}, $\Gamma_{ij,S}$ and $\Gamma_{qq,PS}$.
  For massless quarks it would hold: $\Gamma = Z^{-1}$.
  Here: Γ–matrices apply to parts of the diagrams with massless lines only .

Generic formula for operator renormalization and mass factorization:

$$A_{ij} = Z_{il}^{-1} \hat{A}_{lm} \Gamma_{mj}^{-1}$$

$\Longrightarrow$ $O(\varepsilon)$–terms of the 2–loop OMEs are needed for renormalization at 3–loops.

Calculation Techniques

- Calculation in Mellin-space for space–like q^2, $Q^2 = -q^2$: $\quad 0 \le x \le 1$

$$\mathbf{M}[f](N) := \int_0^1 dz z^{N-1} f(z) \ .$$

- Analytic results for general value of Mellin N are obtained in terms of harmonic sums [J.B., Kurth, 1999; Vermaseren, 1999.]

$$S_{a_1,\dots,a_m}(N) = \sum_{n_1=1}^{N} \sum_{n_2=1}^{n_1} \dots \sum_{n_m=1}^{n_{m-1}} \frac{(\mathrm{sign}(a_1))^{n_1}}{n_1^{|a_1|}} \frac{(\mathrm{sign}(a_2))^{n_2}}{n_2^{|a_2|}} \dots \frac{(\mathrm{sign}(a_m))^{n_m}}{n_m^{|a_m|}} \ ,$$

$$N \in \mathrm{N}, \ \forall\, l, \ a_l \in \mathrm{Z} \setminus 0 \ ,$$

$$S_{-2,1}(N) = \sum_{i=1}^{N} \frac{(-1)^i}{i^2} \sum_{j=1}^{i} \frac{1}{j} \ .$$

- Algebraic and structural simplification of the harmonic sums [J.B., 2003, 2007.].
- Analytic continuation to complex N via analytic relations or integral representations, e.g.

$$\mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N+1) - \zeta_2 \beta(N+1) = (-1)^{N+1}\left[S_{-2,1}(N) + \frac{5}{8}\zeta_3\right] \ .$$

- Harmonic sums appear in many single scale higher order processes.

  Str. Functions, DIS HQ, Fragm. Functions, DY, Hadr. Higgs-Prod., s+v contr. to Bhabha scatt., ...

$O(a_s^2)$ Contributions to $O(\varepsilon)$

- use of generalized hypergeometric functions for general analytic results

  $\Longrightarrow$ allows feasible computation of higher orders in ε & automated check for fixed values of N.

- use of Mellin-Barnes integrals for numerical checks (MB, [Czakon, 2006.])

- Summation of lots of new infinite one-parameter sums into harmonic sums. E.g.:

$$N \sum_{i,j=1}^{\infty} \frac{S_1(i) S_1(i+j+N)}{i(i+j)(j+N)} = 4S_{2,1,1} - 2S_{3,1} + S_1\left(-3S_{2,1} + \frac{4S_3}{3}\right) - \frac{S_4}{2}$$
$$- S_2^2 + S_1^2 S_2 + \frac{S_1^4}{6} + 6S_1\zeta_3 + \zeta_2\left(2S_1^2 + S_2\right) .$$

  use of integral techniques and the Mathematica package SIGMA [Schneider, 2007.], [Bierenbaum, J.B., Klein, Schneider, 2007, 2008.]

- Partial checks for fixed values of N using SUMMER, [Vermaseren, 1999.]

We calculated all 2–loop $O(\varepsilon)$–terms in the unpolarized case

and several 2–loop $O(\varepsilon)$–terms in the polarized case:

$$\overline{a}_{Qg}^{(2)},\ \overline{a}_{Qq}^{(2),\mathbf{PS}},\ \overline{a}_{gg,Q}^{(2)},\ \overline{a}_{gq,Q}^{(2)},\ \overline{a}_{qq,Q}^{(2),\mathbf{NS}}.$$

$$\Delta\overline{a}_{Qg}^{(2)},\ \Delta\overline{a}_{Qq}^{(2),\mathbf{PS}},\ \Delta\overline{a}_{qq,Q}^{(2),\mathbf{NS}}.$$

We verified all corresponding 2–loop $O(\varepsilon^0)$–results by van Neerven et. al.

- A remark on the appearing functions:

  van Neerven et al. to O(1): unpolarized: 48 basic functions; polarized: 24 basic functions.

  O(1): $\{S_1,\ S_2,\ S_3,\ S_{-2}, S_{-3}\}$, $S_{-2,1} \Longrightarrow$ 2 basic objects.

  O(ε): $\{S_1,\ S_2,\ S_3,\ S_4,\ S_{-2}, S_{-3}, S_{-4}\}$, $S_{2,1}$, $S_{-2,1}$, $S_{-3,1}$, $S_{2,1,1}$, $S_{-2,1,1}$

  $\Longrightarrow$ 6 basic objects

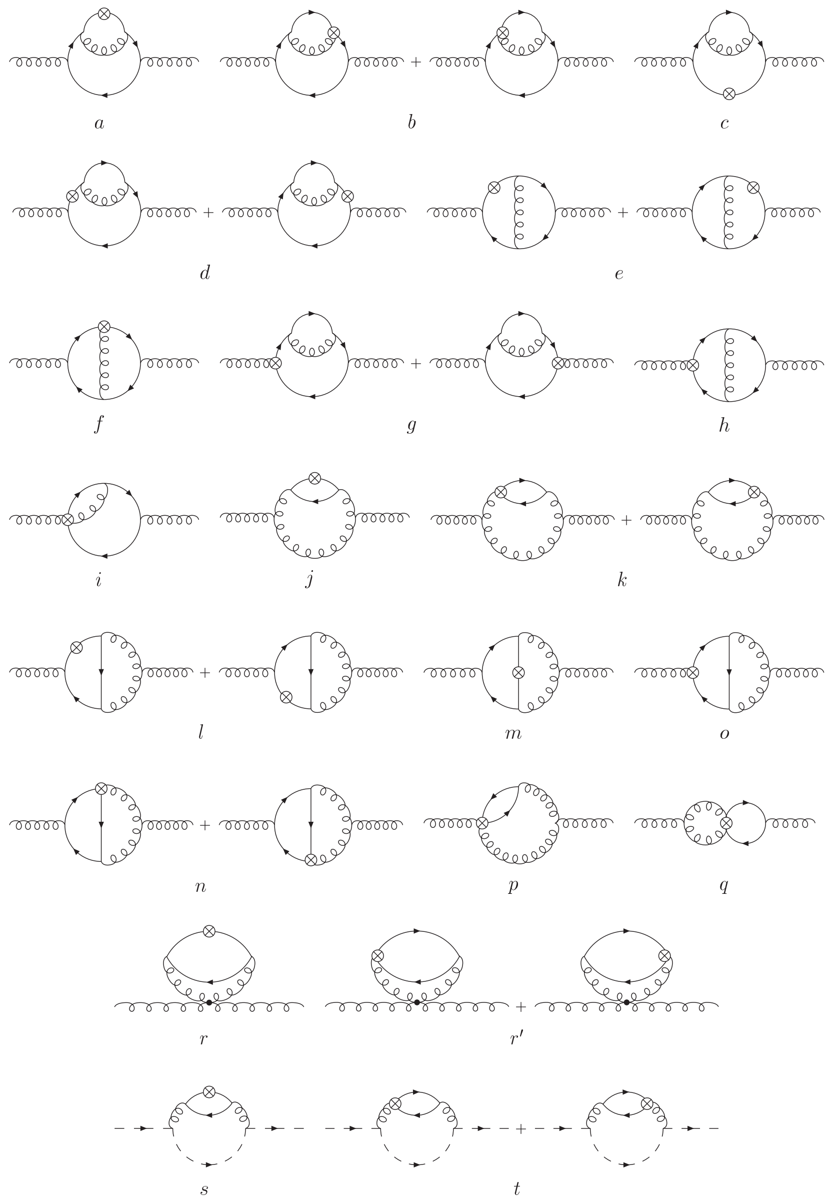
- harmonic sums with index $\{-1\}$ cancel (holds even for each diagram)

  J.B., 2004; J.B., Ravindran, 2005,2006; J.B., Klein, 2007; J.B., Moch in preparation.]

- Expectation for 3–loops: weight 5 (6) harmonic sums

- Diagrams contain two scales: the mass m and the Mellin–parameter N.
- 2–point functions with on–shell external momentum, $p^2 = 0$.
  $\rightarrow$ reduce for $N = 0$ to massive tadpoles.
- E.g. diagrams contributing to the gluonic OME

  $\hat{A}_{Qg}^{(2)} \qquad \Longrightarrow$



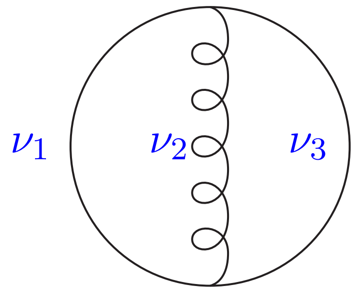
Example: Unpolarized case, Singlet, O(ε)

$$
\begin{aligned}
\overline{a}_{Qg}^{(2)} &= T_F C_F \Biggl\{ \frac{2}{3} \frac{(N^2+N+2)(3N^2+3N+2)}{N^2(N+1)^2(N+2)} \zeta_3 + \frac{P_1}{N^3(N+1)^3(N+2)} S_2 + \frac{N^4-5N^3-32N^2-18N-4}{N^2(N+1)^2(N+2)} S_1^2 \\
&+ \frac{N^2+N+2}{N(N+1)(N+2)} \Bigl(16 S_{2,1,1} - 8 S_{3,1} - 8 S_{2,1} S_1 + 3 S_4 - \frac{4}{3} S_3 S_1 - \frac{1}{2} S_2^2 - S_2 S_1^2 - \frac{1}{6} S_1^4 + 2\zeta_2 S_2 - 2\zeta_2 S_1^2 - \frac{8}{3}\zeta_3 S_1\Bigr) \\
&- 8 \frac{N^2-3N-2}{N^2(N+1)(N+2)} S_{2,1} + \frac{2}{3} \frac{3N+2}{N^2(N+2)} S_1^3 + \frac{2}{3} \frac{3N^4+48N^3+43N^2-22N-8}{N^2(N+1)^2(N+2)} S_3 + 2 \frac{3N+2}{N^2(N+2)} S_2 S_1 + 4 \frac{S_1}{N^2} \zeta_2 \\
&+ \frac{N^5+N^4-8N^3-5N^2-3N-2}{N^3(N+1)^3} \zeta_2 - 2 \frac{2N^5-2N^4-11N^3-19N^2-44N-12}{N^2(N+1)^3(N+2)} S_1 + \frac{P_2}{N^5(N+1)^5(N+2)} \Biggr\} \\
&+ T_F C_A \Biggl\{ \frac{N^2+N+2}{N(N+1)(N+2)} \Bigl(16 S_{-2,1,1} - 4 S_{2,1,1} - 8 S_{-3,1} - 8 S_{-2,2} - 4 S_{3,1} - \frac{2}{3}\beta''' + 9 S_4 - 16 S_{-2,1} S_1 \\
&+ \frac{40}{3} S_1 S_3 + 4\beta'' S_1 - 8\beta' S_2 + \frac{1}{2} S_2^2 - 8\beta' S_1^2 + 5 S_1^2 S_2 + \frac{1}{6} S_1^4 - \frac{10}{3} S_1 \zeta_3 - 2 S_2 \zeta_2 - 2 S_1^2 \zeta_2 - 4\beta' \zeta_2 - \frac{17}{5}\zeta_2^2\Bigr) \\
&+ \frac{4(N^2-N-4)}{(N+1)^2(N+2)^2} \Bigl(-4 S_{-2,1} + \beta'' - 4\beta' S_1\Bigr) - \frac{2}{3} \frac{N^3+8N^2+11N+2}{N(N+1)^2(N+2)^2} S_1^3 + 8 \frac{N^4+2N^3+7N^2+22N+20}{(N+1)^3(N+2)^3} \beta' \\
&+ 2 \frac{3N^3-12N^2-27N-2}{N(N+1)^2(N+2)^2} S_2 S_1 - \frac{16}{3} \frac{N^5+10N^4+9N^3+3N^2+7N+6}{(N-1)N^2(N+1)^2(N+2)^2} S_3 - 8 \frac{N^2+N-1}{(N+1)^2(N+2)^2} \zeta_2 S_1 \\
&- \frac{2}{3} \frac{9N^5-10N^4-11N^3+68N^2+24N+16}{(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 - \frac{P_3}{(N-1)N^3(N+1)^3(N+2)^3} S_2 - \frac{2P_4}{(N-1)N^3(N+1)^3(N+2)^2} \zeta_2 \\
&- \frac{P_5}{N(N+1)^3(N+2)^3} S_1^2 + \frac{2P_6}{N(N+1)^4(N+2)^4} S_1 - \frac{2P_7}{(N-1)N^5(N+1)^5(N+2)^5} \Biggr\} .
\end{aligned}
$$

Use of hypergeometric functions for general analytic results

$$ {}_PF_Q\left[\begin{array}{c}(a_1)...(a_P)\\(b_1)...(b_Q)\end{array};z\right] = \sum_{i=0}^{\infty}\frac{(a_1)_i...(a_P)_i}{(b_1)_i...(b_Q)_i}\frac{z^i}{\Gamma(i+1)}\,, \qquad {}_1F_0[a;z] = \frac{1}{(1-z)^a}\,. $$

Consider the massive 2–loop tadpole diagram with arbitrary exponents ν_i and $\nu_{i...j} := \nu_i + ... + \nu_j$ etc.



Using Feynman–parameters, this integral can be cast into the general form

$$ I_1 = C_1 \iint_0^1 dxdy\,\frac{x^a(1-x)^b y^c(1-y)^d}{(1-xy)^e}\,. $$

Thus one obtains

$$ I_1 = C_1\Gamma\left[\begin{array}{c}\nu_{123}-4-\varepsilon,\varepsilon/2-\nu_2,\nu_{23}-2-\varepsilon/2,\nu_{12}-2-\varepsilon/2\\ \nu_1,\nu_2,\nu_3,\nu_{123}-2-\varepsilon/2\end{array}\right]{}_3F_2\left[\begin{array}{c}\nu_{123}-4-\varepsilon,\varepsilon/2+2-\nu_2,\nu_3\\ \nu_3,\nu_{123}-2-\varepsilon/2\end{array};1\right]. $$

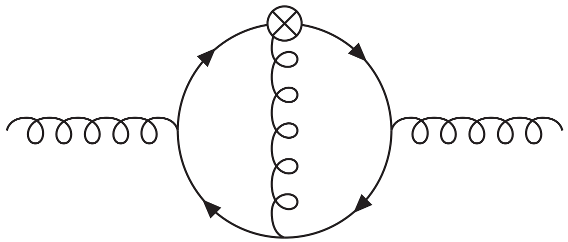
For any diagram deriving from the 2–loop tadpole topology, one obtains as a general integral

$$I_2 \quad = \quad C_2 \iint_0^1 \; dx dy \; \frac{x^a (1-x)^b y^c (1-y)^d}{(1-xy)^e} \int_0^1 dz_1 ... \int_0^1 dz_i \; \mathsf{P}\Big(x, y, z_1, ..., z_i, N\Big) \; .$$

Here P is a rational function of x, y and possibly more parameters $z_1 ... z_i$. N is the Mellin–parameter and occurs in some exponents.

$\Longrightarrow$ for fixed values of N, one obtains for all diagrams a finite sum over integrals of the type I_1.

Consider e.g the scalar Integral of the Diagram



$$\begin{aligned} I_3 \quad &= \quad C_3 \exp\Biggl\{\sum_{l=2}^{\infty} \frac{\zeta_l}{l} \varepsilon^l \Biggr\} \frac{2\pi}{N \sin(\frac{\pi}{2}\varepsilon)} \sum_{j=1}^{N} \Biggl\{ \binom{N}{j} (-1)^j + \delta_{j,N} \Biggr\} \\ &\quad \times \Biggl\{ \frac{\Gamma(j)\Gamma(j+1-\frac{\varepsilon}{2})}{\Gamma(j+2-\varepsilon)\Gamma(j+1+\frac{\varepsilon}{2})} - \frac{B(1-\frac{\varepsilon}{2}, 1+j)}{j} \; {}_3F_2\Bigl[1-\varepsilon, \frac{\varepsilon}{2}, j+1; 1, j+2-\frac{\varepsilon}{2}; 1\Bigr] \Biggr\} \\ &= \quad C_3 \Biggl\{ \frac{4}{N} \Biggl[S_2 - \frac{S_1}{N} \Biggr] + \frac{\varepsilon}{N} \Biggl[-2 S_{2,1} + 2 S_3 + \frac{4N+1}{N} S_2 - \frac{S_1^2}{N} - \frac{4}{N} S_1 \Biggr] \Biggr\} + O(\varepsilon^2) \; . \end{aligned}$$

4. Polarized Heavy Flavor

• DIS QCD analyzes of the polarized world data have been carried out without reference of heavy quark contributions so far, despite heavy flavor being produced in the final states (NMC).

Leading Order : $g_1(x,Q^2)$ [Watson, 1982; Glück, Reya, Vogelsang, 1991; Vogelsang, 1991]

$$\begin{aligned} g_1(x,Q^2) &= 4e_Q^2 a_s(Q^2)\int_{ax}^1 \frac{dy}{y} C_{g_1}^{(1)}\left(\frac{x}{y}, m_Q^2, Q^2\right)\Delta G(y,Q^2) \\ C_{g_1}^{(1)}(z,m_Q^2,Q^2) &= \frac{1}{2}\left[\beta(3-4z)-(1-2z)\ln\left|\frac{1+\beta}{1-\beta}\right|\right] \end{aligned}$$

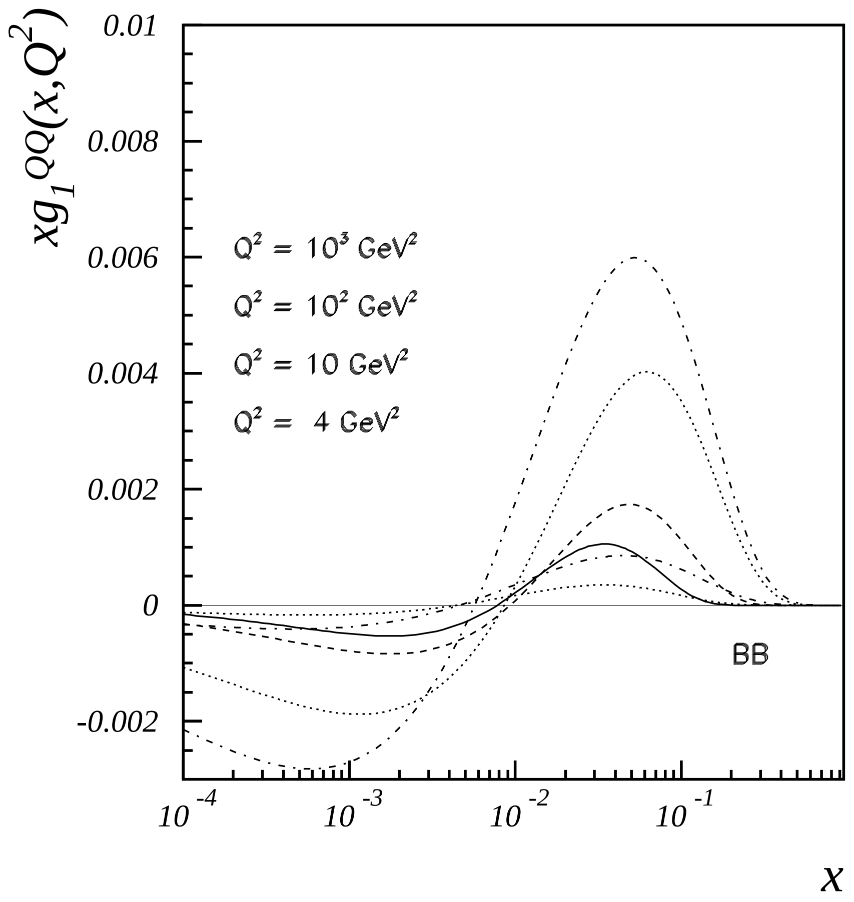
Leading Order : $g_2(x,Q^2)$ [J.B., Ravindran, van Neerven, 2003] $\Longrightarrow$ holds to all orders

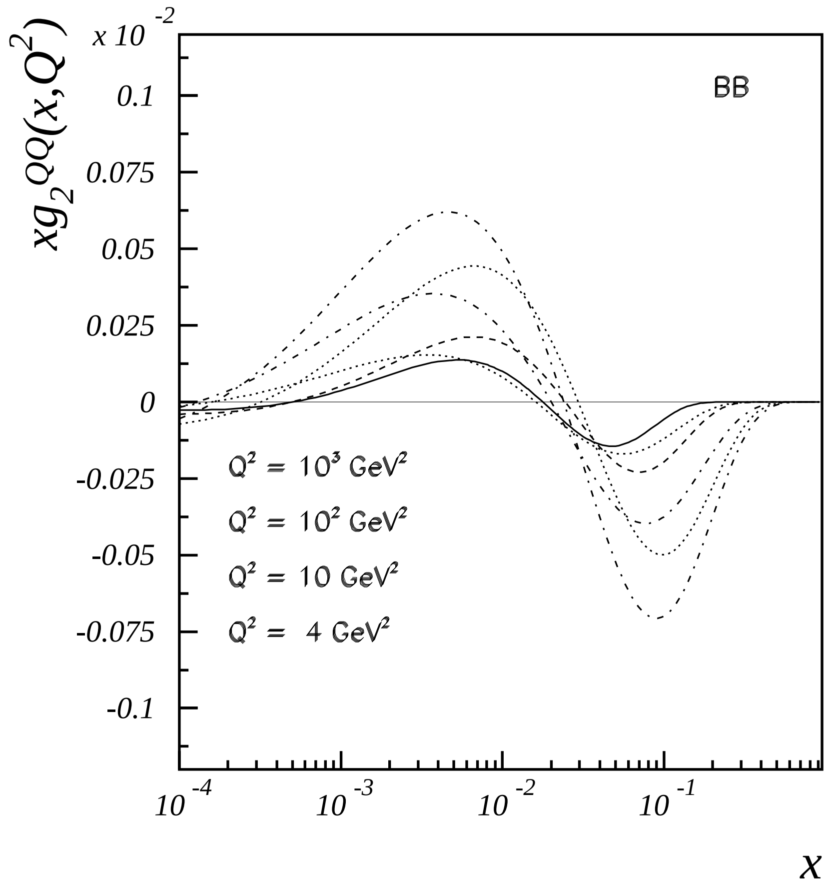
$$g_2(x,Q^2) = -g_1(x,Q^2) + \int_x^1 \frac{dy}{y} g_1(y,Q^2)$$

The Wandzura-Wilczek relation follows from the covariant parton model here.

$$\int_0^{1/a} dz C_{g_1}^{(1)}(z,m_Q^2,Q^2) = 0$$

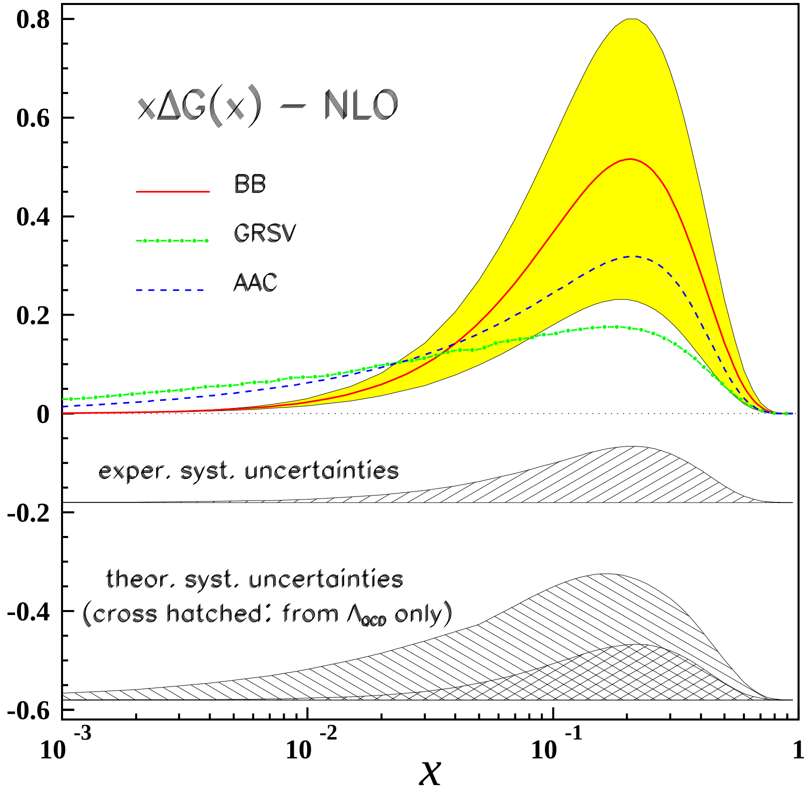
Polarized Heavy Flavor



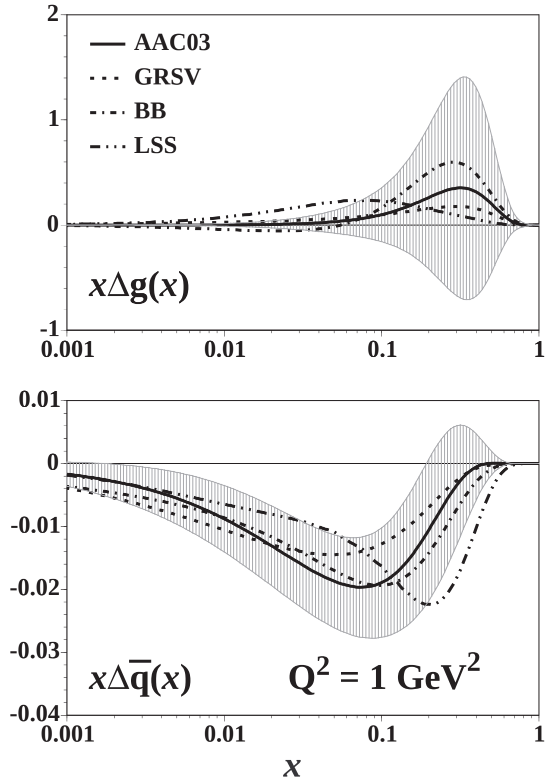


[J.B., Ravindran, van Neerven, 2003]

Polarized Gluon Density at Present



J.B., H. Böttcher (2002)



AAC

$\Longrightarrow$ Currently slight move of ΔG towards lower values

Polarized Heavy Flavor

Next-to-Leading Order : $g_1(x, Q^2)$ only asymptotic results $Q^2 \gg m_Q^2$ i.e.
$Q^2 \gtrsim 10 m_Q^2$ [M. Buza, Y. Matiounine, J. Smith and W. L. van Neerven, 1996, Bierenbaum, J.B., Klein, 2008]

$$\int_0^{1/a} dz C_{g_1}^{(2)}(z, m_Q^2, Q^2) = 0$$

Conjecture: holds for even higher orders.

$O(a_s^2 \varepsilon)$ terms: [Bierenbaum, J.B., Klein, 2008]

- NS OME's are the same as in unpolarized case (Ward Identity).

Polarized Heavy Flavor

$$
\begin{aligned}
\Delta a_{Qg}^{(2)}(N) &= T_F C_F \Biggl\{ 4\frac{N-1}{3N(N+1)}\Bigl(-4S_3(N)+S_1^3(N)+3S_1(N)S_2(N)+6S_1(N)\zeta_2\Bigr) \\
&-4\frac{N^4+17N^3+43N^2+33N+2}{N^2(N+1)^2(N+2)}S_2(N)-4\frac{3N^2+3N-2}{N^2(N+1)(N+2)}S_1^2(N) \\
&-2\frac{(N-1)(3N^2+3N+2)}{N^2(N+1)^2}\zeta_2-4\frac{N^3-2N^2-22N-36}{N^2(N+1)(N+2)}S_1(N) \\
&-\frac{2P_1(N)}{N^4(N+1)^4(N+2)}\Biggr\} \\
&+T_F C_A \Biggl\{ 4\frac{N-1}{3N(N+1)}\Bigl(12\mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N+1)+3\beta''(N+1)-8S_3(N) \\
&-S_1^3(N)-9S_1(N)S_2(N)-12S_1(N)\beta'(N+1)-12\beta(N+1)\zeta_2-3\zeta_3\Bigr) \\
&-16\frac{N-1}{N(N+1)^2}\beta'(N+1)+4\frac{N^2+4N+5}{N(N+1)^2(N+2)}S_1^2(N) \\
&+4\frac{7N^3+24N^2+15N-16}{N^2(N+1)^2(N+2)}S_2(N)+8\frac{(N-1)(N+2)}{N^2(N+1)^2}\zeta_2 \\
&+4\frac{N^4+4N^3-N^2-10N+2}{N(N+1)^3(N+2)}S_1(N)-\frac{4P_2(N)}{N^4(N+1)^4(N+2)}\Biggr\}
\end{aligned}
$$

+ known finite renormalization.

5. Trading Final- for Initial States

- There are no genuine heavy flavor parton densities.

$\Longrightarrow$ Power corrections in Wilson coefficients.

$\Longrightarrow$ Heavy quarks are not produced collinearly

$\Longrightarrow$ Their finite mass implies a finite & short lifetime.

- Do heavy quarks become light ? - will be discussed later.
- Simplifying matters with a method à la Fermi-Williams-Weizsäcker.
- Produce a Heavy Quark of large enough lifetime.

$\Longrightarrow$ Its further fate decouples (factorizes) from its past in time.

- Heavy Quark Initial State.

$\Longrightarrow$ Inapplicable to power corrections !

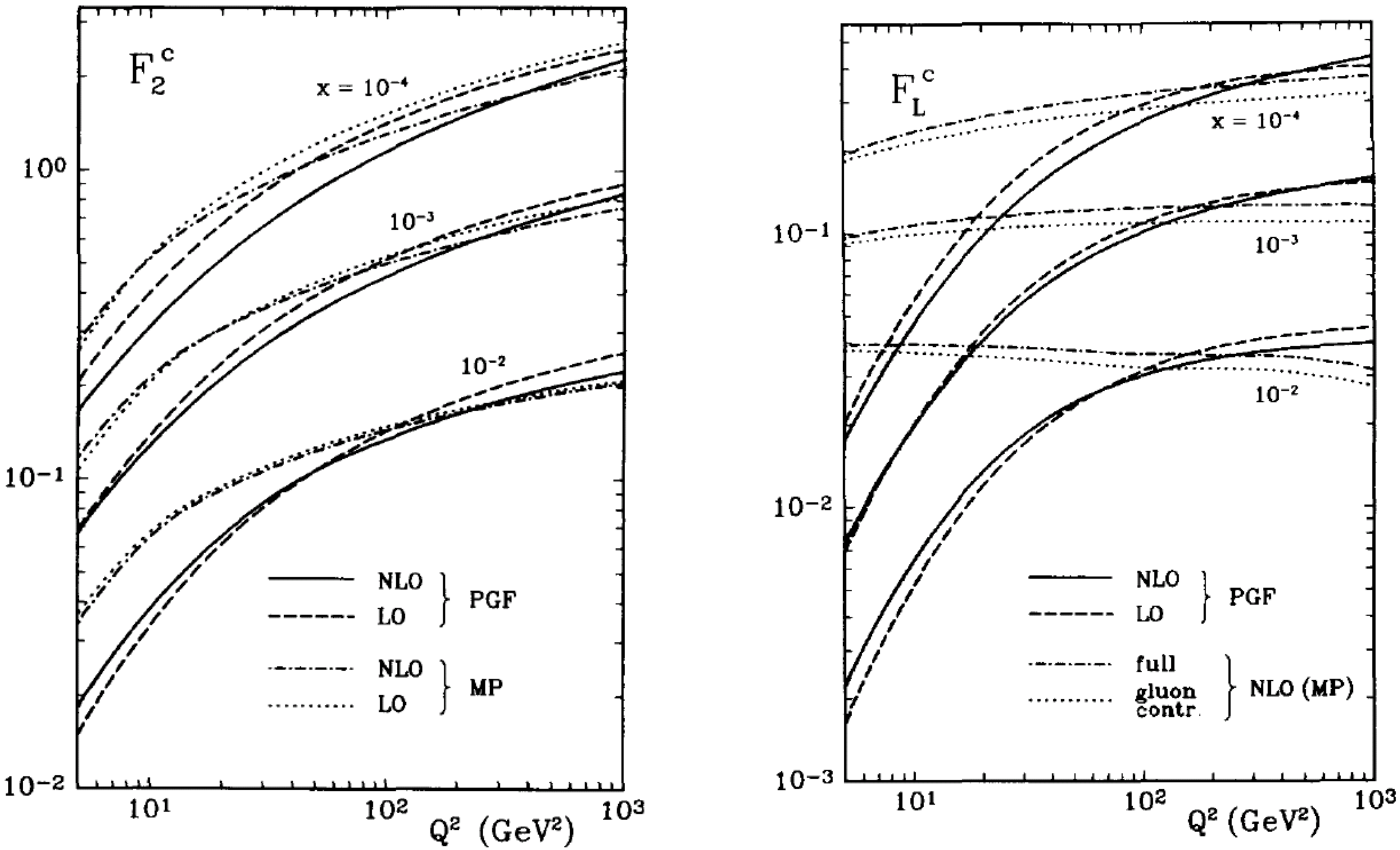
$\Longrightarrow$ Choose a physical quantity and kinematic region for which power corrections are fairly unimportant.

- Opportunities:

$\Longrightarrow$ Study of decoupling regimes in QCD.

$\Longrightarrow$ Safe one loop for certain Standard Model and "BSM" processes.

Wilson Coefficient vs Massles Parton Approach



[Glück, Reya, Stratmann, 1994]

- Even at high values of Q^2 and W^2 the massless charm approach becomes not effective; it can even lead to misleading results.

FFNS:

- Fixed order perturbation theory and Fixed number of light partons in the proton.
- The heavy quarks are produced extrinsically only.
- The large logarithmic terms in the heavy quark coefficient functions entirely determine the charm component of the structure function for large values of Q^2.

VFNS:

- Define a threshold above which the heavy quark is treated as light, thereby obtaining a parton density.
- Phenomenological interesting to remove the mass singular terms from the asymptotic heavy quark coefficient functions and absorb them into parton densities.
- Heavy Flavor initial state parton densities for the LHC. E.g. for $c\,\bar{s} \rightarrow W^+$.

The VFNS is derived from the FFNS directly. New parton density appears corresponding to the heavy quark, which is now treated as light (massless). $\Longrightarrow$ Relations between parton densities for n_f and $n_f + 1$ flavors.

The RGE does not leave room for other scenarios.

$$
\begin{aligned}
f_k(n_f+1,\mu^2)+f_{\overline{k}}(n_f+1,\mu^2) &= A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes\left[f_k(n_f,\mu^2)+f_{\overline{k}(n_f,\mu^2)}\right] \\
&\quad +\tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes\Sigma(n_f,\mu^2) \\
&\quad +\tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes G(n_f,\mu^2) \\
f_{Q+\bar{Q}}(n_f+1,\mu^2) &= \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes\Sigma(n_f,\mu^2)+\tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes G(n_f,\mu^2)\ . \\
G(n_f+1,\mu^2) &= A_{gq,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes\Sigma(n_f,\mu^2)+A_{gg,Q}^{S}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes G(n_f,\mu^2)\ . \\
\Sigma(n_f+1,\mu^2) &= \sum_{k=1}^{n_f+1}\left[f_k(n_f+1,\mu^2)+f_{\overline{k}}(n_f+1,\mu^2)\right] \\
&= \left[A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right)+n_f\tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)+\tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)\right] \\
&\quad \otimes\Sigma(n_f,\mu^2) \\
&\quad +\left[n_f\tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)+\tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)\right]\otimes G(n_f,\mu^2)
\end{aligned}
$$

Polarized Case :

$$f_i \to \Delta f_i, \qquad\qquad A_{ij}\to \Delta A_{ij}$$

6. Towards 3-Loop Precision

Need for the calculation:

- Heavy flavor (charm) contributions to DIS structure functions are rather large [20–40 % at lower values of x] .
- Increase in accuracy of the perturbative description of DIS structure functions.
- $\Longleftrightarrow$ QCD analysis and determination of $\Lambda_{\rm QCD}$, resp. $\alpha_s(M_Z^2)$, from DIS data: $\delta\alpha_s/\alpha_s \; < 1$ %.
- $\Longleftrightarrow$ Precise determination of the gluon and sea quark distributions.
- $\Longleftrightarrow$ Derivation of variable flavor number scheme for heavy quark production to $O(a_s^3)$.

Goal:

- Calculation of the heavy flavor Wilson coefficients to higher orders for $Q^2 \geq 25\,{\rm GeV}^2$ [sufficient in many applications] .
- First recalculation of the fermionic contributions to the NNLO anomalous dimensions.

Fixed moments at 3–Loop: $F_2^{Q\overline{Q}}$

Contributing OMEs:

Singlet	A_{Qg}	$A_{qg,Q}$	$A_{gg,Q}$	$A_{gq,Q}$	} mixing
Pure–Singlet		$A_{Qq}^{\rm PS}$	$A_{qq,Q}^{\rm PS}$		
Non–Singlet	$A_{qq,Q}^{\rm NS,+}$				

- All 2–loop $O(\varepsilon)$–terms in the unpolarized case are known:
- Unpolarized anomalous dimensions are known up to $O(a_s^3)$ [Moch, Vermaseren, Vogt, 2004.]
  $\Longrightarrow$ All terms needed for the renormalization of unpolarized 3–Loop heavy OMEs are present.
  $\Longrightarrow$ Calculation will provide first independent checks on $\gamma_{qg}^{(2)}$, $\gamma_{qq}^{(2),\rm PS}$ and on respective color projections of $\gamma_{qq}^{(2),\rm NS\pm,v}$, $\gamma_{gg}^{(2)}$ and $\gamma_{gq}^{(2)}$.
- Calculation proceeds in the same way in the polarized case.
- Independent checks provided by pole terms (anomalous dimensions) and sum rules for $N = 2$.

Fixed moments using MATAD

- three–loop “self-energy” type diagrams with an operator insertion
- Extension: additional scale compared to massive propagators: Mellin variable N
- Genuine tensor integrals due to

$$\Delta^{\mu_1}...\Delta^{\mu_n} \langle p|O_{\mu_1...\mu_n}|p\rangle = \Delta^{\mu_1}...\Delta^{\mu_n} \langle p|S\, \bar{\Psi}\gamma_{\mu_1} D_{\mu_2}...D_{\mu_n}\Psi|p\rangle = A(N)\cdot(\Delta p)^N$$

$$D_\mu = \partial_\mu - igt_a A_\mu^a \quad , \qquad \Delta^2 = 0.$$

- Construction of a projector to obtain the desired moment in N [undo Δ-contraction]
- 3–loop OMEs are generated with QGRAF [Nogueira, 1993.]
- Color factors are calculated with [van Ritbergen, Schellekens, Vermaseren, 1998.]
- Translation to suitable input for MATAD [Steinhauser, 2001.]

Tests performed:

- Various 2–loop calculations for $N = 2, 4, 6, ...$ were repeated
  → agreement with our previous calculation.
- Several non–trivial scalar 3–loop diagrams were calculated using Feynman–parameters for all N
  → agreement with MATAD.

General structure of the result: the PS –case

$$
\begin{aligned}
A_{Qq}^{(3),\mathsf{PS}} + A_{qq,Q}^{(3),\mathsf{PS}} = & \frac{\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}}{48}\Biggl\{\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 4(n_f+1)\beta_{0,Q} + 6\beta_0\Biggr\}\ln^3\left(\frac{m^2}{\mu^2}\right) \\
& +\Biggl\{\frac{\hat{\gamma}_{qq}^{(1),\mathsf{PS}}}{2}\Bigl((n_f+1)\beta_{0,Q} - \beta_0\Bigr) + \frac{\hat{\gamma}_{qg}^{(0)}}{8}\Bigl((n_f+1)\hat{\gamma}_{gq}^{(1)} - \gamma_{gq}^{(1)}\Bigr) - \frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(1)}}{8}\Biggr\}\ln^2\left(\frac{m^2}{\mu^2}\right) \\
& +\Biggl\{\frac{\hat{\gamma}_{qq}^{(2),\mathsf{PS}}}{2} - \zeta_2\frac{\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}}{16}\Bigl(\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 4(n_f+1)\beta_{0,Q} + 6\beta_0\Bigr) - 2a_{Qq}^{(2),\mathsf{PS}}\beta_0 \\
& +\frac{n_f+1}{2}\hat{\gamma}_{qg}^{(0)}a_{gq,Q}^{(2)} - \frac{\gamma_{gq}^{(0)}}{2}a_{Qg}^{(2)}\Biggr\}\ln\left(\frac{m^2}{\mu^2}\right) + \zeta_3\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}}{48}\Bigl(\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 4n_f\beta_{0,Q} + 6\beta_0\Bigr) \\
& +\frac{\zeta_2}{16}\Bigl(-4n_f\beta_{0,Q}\hat{\gamma}_{qq}^{(1),\mathsf{PS}} + \hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(1)}\Bigr) + 4(\beta_0 + \beta_{0,Q})\overline{a}_{Qq}^{(2),\mathsf{PS}} + \gamma_{gq}^{(0)}\overline{a}_{Qg}^{(2)} - (n_f+1)\hat{\gamma}_{qg}^{(0)}\overline{a}_{gq,Q}^{(2)} \\
& +C_F\Bigl(-(4+\frac{3}{4}\zeta_2)\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)} - 4\hat{\gamma}_{qq}^{(1),\mathsf{PS}} + 12a_{Qq}^{(2),\mathsf{PS}}\Bigr) + a_{Qq}^{(3),\mathsf{PS}} + a_{qq,Q}^{(3),\mathsf{PS}}\ .
\end{aligned}
$$

- n_f–dependence non–trivial. Take all quantities at n_f flavors and adopt notation

$$\hat{\gamma}_{ij} \equiv \gamma_{ij}(n_f+1) - \gamma_{ij}(n_f)\ , \quad \beta_{0,Q} \equiv \beta_0(n_f+1) - \beta_0(n_f)\ .$$

- There are similar formulas for the remaining OMEs.

- Number of Diagrams to be calculated:
$A_{Q(q)q}^{(3),\mathsf{PS}}$: 132, $A_{qq}^{(3),\mathsf{NS}}$: 128, $A_{gq}^{(3)}$: 89, $A_{Qg}^{(3)}$: 1498, $A_{gg,Q}^{(3)}$: 865.

- We calculated the terms

$$A_{Qq}^{(3),\mathsf{PS}} + A_{qq,Q}^{(3),\mathsf{PS}} , \quad A_{qq,Q}^{(3),\mathsf{NS}} , \quad A_{gq,Q}^{(3)} .$$

for $N = 2, 4, 6, 8, 10, 12$ using MATAD and find agreement of the pole terms with the prediction obtained from renormalization.

- An additional check is provided by the sum rule

$$A_{qq,Q}^{(3),\mathsf{NS}}\Big|_{N=2} + A_{qq,Q}^{(3),\mathsf{PS}}\Big|_{N=2} + A_{Qq}^{(3),\mathsf{PS}}\Big|_{N=2} + A_{gq,Q}^{(3)}\Big|_{N=2} = 0 ,$$

which is full filled by our result.

- All terms proportional to ζ_2 cancel in the renormalized result.

- We observe the number

$$\mathsf{B4} = -4\zeta_2 \ln^2 2 + \frac{2}{3}\ln^4 2 - \frac{13}{2}\zeta_4 + 16\mathsf{Li}_4\left(\frac{1}{2}\right) = -8\sigma_{-3,-1} + \frac{11}{2}\zeta_4 .$$

- The term B4 appears as

$$T_F C_F \left(C_F - \frac{C_A}{2}\right)\mathsf{B4} .$$

Result for the renormalized PS–term for $N = 4$.

$$A_{Qq}^{(3),\mathsf{PS}} + A_{qq,Q}^{(3),\mathsf{PS}}\Big|_{N=4} = \Big\{-\frac{484}{2025}C_F{T_F}^2(2n_f+1) + \frac{4598}{3375}C_FC_AT_F - \frac{18997}{40500}{C_F}^2T_F\Big\}\ln^3\Big(\frac{m^2}{\mu^2}\Big)$$
$$+\Big\{-\frac{16}{125}C_F{T_F}^2 + \frac{36751}{202500}C_FC_AT_F - \frac{697631}{405000}{C_F}^2T_F\Big\}\ln^2\Big(\frac{m^2}{\mu^2}\Big) + \Big\{-\frac{2131169}{303750}C_F{T_F}^2n_f$$
$$-\frac{427141}{121500}C_F{T_F}^2 + \Big(-\frac{484}{75}\zeta_3 + \frac{24888821}{2700000}\Big)C_FC_AT_F + \Big(\frac{484}{75}\zeta_3 + \frac{63582197}{16200000}\Big){C_F}^2T_F\Big\}\ln\Big(\frac{m^2}{\mu^2}\Big)$$
$$+\Big(\frac{7744}{2025}\zeta_3 - \frac{143929913}{27337500}\Big)C_F{T_F}^2n_f + \Big(-\frac{13552}{2025}\zeta_3 + \frac{218235943}{54675000}\Big)C_F{T_F}^2 + \Big(\frac{242}{225}\mathsf{B4} - \frac{242}{25}\zeta_4$$
$$+\frac{86833}{13500}\zeta_3 + \frac{4628174}{1265625}\Big)C_FC_AT_F + \Big(-\frac{484}{225}\mathsf{B4} + \frac{242}{25}\zeta_4 + \frac{298363}{20250}\zeta_3 - \frac{57518389433}{2187000000}\Big){C_F}^2T_F\ .$$

We obtain for the moments of the PS, NS and $\hat{\gamma}_{gq}^{(2)}$ anomalous dimensions

N	$\hat{\gamma}_{qq}^{(2),\mathrm{PS}}/T_F/C_F$
2	$-\frac{5024}{243}T_F(1+2n_f)+\frac{256}{3}\Big(C_F-C_A\Big)\zeta_3+\frac{10136}{243}C_A-\frac{14728}{243}C_F$
4	$-\frac{618673}{151875}T_F(1+2n_f)+\frac{968}{75}\Big(C_F-C_A\Big)\zeta_3+\frac{2485097}{506250}C_A-\frac{2217031}{675000}C_F$
6	$-\frac{126223052}{72930375}T_F(1+2n_f)+\frac{3872}{735}\Big(C_F-C_A\Big)\zeta_3+\frac{1988624681}{4084101000}C_A+\frac{11602048711}{10210252500}C_F$
8	$-\frac{13131081443}{13502538000}T_F(1+2n_f)+\frac{2738}{945}\Big(C_F-C_A\Big)\zeta_3-\frac{343248329803}{648121824000}C_A+\frac{39929737384469}{22684263840000}C_F$
10	$-\frac{265847305072}{420260754375}T_F(1+2n_f)+\frac{50176}{27225}\Big(C_F-C_A\Big)\zeta_3-\frac{1028766412107043}{1294403123475000}C_A+\frac{839864254987192}{485401171303125}C_F$
12	$-\frac{2566080055386457}{5703275664286200}T_F(1+2n_f)+\frac{49928}{39039}\Big(C_F-C_A\Big)\zeta_3-\frac{6969748954384649691}{83039693672007072000}C_A$ $+\frac{86033255402443256197}{54806197823524667520}C_F$

N	$\hat{\gamma}_{gq}^{(2)}/T_F/C_F$
2	$\frac{2272}{81}T_F(1+2n_f)+\frac{512}{3}\Big(C_A-C_F\Big)\zeta_3+\frac{88}{9}C_A+\frac{28376}{243}C_F$
4	$\frac{109462}{10125}T_F(1+2n_f)+\frac{704}{15}\Big(C_A-C_F\Big)\zeta_3-\frac{799}{12150}C_A+\frac{14606684}{759375}C_F$
6	$\frac{22667672}{3472875}T_F(1+2n_f)+\frac{2816}{105}\Big(C_A-C_F\Big)\zeta_3-\frac{253841107}{145860750}C_A+\frac{20157323311}{2552563125}C_F$
8	$\frac{339184373}{75014100}T_F(1+2n_f)+\frac{1184}{63}\Big(C_A-C_F\Big)\zeta_3-\frac{3105820553}{1687817250}C_A+\frac{8498139408671}{2268426384000}C_F$
10	$\frac{1218139408}{363862125}T_F(1+2n_f)+\frac{7168}{495}\Big(C_A-C_F\Big)\zeta_3-\frac{18846629176433}{11767301122500}\Big)C_A+\frac{529979902254031}{323600780868750}C_F$
12	$\frac{13454024393417}{5222779912350}T_F(1+2n_f)+\frac{5056}{429}\Big(C_A-C_F\Big)\zeta_3-\frac{64190493078139789}{48885219979596000}C_A+\frac{1401404001326440151}{3495293228541114000}C_F$

N	$\hat{\gamma}_{qq}^{(2),\mathrm{NS},+}/T_F/C_F$
2	$-\frac{1792}{243}T_F(1+2n_f)+\frac{256}{3}\Big(C_F-C_A\Big)\zeta_3-\frac{12512}{243}C_A-\frac{13648}{243}C_F$
4	$-\frac{384277}{30375}T_F(1+2n_f)+\frac{2512}{15}\Big(C_F-C_A\Big)\zeta_3-\frac{8802581}{121500}C_A-\frac{165237563}{1215000}C_F$
6	$-\frac{160695142}{10418625}T_F(1+2n_f)+\frac{22688}{105}\Big(C_F-C_A\Big)\zeta_3-\frac{13978373}{171500}C_A-\frac{44644018231}{243101250}C_F$
8	$-\frac{38920977797}{2250423000}T_F(1+2n_f)+\frac{79064}{315}\Big(C_F-C_A\Big)\zeta_3-\frac{1578915745223}{18003384000}C_A-\frac{91675209372043}{420078960000}C_F$
10	$-\frac{27995901056887}{1497656506500}T_F(1+2n_f)+\frac{192880}{693}\Big(C_F-C_A\Big)\zeta_3-\frac{9007773127403}{97250422500}C_A-\frac{75522073210471127}{307518802668000}C_F$
12	$-\frac{65155853387858071}{3290351344780500}T_F(1+2n_f)+\frac{13549568}{45045}\Big(C_F-C_A\Big)\zeta_3-\frac{25478252190337435009}{263228107582440000}C_A$ $-\frac{35346062280941906036867}{131745667845011220000}C_F$

$\Longrightarrow$ Agreement for the terms $\propto\ T_F$ with

[Larin, Nogueira, Ritbergen, Vermaseren, 1997; Moch, Vermaseren, Vogt, 2004.]

• How far can we go ?

$N = 14$ in some cases; generally: $N = 12$. $\Longrightarrow$ Phenomenology

CPU time: various months; up to 64 Gb processors needed.

Unfortunately not enough to perform the automatic
fixed moments $\rightarrow$ all moments turn. [J.B., Kauers, Klein, Schneider, 2008].

3-Loop OME Calculations at General N

At 3-loop order known:

- $A_{qq,Q}^{\rm PS}$, $A_{qg,Q}$: complete;
- A_{Qg}, $A_{Qq}^{\rm PS}$, $A_{qq,Q}^{\rm NS}A_{qq,Q}^{\rm NS,TR}$: all terms of $O(n_f T_F^2 C_{A/F})$
- $A_{Qq}^{\rm PS}$, $A_{qq,Q}^{\rm NS}A_{qq,Q}^{\rm NS,TR}$: all terms of $O(T_F^2 C_{A/F})$
- $A_{gq,Q}$, $A_{gg,Q}$: see this talk $\longrightarrow$ all terms of $O(n_f T_F^2 C_{A/F})$
- Ladder topologies with a single massive line: first results this talk.
- First results on Benz topologies with a single massive line: first results this talk.

The $O(n_f T_F^2 \alpha_s^3)$ contributions to $A_{gg,Q}$

$$
\begin{aligned}
A_{ggQ}^{n_f T_f^2 1\mathrm{PI}} &= S_\varepsilon^3 a_s^3 n_f T_F^2 \frac{1+(-1)^N}{2}\left(\frac{m^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon}\Biggl\{\frac{1}{\varepsilon^3}\Biggl(\mathbf{C_A}\left[\frac{512}{27}S_1 - \frac{64\left(3N^4+6N^3+13N^2+10N+16\right)}{27(N-1)N(N+1)(N+2)}\right] \\
&- \mathbf{C_F}\frac{512\left(N^2+N+2\right)^2}{9(N-1)N^2(N+1)^2(N+2)}\Biggr) + \frac{1}{\varepsilon^2}\Biggl(\mathbf{C_A}\left[\frac{1280}{81}S_1 - \frac{16P_1}{81(N-1)N^2(N+1)^2(N+2)}\right] \\
&+ \mathbf{C_F}\frac{1}{(N-1)(N+2)}\left[\frac{128\left(N^2+N+2\right)^2}{9N^2(N+1)^2}S_1 - \frac{128P_2}{27N^3(N+1)^3}\right]\Biggr) \\
&+ \frac{1}{\varepsilon}\Biggl(\mathbf{C_A}\frac{1}{(N-1)(N+2)}\Biggl[-\frac{4P_8}{81N^3(N+1)^3} - \frac{8\left(3N^4+6N^3+13N^2+10N+16\right)}{9N(N+1)}\zeta_2 \\
&+ \frac{32P_9}{27N^2(N+1)^2}S_1 + \frac{64}{9}(N-1)(N+2)\zeta_2 S_1\Biggr] + \mathbf{C_F}\frac{1}{(N-1)(N+2)}\Biggl[-\frac{160\left(N^2+N+2\right)^2}{9N^2(N+1)^2}S_1^2 \\
&- \frac{64\left(N^2+N+2\right)^2}{3N^2(N+1)^2}\zeta_2 + \frac{32\left(N^2+N+2\right)^2}{3N^2(N+1)^2}S_2 - \frac{64P_{10}}{81N^4(N+1)^4} + \frac{64P_{11}}{27N^3(N+1)^3}S_1\Biggr]\Biggr) \\
&+ \mathbf{C_A}\frac{1}{(N-1)(N+2)}\Biggl[\frac{4P_3}{27N^2(N+1)^2}S_1^2 + \frac{8P_4}{729N^3(N+1)^3}S_1 + \frac{160}{27}(N-1)(N+2)\zeta_2 S_1 \\
&- \frac{448}{27}(N-1)(N+2)\zeta_3 S_1 + \frac{P_5}{729N^4(N+1)^4} - \frac{2P_6}{27N^2(N+1)^2}\zeta_2 - \frac{4P_7}{27N^2(N+1)^2}S_2 \\
&+ \frac{56\left(3N^4+6N^3+13N^2+10N+16\right)}{27N(N+1)}\zeta_3\Biggr] + \mathbf{C_F}\frac{1}{(N-1)(N+2)}\Biggl[\frac{112\left(N^2+N+2\right)^2}{27N^2(N+1)^2}S_1^3 \\
&- \frac{16P_{12}}{27N^3(N+1)^3}S_1^2 + \frac{32P_{13}}{81N^4(N+1)^4}S_1 + \frac{16\left(N^2+N+2\right)^2}{3N^2(N+1)^2}\zeta_2 S_1 + \frac{16\left(N^2+N+2\right)^2}{3N^2(N+1)^2}S_2 S_1 \\
&- \frac{32P_{14}}{243N^5(N+1)^5} - \frac{16P_2}{9N^3(N+1)^3}\zeta_2 + \frac{448\left(N^2+N+2\right)^2}{9N^2(N+1)^2}\zeta_3 + \frac{16P_{15}}{9N^3(N+1)^3}S_2 - \frac{160\left(N^2+N+2\right)^2}{27N^2(N+1)^2}S_3\Biggr]\Biggr\}
\end{aligned}
$$

The structure of the unrenormalized OME: [Bierenbaum, Blümlein, Klein 2009 Nucl.Phys.B]

$$
\begin{aligned}
\hat{\hat{A}}_{gg,Q}^{(3)} = & \left(\frac{\hat{m}^2}{\mu^2}\right)^{3\varepsilon/2}\Bigg[\frac{1}{\varepsilon^3}\Bigg(-\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}}{6}\Big[\gamma_{gg}^{(0)}-\gamma_{qq}^{(0)}+6\beta_0+4n_f\beta_{0,Q}+10\beta_{0,Q}\Big]-\frac{2\gamma_{gg}^{(0)}\beta_{0,Q}}{3}\Big[2\beta_0+7\beta_{0,Q}\Big] \\
& -\frac{4\beta_{0,Q}}{3}\Big[2\beta_0^2+7\beta_{0,Q}\beta_0+6\beta_{0,Q}^2\Big]\Bigg)+\frac{1}{\varepsilon^2}\Bigg(\frac{\hat{\gamma}_{qg}^{(0)}}{6}\Big[\gamma_{gq}^{(1)}-(2n_f-1)\hat{\gamma}_{gq}^{(1)}\Big]+\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(1)}}{3}-\frac{\hat{\gamma}_{gg}^{(1)}}{3}\Big[4\beta_0+7\beta_{0,Q}\Big] \\
& +\frac{2\beta_{0,Q}}{3}\Big[\gamma_{gg}^{(1)}+\beta_1+\beta_{1,Q}\Big]+\frac{2\gamma_{gg}^{(0)}\beta_{1,Q}}{3}+\delta m_1^{(-1)}\Big[-\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}-2\beta_{0,Q}\gamma_{gg}^{(0)}-10\beta_{0,Q}^2-6\beta_{0,Q}\beta_0\Big]\Bigg) \\
& +\frac{1}{\varepsilon}\Bigg(\frac{\hat{\gamma}_{gg}^{(2)}}{3}-2(2\beta_0+3\beta_{0,Q})\mathbf{a}_{\mathbf{gg,Q}}^{(\mathbf{2})}-n_f\hat{\gamma}_{qg}^{(0)}\mathbf{a}_{\mathbf{gq,Q}}^{(\mathbf{2})}+\gamma_{gq}^{(0)}\mathbf{a}_{\mathbf{Qg}}^{(\mathbf{2})}+\beta_{1,Q}^{(1)}\gamma_{gg}^{(0)}+\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\zeta_2}{16}\Big[\gamma_{gg}^{(0)}-\gamma_{qq}^{(0)} \\
& +2(2n_f+1)\beta_{0,Q}+6\beta_0\Big]+\frac{\beta_{0,Q}\zeta_2}{4}\Big[\gamma_{gg}^{(0)}\{2\beta_0-\beta_{0,Q}\}+4\beta_0^2-2\beta_{0,Q}\beta_0-12\beta_{0,Q}^2\Big] \\
& +\delta m_1^{(-1)}\Big[-3\delta m_1^{(-1)}\beta_{0,Q}-2\delta m_1^{(0)}\beta_{0,Q}-\hat{\gamma}_{gg}^{(1)}\Big]+\delta m_1^{(0)}\Big[-\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}-2\gamma_{gg}^{(0)}\beta_{0,Q}-4\beta_{0,Q}\beta_0-8\beta_{0,Q}^2\Big] \\
& +2\delta m_2^{(-1)}\beta_{0,Q}\Bigg)+a_{gg,Q}^{(3)}\Bigg].
\end{aligned}
$$

$\longrightarrow$ use for checking the ε singular parts

We confirm the $n_f T_F^2$ part of the 3-Loop anomalous dimension:

[Moch, Vermaseren, Vogt 2004 Nucl.Phys.B]

$$
\begin{aligned}
\hat{\gamma}_{gg}^{(2)} &= n_f T_F^2 \mathbf{C_A} \Bigg[-\frac{32\left(8N^6+24N^5-19N^4-78N^3-253N^2-210N-96\right)}{27(N-1)N^2(N+1)^2(N+2)} S_1 \\
&\quad -\frac{8\left(87N^8+348N^7+848N^6+1326N^5+2609N^4+3414N^3+2632N^2+1088N+192\right)}{27(N-1)N^3(N+1)^3(N+2)} \Bigg] \\
&\quad + n_f T_F^2 \mathbf{C_F} \Bigg[\frac{64\left(N^2+N+2\right)^2}{3(N-1)N^2(N+1)^2(N+2)} \left(S_1^2 - 3S_2\right) - \frac{16P_1}{27(N-1)N^4(N+1)^4(N+2)} \\
&\quad + \frac{128\left(4N^6+3N^5-50N^4-129N^3-100N^2-56N-24\right)}{9(N-1)N^3(N+1)^3(N+2)} S_1 \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
P_1 &= 33N^{10}+165N^9+256N^8-542N^7-3287N^6-8783N^5-11074N^4-9624N^3 \\
&\quad -5960N^2-2112N-288
\end{aligned}
$$

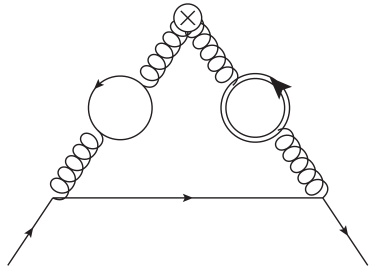
First diagrammatic recalculation

$$
\begin{aligned}
A_{gg,Q}^{(3),\overline{\mathsf{MS}}} = \; & \frac{1}{48}\Biggl\{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\Bigl(\gamma_{qq}^{(0)}-\gamma_{gg}^{(0)}-6\beta_0-4n_f\beta_{0,Q}-10\beta_{0,Q}\Bigr)-4\Bigl(\gamma_{gg}^{(0)}\Bigl[2\beta_0+7\beta_{0,Q}\Bigr]+4\beta_0^2+14\beta_{0,Q}\beta_0 \\
& +12\beta_{0,Q}^2\Bigr)\beta_{0,Q}\Biggr\}\ln^3\Bigl(\frac{\mathbf{m^2}}{\mu^2}\Bigr)+\frac{1}{8}\Biggl\{\hat{\gamma}_{qg}^{(0)}\Bigl(\gamma_{gq}^{(1)}+(1-n_f)\hat{\gamma}_{gq}^{(1)}\Bigr)+\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(1)}-4\hat{\gamma}_{gg}^{(1)}[\beta_0+2\beta_{0,Q}] \\
& +4\gamma_{gg}^{(1)}\beta_{0,Q}+4[\beta_1+\beta_{1,Q}]\beta_{0,Q}+2\gamma_{gg}^{(0)}\beta_{1,Q}\Biggr\}\ln^2\Bigl(\frac{\mathbf{m^2}}{\mu^2}\Bigr)+\frac{1}{16}\Biggl\{8\hat{\gamma}_{gg}^{(2)}-8n_f a_{gq,Q}^{(2)}\hat{\gamma}_{qg}^{(0)}+8\gamma_{gq}^{(0)}a_{Qg}^{(2)} \\
& -16a_{gg,Q}^{(2)}(2\beta_0+3\beta_{0,Q})+8\gamma_{gg}^{(0)}\beta_{1,Q}^{(1)}+\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\zeta_2\Bigl(\gamma_{gg}^{(0)}-\gamma_{qq}^{(0)}+6\beta_0+4n_f\beta_{0,Q}+6\beta_{0,Q}\Bigr) \\
& +4\beta_{0,Q}\zeta_2\Bigl(\gamma_{gg}^{(0)}+2\beta_0\Bigr)\Bigl(2\beta_0+3\beta_{0,Q}\Bigr)\Biggr\}\ln\Bigl(\frac{\mathbf{m^2}}{\mu^2}\Bigr)+2(2\beta_0+3\beta_{0,Q})\overline{\mathbf{a}}_{\mathbf{gg,Q}}^{(\mathbf{2})}+n_f\hat{\gamma}_{qg}^{(0)}\overline{\mathbf{a}}_{\mathbf{gq,Q}}^{(\mathbf{2})}-\gamma_{gq}^{(0)}\overline{\mathbf{a}}_{\mathbf{Qg}}^{(\mathbf{2})} \\
& -\beta_{1,Q}^{(2)}\gamma_{gg}^{(0)}+\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\zeta_3}{48}\Bigl(\gamma_{qq}^{(0)}-\gamma_{gg}^{(0)}-2[2n_f+1]\beta_{0,Q}-6\beta_0\Bigr)+\frac{\beta_{0,Q}\zeta_3}{12}\Bigl([\beta_{0,Q}-2\beta_0]\gamma_{gg}^{(0)} \\
& +2[\beta_0+6\beta_{0,Q}]\beta_{0,Q}-4\beta_0^2\Bigr)-\frac{\hat{\gamma}_{qg}^{(0)}\zeta_2}{16}\Bigl(\gamma_{gq}^{(1)}+\hat{\gamma}_{gq}^{(1)}\Bigr)+\frac{\beta_{0,Q}\zeta_2}{8}\Bigl(\hat{\gamma}_{gg}^{(1)}-2\gamma_{gg}^{(1)}-2\beta_1-2\beta_{1,Q}\Bigr) \\
& +\frac{\delta m_1^{(-1)}}{4}\Bigl(8a_{gg,Q}^{(2)}+24\delta m_1^{(0)}\beta_{0,Q}+8\delta m_1^{(1)}\beta_{0,Q}+\zeta_2\beta_{0,Q}\beta_0+9\zeta_2\beta_{0,Q}^2\Bigr)+\delta m_1^{(0)}\Bigl(\beta_{0,Q}\delta m_1^{(0)}+\hat{\gamma}_{gg}^{(1)}\Bigr) \\
& +\delta m_1^{(1)}\Bigl(\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}+2\beta_{0,Q}\gamma_{gg}^{(0)}+4\beta_{0,Q}\beta_0+8\beta_{0,Q}^2\Bigr)-2\delta m_2^{(0)}\beta_{0,Q}+a_{gg,Q}^{(3)}
\end{aligned}
$$

The final renormalized contribution with the -mass $\bar{m}$:

$$
\begin{aligned}
A_{gg,Q}^{(3),n_f T_F^2,\overline{\mathsf{MS}}} = n_f T_F^2 \Bigg\{ & \Bigg(\mathbf{C_F} \frac{64\left(N^2+N+2\right)^2}{9(N-1)N^2(N+1)^2(N+2)} + \mathbf{C_A} \left[\frac{128\left(N^2+N+1\right)}{27(N-1)N(N+1)(N+2)} - \frac{64}{27} S_1 \right] \Bigg) \mathbf{ln^3} \left(\frac{\mathbf{\bar{m}^2}}{\boldsymbol{\mu}^2} \right) \\
& - \mathbf{C_F} \frac{16}{3} \mathbf{ln^2} \left(\frac{\mathbf{\bar{m}^2}}{\boldsymbol{\mu}^2} \right) + \Bigg(\mathbf{C_A} \frac{1}{(N-1)(N+2)} \left[-\frac{4P_1}{81N^3(N+1)^3} - \frac{16P_2}{81N^2(N+1)^2} S_1 \right] \\
& + \mathbf{C_F} \frac{1}{(N-1)(N+2)} \left[\frac{16\left(N^2+N+2\right)^2}{N^2(N+1)^2} \left(S_1^2 - \frac{5}{3} S_2 \right) - \frac{4P_3}{9N^4(N+1)^4} - \frac{32P_4}{3N^3(N+1)^3} S_1 \right] \Bigg) \mathbf{ln} \left(\frac{\mathbf{\bar{m}^2}}{\boldsymbol{\mu}^2} \right) \\
& + \mathbf{C_A} \frac{1}{(N-1)(N+2)} \Bigg[-\frac{4P_5}{27N^2(N+1)^2} S_1^2 - \frac{8P_6}{729N^3(N+1)^3} S_1 + \frac{512}{27}(N-1)(N+2)\zeta_3 S_1 \\
& - \frac{2P_7}{729N^4(N+1)^4} - \frac{1024\left(N^2+N+1\right)}{27N(N+1)} \zeta_3 + \frac{4P_8}{27N^2(N+1)^2} S_2 \Bigg] \\
& + \mathbf{C_F} \frac{1}{(N-1)(N+2)} \Bigg[\frac{64\left(N^2+N+2\right)^2}{9N^2(N+1)^2} \left(-\frac{1}{3} S_1^3 - 8\zeta_3 + \frac{4}{3} S_3 \right) + \frac{32P_9}{27N^3(N+1)^3} S_1^2 \\
& - \frac{64P_{10}}{81N^4(N+1)^4} S_1 - \frac{32P_{11}}{243N^5(N+1)^5} - \frac{32P_{12}}{3N^3(N+1)^3} S_2 \Bigg] \Bigg\}
\end{aligned}
$$

The $O(n_f T_F^2 \alpha_s^3)$ contributions to $A_{gq,Q}$



The all-ε result constituting the color factor $T_F^2 n_f C_F$

$$\hat{\hat{A}}^{(3)}_{gq,T_F^2 n_f} = -96 a_s^3 T_F^2 n_f C_F \left(\frac{m^2}{\mu^2}\right)^{\frac{3\varepsilon}{2}} S_\varepsilon^3 \frac{1+(-1)^N}{2} e^{-\frac{3\varepsilon}{2}\gamma} \frac{(\varepsilon-1)^2(\varepsilon+2)\left(\varepsilon+N^2+N+2\right)}{\varepsilon(\varepsilon+1)(\varepsilon+3)}$$
$$\times\Gamma(1-\varepsilon)^2\Gamma\left(-\frac{\varepsilon}{2}-4\right)\Gamma\left(\frac{\varepsilon}{2}+2\right)\frac{\Gamma\left(\frac{\varepsilon}{2}+5\right)\Gamma\left(-\frac{3\varepsilon}{2}\right)\Gamma(N-1)}{\Gamma(4-2\varepsilon)\Gamma\left(\frac{\varepsilon}{2}+N+2\right)}$$

yields the renormalized contribution

$$A_{gq,Q}^{(3),n_f T_F^2,\overline{\mathsf{MS}}} = n_f T_F^2 \frac{1+(-1)^N}{2}\Biggl\{\mathbf{C_F}\frac{32\left(N^2+N+2\right)}{9(N-1)N(N+1)}\mathbf{ln^3}\left(\frac{\bar{\mathbf{m}}^2}{\boldsymbol{\mu}^2}\right)+\mathbf{C_F}\Biggl[-\frac{16\left(N^2+N+2\right)}{3(N-1)N(N+1)}\left(S_1^2+S_2\right)$$
$$+\frac{32\left(8N^3+13N^2+27N+16\right)}{9(N-1)N(N+1)^2}S_1+\frac{32\left(19N^4+81N^3+86N^2+80N+38\right)}{27(N-1)N(N+1)^3}\Biggr]\mathbf{ln}\left(\frac{\bar{\mathbf{m}}^2}{\boldsymbol{\mu}^2}\right)$$
$$+\mathbf{C_F}\Biggl[\frac{32\left(N^2+N+2\right)}{27(N-1)N(N+1)}\left(S_1^3+3S_2S_1+2S_3-24\zeta_3\right)-\frac{32\left(8N^3+13N^2+27N+16\right)}{27(N-1)N(N+1)^2}\left(S_1^2+S_2\right)$$
$$+\frac{64\left(4N^4+4N^3+23N^2+25N+8\right)}{27(N-1)N(N+1)^3}S_1+\frac{64\left(197N^5+824N^4+1540N^3+1961N^2+1388N+394\right)}{243(N-1)N(N+1)^4}\Biggr]\Biggr\}$$

Here we confirm the n_f contribution to the anomalous dimension:
[Moch, Vermaseren, Vogt 2004 Nucl.Phys.B]

$$\hat{\gamma}_{gq}^{(2),n_f} = n_f T_F^2 C_F \left(\frac{64\,(N^2+N+2)}{3(N-1)N(N+1)} - \left(S_1^2 + S_2\right) + \frac{128\,(8N^3+13N^2+27N+16)}{9(N-1)N(N+1)^2} S_1 - \frac{128\,(4N^4+4N^3+23N^2+25N+8)}{9(N-1)N(N+1)^3} \right)$$

in an independent calculation.

Furthermore we are able to check a result for the combination

$$\tilde{\gamma}_{gg}^{(2)} + \frac{\tilde{\gamma}_{gq}^{(2)} \gamma_{qg}^{(0)}}{\tilde{\gamma}_{gg}^{(0)} n_f}$$

of 3-loop anomalous dimensions, derived from the large n_f expansion in QCD
by [Bennett, Gracey 1997]; where we denote with $\tilde{\gamma}_{ij}^{(k)}$ the leading n_f coefficient of $\gamma_{ij}^{(k)}$.

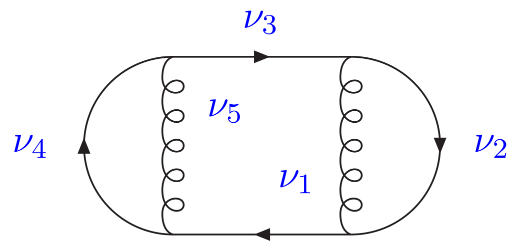
Graphs with m_c and m_b

$$
\begin{aligned}
a_{Qg}^{(3)}(N=6) \quad = \quad & T_F^2 C_A \Big\{ \frac{69882273800453}{367569090000} - \frac{395296}{19845}\zeta_3 + \frac{1316809}{39690}\zeta_2 + \frac{832369820129}{14586075000}x + \frac{1511074426112}{624023544375}x^2 - \frac{84840004938801319}{690973782403905000}x^3 \\
& + \ln\Big(\frac{m_2^2}{\mu^2}\Big)\Big[\frac{11771644229}{194481000} + \frac{78496}{2205}\zeta_2 - \frac{1406143531}{69457500}x - \frac{105157957}{180093375}x^2 + \frac{2287164970759}{7669816654500}x^3\Big] \\
& + \ln^2\Big(\frac{m_2^2}{\mu^2}\Big)\Big[\frac{2668087}{79380} + \frac{112669}{661500}x - \frac{49373}{51975}x^2 - \frac{31340489}{34054020}x^3\Big] + \ln^3\Big(\frac{m_2^2}{\mu^2}\Big)\frac{324148}{19845} + \ln^2\Big(\frac{m_2^2}{\mu^2}\Big)\ln\Big(\frac{m_1^2}{\mu^2}\Big)\frac{156992}{6615} \\
& + \ln\Big(\frac{m_2^2}{\mu^2}\Big)\ln\Big(\frac{m_1^2}{\mu^2}\Big)\Big[\frac{128234}{3969} - \frac{112669}{330750}x + \frac{98746}{51975}x^2 + \frac{31340489}{17027010}x^3\Big] + \ln\Big(\frac{m_2^2}{\mu^2}\Big)\ln^2\Big(\frac{m_1^2}{\mu^2}\Big)\frac{68332}{6615} \\
& + \ln\Big(\frac{m_1^2}{\mu^2}\Big)\Big[\frac{83755534727}{583443000} + \frac{78496}{2205}\zeta_2 + \frac{1406143531}{69457500}x + \frac{105157957}{180093375}x^2 - \frac{2287164970759}{7669816654500}x^3\Big] \\
& + \ln^2\Big(\frac{m_1^2}{\mu^2}\Big)\Big[\frac{2668087}{79380} + \frac{112669}{661500}x - \frac{49373}{51975}x^2 - \frac{31340489}{34054020}x^3\Big] + \ln^3\Big(\frac{m_1^2}{\mu^2}\Big)\frac{412808}{19845}\Big\} \\
& + T_F^2 C_F \Big\{ - \frac{3161811182177}{71471767500} + \frac{447392}{19845}\zeta_3 + \frac{9568018}{4862025}\zeta_2 - \frac{64855635472}{2552563125}x + \frac{1048702178522}{97070329125}x^2 + \frac{198056606982672}{2467763508585375}x^3 \\
& + \ln\Big(\frac{m_2^2}{\mu^2}\Big)\Big[\frac{1786067629}{204205050} - \frac{111848}{15435}\zeta_2 - \frac{128543024}{24310125}x - \frac{22957168}{3361743}x^2 - \frac{2511536080}{2191376187}x^3\Big] \\
& + \ln^2\Big(\frac{m_2^2}{\mu^2}\Big)\Big[\frac{3232799}{4862025} + \frac{752432}{231525}x + \frac{177944}{40425}x^2 + \frac{127858928}{42567525}x^3\Big] - \ln^3\Big(\frac{m_2^2}{\mu^2}\Big)\frac{111848}{19845} - \ln^2\Big(\frac{m_2^2}{\mu^2}\Big)\ln\Big(\frac{m_1^2}{\mu^2}\Big)\frac{223696}{46305} \\
& + \ln\Big(\frac{m_2^2}{\mu^2}\Big)\ln\Big(\frac{m_1^2}{\mu^2}\Big)\Big[\frac{22238456}{4862025} - \frac{1504864}{231525}x - \frac{355888}{40425}x^2 - \frac{255717856}{42567525}x^3\Big] + \ln\Big(\frac{m_2^2}{\mu^2}\Big)\ln^2\Big(\frac{m_1^2}{\mu^2}\Big)\frac{223696}{46305} \\
& + \ln\Big(\frac{m_1^2}{\mu^2}\Big)\Big[-\frac{24797875607}{1021025250} - \frac{111848}{15435}\zeta_2 + \frac{128543024}{24310125}x + \frac{22957168}{3361743}x^2 + \frac{2511536080}{2191376187}x^3\Big] \\
& + \ln^2\Big(\frac{m_1^2}{\mu^2}\Big)\Big[\frac{3232799}{4862025} + \frac{752432}{231525}x + \frac{177944}{40425}x^2 + \frac{127858928}{42567525}x^3\Big] - \ln^3\Big(\frac{m_1^2}{\mu^2}\Big)\frac{1230328}{138915}\Big\} + O\left(x^4 \ln^3(x)\right)
\end{aligned}
$$

These moments have been calculated referring **qexp** by Steinhauser et al. with operator insertions. Despite being universal, these contribution do not belong to the charm or bottom PDF. This is then the end of the VFNF.

Fixed moments using Feynman–parameters

Consider e.g the 3–loop tadpole diagram



Using Feynman–parameters, one obtains a representation in terms of a double sum

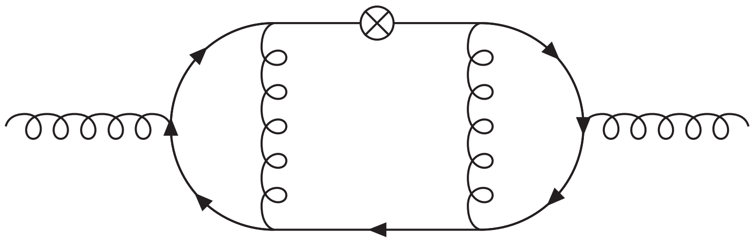
$$I \quad = \quad C\Gamma\left[\begin{matrix}2+\varepsilon/2-\nu_1, 2+\varepsilon/2-\nu_5, \nu_{12}-2-\varepsilon/2, \nu_{45}-2-\varepsilon/2, \nu_{1345}-4-\varepsilon, \nu_{12345}-6-3/2\varepsilon \\ \nu_1, \nu_2, \nu_4, 2+\varepsilon/2, \nu_{345}-2-\varepsilon/2, \nu_{12345}-4-\varepsilon\end{matrix}\right]$$

$$\sum_{m,n=0}^{\infty} \frac{(\nu_{345}-2-\varepsilon/2)_{n+m}(\nu_{12345}-6-3/2\varepsilon)_m(2+\varepsilon/2-\nu_1)_m(2+\varepsilon/2-\nu_5)_n(\nu_{45}-2-\varepsilon/2)_n}{m!n!(\nu_{12345}-4-\varepsilon)_{n+m}(\nu_{345}-2-\varepsilon/2)_m(\nu_{345}-2-\varepsilon/2)_n},$$

which derives from a Appell–function of the first kind, F_1 .

First all–N results

As in the 2–loop case, for any diagram deriving from the tadpole–ladder topology, one obtains for fixed values of N a finite sum over double sums of the same type. Consider e.g. the scalar diagram



For the above diagram, we obtained a result for arbitrary N using similar techniques as in the 2–loop case and the package SIGMA.

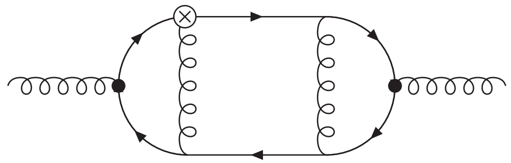
$$I_1 = -\frac{4(N+1)S_1+4}{(N+1)^2(N+2)}\zeta_3 + \frac{2S_{2,1,1}}{(N+2)(N+3)} + \frac{1}{(N+1)(N+2)(N+3)}\Bigg\{-2(3N+5)S_{3,1} - \frac{{S_1}^4}{4}$$
$$+\frac{4(N+1)S_1-4N}{N+1}S_{2,1} + 2\left((2N+3)S_1 + \frac{5N+6}{N+1}\right)S_3 + \frac{9+4N}{4}{S_2}^2 + \left(\frac{5N}{N+1}S_1 + 2\frac{7N+11}{(N+1)(N+2)}\right.$$
$$\left.-\frac{5}{2}{S_1}^2\right)S_2 + \frac{N}{N+1}{S_1}^3 + \frac{2(3N+5){S_1}^2}{(N+1)(N+2)} + \frac{4(2N+3)S_1}{(N+1)^2(N+2)} - \frac{(2N+3)S_4}{2} + 8\frac{2N+3}{(N+1)^3(N+2)}\Bigg\} + O(\varepsilon)\ .$$

For fixed N, this formula agrees with the result we obtain using MATAD.

Calculation of Convergent Massive 3-Loop Graphs

- Aim:
  - Compute fixed Mellin moments of convergent 3-loop diagrams
  - Find general N representations for all convergent 3-loop topologies
- Here we work in the α-representation to calculate the integrals.
- The corresponding graph polynomials of a graph G are given by
  - $U = \sum_T \prod_{l \notin T} \alpha_l$, where T denotes the spanning trees of G
  - $V = \sum_{l \in massive} \alpha_l$
  - different Dodgson polynomials, which can be derived from the corresponding tadpole diagram, for the operator insertions

Calculation of Convergent Massive 3-Loop Graphs



$$
\begin{aligned}
I_4(N) &= \int \cdots \int d\alpha_1\ d\alpha_2\ d\alpha_3\ d\alpha_4\ d\alpha_5\ d\alpha_6\ d\alpha_7\ d\alpha_8\ \frac{\sum_{j=0}^{N} T_{4\alpha}^{N-j} T_{4b}^{j}}{U^2 V^2} \\
T_{4\alpha} &= \alpha_5\alpha_7\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_2\alpha_5\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_2\alpha_5\alpha_8 + \alpha_8\alpha_5\alpha_4 + \alpha_5\alpha_7\alpha_8 + \alpha_2\alpha_3\alpha_8 \\
&\quad +\alpha_7\alpha_2\alpha_8 + \alpha_6\alpha_2\alpha_8 + \alpha_3\alpha_7\alpha_2 + \alpha_2\alpha_3\alpha_6 + \alpha_4\alpha_2\alpha_8 + \alpha_2\alpha_6\alpha_4 + \alpha_4\alpha_7\alpha_2 \\
T_{4b} &= +\alpha_2\alpha_5\alpha_4 + \alpha_4\alpha_2\alpha_8 + \alpha_4\alpha_7\alpha_2 + \alpha_2\alpha_5\alpha_8 + \alpha_2\alpha_3\alpha_5 + \alpha_7\alpha_2\alpha_8 + \alpha_3\alpha_7\alpha_2 + \alpha_8\alpha_5\alpha_4 \\
&\quad +\alpha_5\alpha_7\alpha_4 + \alpha_4\alpha_1\alpha_8 + \alpha_1\alpha_7\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_5\alpha_7\alpha_8 + \alpha_8\alpha_1\alpha_7 + \alpha_1\alpha_3\alpha_7 \\
U &= \alpha_2\alpha_5\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_1\alpha_3\alpha_5 + \alpha_5\alpha_7\alpha_4 + \alpha_1\alpha_6\alpha_4 + \alpha_1\alpha_3\alpha_6 + \alpha_2\alpha_3\alpha_6 + \alpha_2\alpha_6\alpha_4 \\
&\quad +\alpha_5\alpha_6\alpha_4 + \alpha_1\alpha_5\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_1\alpha_3\alpha_7 + \alpha_1\alpha_7\alpha_4 + \alpha_3\alpha_7\alpha_2 + \alpha_4\alpha_7\alpha_2 + \alpha_3\alpha_5\alpha_6 \\
&\quad +\alpha_2\alpha_3\alpha_8 + \alpha_2\alpha_5\alpha_8 + \alpha_5\alpha_7\alpha_8 + \alpha_8\alpha_5\alpha_4 + \alpha_8\alpha_5\alpha_6 + \alpha_5\alpha_3\alpha_8 + \alpha_1\alpha_8\alpha_5 + \alpha_1\alpha_8\alpha_6 \\
&\quad +\alpha_6\alpha_2\alpha_8 + \alpha_1\alpha_8\alpha_3 + \alpha_4\alpha_1\alpha_8 + \alpha_4\alpha_2\alpha_8 + \alpha_7\alpha_2\alpha_8 + \alpha_8\alpha_1\alpha_7 \\
V &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7
\end{aligned}
$$

- The integral above is a projective integral, one α-parameter may be set 1
- The operators sit on on-shell diagrams which obey specific symmetries. These are generally not obeyed by the operator insertion.
- For the above example : after applying symmetry transformations $\alpha_1 \to x_1 - \alpha_2$, $\alpha_3 \to x_2 - \alpha_4$, $\alpha_5 \to x_5 - \alpha_6$ $\alpha_2, \alpha_4, \alpha_6$ are only contained in the operator polynomials and may be integrated out at this stage.

General Values of N

- Due to the operator-insertions leading to power-type functions, the integrals do not fit directly into the framework of the algorithm for general values of N.

- In order to use the algorithm also on integrals with general values of N, a generating function is constructed e.g. by the mapping

$$p\left(\alpha_1, \cdots, \alpha_n\right)^N \to \frac{1}{1 - x\; p\left(\alpha_1, \cdots, \alpha_n\right)} \; .$$

- Performing the Feynman-parameter integrations then leads to an expression which contains hyperlogarithms $L_w(x)$ in the variable x.

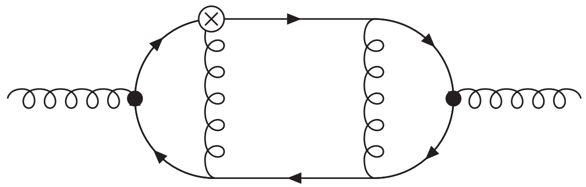
- Finally the Nth coefficient of this expression in x has to be extracted analytically. This has been done with the package `HarmonicSums` by J.Ablinger.

- Generalized harmonic sums occur:

$$S_{n_1,...,n_k}(a_1, ..., a_k)(N), \quad n_i \in \mathrm{N}, a_i \in \mathrm{Q}$$

.

Six Massive Lines & Vertex Insertion



$$
\begin{aligned}
\hat{I}_4 \;=\; & \frac{Q_1(N)}{2(1+N)^5(2+N)^5(3+N)^5} + \frac{Q_2(N)}{(1+N)^2(2+N)^2(3+N)^2}\zeta_3 + \frac{(-1)^N\left(65+101N+56N^2+13N^3+N^4\right)}{2(1+N)^2(2+N)^2(3+N)^2}S_{-3} \\
& + \frac{(-24-5N+2N^2)}{12(2+N)^2(3+N)^2}S_1^3 - \frac{1}{2(1+N)(2+N)(3+N)}S_2^2 + \frac{1}{(2+N)(3+N)}S_1^2 S_2 \\
& + \frac{Q_4(N)}{4(1+N)^3(2+N)^2(3+N)^2}S_1^2 - \frac{3}{2}S_5 - \frac{Q_5(N)}{6(1+N)^2(2+N)^2(3+N)^2}S_3 - 2S_{-2,-3} - 2\zeta_3 S_{-2} - S_{-2,1}S_{-2} \\
& + \frac{(-1)^N\left(65+101N+56N^2+13N^3+N^4\right)}{(1+N)^2(2+N)^2(3+N)^2}S_{-2,1} + \frac{(59+42N+6N^2)}{2(1+N)(2+N)(3+N)}S_4 + \frac{(5+N)}{(1+N)(3+N)}\zeta_3 S_1(2) \\
& - \frac{Q_6(N)}{4(1+N)^3(2+N)^2(3+N)^2}S_2 - \zeta_3 S_2 - \frac{3}{2}S_3 S_2 - 2S_{2,1}S_2 + \frac{(99+225N+190N^2+65N^3+7N^4)}{2(1+N)^2(2+N)^2(3+N)}S_{2,1} \\
& + \frac{Q_3(N)}{(1+N)^4(2+N)^4(3+N)^4}S_1 - \frac{(11+5N)}{(1+N)(2+N)(3+N)}\zeta_3 S_1 - \frac{Q_7(N)}{4(1+N)^2(2+N)^2(3+N)^2}S_2 S_1 - S_{2,3} \\
& + \frac{(53+29N)}{2(1+N)(2+N)(3+N)}S_3 S_1 - \frac{3(3+2N)}{(1+N)(2+N)(3+N)}S_1 S_{2,1} + \frac{(-79-40N+N^2)}{2(1+N)(2+N)(3+N)}S_{3,1} - 3S_{4,1} \\
& + S_{-2,1,-2} + \frac{\mathbf{2^{N+1}}\left(-28-25N-4N^2+N^3\right)}{(1+N)^2(2+N)(3+N)^2}S_{1,2}\left(\frac{1}{2},1\right) - \frac{(-7+2N^2)}{(1+N)(2+N)(3+N)}S_{2,1,1} \\
& + 5S_{2,2,1} + 6S_{3,1,1} + \frac{\mathbf{2^{N}}\left(-28-25N-4N^2+N^3\right)}{(1+N)^2(2+N)(3+N)^2}S_{1,1,1}\left(\frac{1}{2},1,1\right) \\
& - \frac{(5+N)}{(1+N)(3+N)}S_{1,1,2}\left(2,\frac{1}{2},1\right) - \frac{(5+N)}{2(1+N)(3+N)}S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)
\end{aligned}
$$

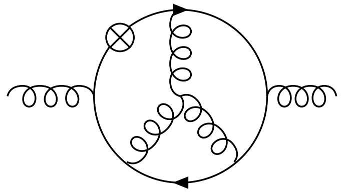
We encounter several limits of S-Sums:

$$\begin{aligned}
S_1\left(\frac{1}{2};\infty\right) &= \ln(2) \\
S_2\left(\frac{1}{2};\infty\right) &= \frac{1}{2}\left(\zeta_2-\ln^2(2)\right) \\
S_{1,1}\left(\frac{1}{2},1;\infty\right) &= \frac{\zeta_2}{2} \\
S_3\left(\frac{1}{2};\infty\right) &= \frac{\ln^3(2)}{6}-\frac{1}{2}\zeta_2\ln(2) \\
S_{1,2}\left(\frac{1}{2},1;\infty\right) &= \frac{5\zeta_3}{8} \\
S_{2,1}\left(1,\frac{1}{2};\infty\right) &= \frac{1}{12}\left(6\zeta_2\ln(2)+3\zeta_3+2\ln^3(2)\right) \\
S_{2,1}\left(\frac{1}{2},1;\infty\right) &= \zeta_3-\frac{1}{2}\zeta_2\ln(2) \\
S_{2,1}\left(\frac{1}{2},2;\infty\right) &= \frac{3}{8}(7\zeta_3-4\zeta_2\ln(2)) \\
S_{1,2}\left(\frac{1}{2},2;\infty\right) &= \frac{3}{2}\zeta_2\ln(2) \\
S_{1,1,1}\left(\frac{1}{2},1,1;\infty\right) &= \frac{3\zeta_3}{4}
\end{aligned}$$

The 2^N factors cancel in the large N limit:

$$
\begin{aligned}
\hat{I}_4 \approx\ & \zeta_2^2\left[\frac{1115231}{20N^{10}}-\frac{74121}{4N^9}+\frac{122951}{20N^8}-\frac{40677}{20N^7}+\frac{13391}{20N^6}-\frac{873}{4N^5}+\frac{1391}{20N^4}-\frac{417}{20N^3}+\frac{101}{20N^2}\right]\\
&+\zeta_3\left[\left(-\frac{95855}{2N^{10}}+\frac{31525}{2N^9}-\frac{10295}{2N^8}+\frac{3325}{2N^7}-\frac{1055}{2N^6}+\frac{325}{2N^5}-\frac{95}{2N^4}+\frac{25}{2N^3}-\frac{5}{2N^2}\right)\ln(N)\right.\\
&\left.-\frac{23280115}{2016N^{10}}+\frac{2093041}{1008N^9}-\frac{177251}{1008N^8}-\frac{25843}{336N^7}+\frac{2569}{48N^6}-\frac{155}{8N^5}+\frac{91}{24N^4}+\frac{2}{3N^3}-\frac{11}{12N^2}\right]\\
&+\zeta_2\left[\left(\frac{19171}{N^{10}}-\frac{6305}{N^9}+\frac{2059}{N^8}-\frac{665}{N^7}+\frac{211}{N^6}-\frac{65}{N^5}+\frac{19}{N^4}-\frac{5}{N^3}+\frac{1}{N^2}\right)\ln^2(N)\right.\\
&+\left(\frac{103016863}{2520N^{10}}-\frac{3091261}{315N^9}+\frac{2571839}{1260N^8}-\frac{6215}{21N^7}-\frac{293}{20N^6}+\frac{2071}{60N^5}-\frac{103}{6N^4}+\frac{67}{12N^3}-\frac{1}{N^2}\right)\ln(N)\\
&\left.+\frac{292993001621}{302400N^{10}}-\frac{4402272031}{30240N^9}+\frac{22261739}{840N^8}-\frac{78507473}{14112N^7}+\frac{180961}{144N^6}-\frac{111807}{400N^5}+\frac{629}{12N^4}-\frac{319}{72N^3}-\frac{7}{4N^2}\right]\\
&+\left(\frac{249223}{6N^{10}}-\frac{145015}{12N^9}+\frac{10295}{3N^8}-\frac{11305}{12N^7}+\frac{1477}{6N^6}-\frac{715}{12N^5}+\frac{38}{3N^4}-\frac{25}{12N^3}+\frac{1}{6N^2}\right)\mathbf{\ln^3(N)}\\
&+\left(\frac{193493767}{10080N^{10}}+\frac{210658237}{10080N^9}-\frac{21541697}{2520N^8}+\frac{243269}{96N^7}-\frac{30539}{48N^6}+\frac{2123}{16N^5}-\frac{59}{3N^4}+\frac{5}{8N^3}+\frac{1}{2N^2}\right)\ln^2(N)\\
&+\left(-\frac{2207364771673}{4233600N^{10}}+\frac{1390655509}{352800N^9}+\frac{285594061}{22050N^8}-\frac{67234111}{14400N^7}+\frac{8617073}{7200N^6}-\frac{35209}{144N^5}+\frac{116}{3N^4}-\frac{119}{24N^3}+\frac{1}{N^2}\right)\ln(N)\\
&+\frac{1344226725047831}{889056000N^{10}}-\frac{165849841805771}{889056000N^9}+\frac{808151260279}{27783000N^8}-\frac{708430537}{120960N^7}+\frac{304474703}{216000N^6}\\
&-\frac{606811}{1728N^5}+\frac{1867}{24N^4}-\frac{1813}{144N^3}+\frac{1}{N^2}+O(N^{-11})
\end{aligned}
$$

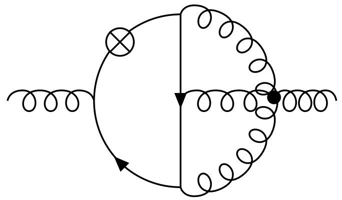
General Values of N: Higher Topologies



$$I(x) = \frac{1}{(1+N)(2+N)x}\Big\{\zeta_3\Big[2L(\{-1\},x) - 2(-1+2x)L(\{1\},x) - 4L(\{1,1\},x)\Big] - 3L(\{-1,0,0,1\},x)$$
$$+2L(\{-1,0,1,1\},x) - 2xL(\{0,0,1,1\},x) + 3xL(\{0,1,0,1\},x) - xL(\{0,1,1,1\},x)$$
$$+(-3+2x)L(\{1,0,0,1\},x) + 2xL(\{1,0,1,1\},x) - (-1+5x)L(\{1,1,0,1\},x) + xL(\{1,1,1,1\},x)$$
$$-2L(\{1,0,0,1,1\},x) + 3L(\{1,0,1,0,1\},x) - L(\{1,0,1,1,1\},x) + 2L(\{1,1,0,0,1\},x)$$
$$+2L(\{1,1,0,1,1\},x) - 5L(\{1,1,1,0,1\},x) + L(\{1,1,1,1,1\},x)\Big\}$$

$$I(N) = \frac{1}{(N+1)(N+2)(N+3)}\Big\{\frac{648+1512N+1458N^2+744N^3+212N^4+32N^5+2N^6}{(1+N)^3(2+N)^3(3+N)^3}$$
$$-\frac{2\left(-1+(-1)^N+N+(-1)^N N\right)}{(1+N)}\zeta_3 - (-1)^N S_{-3} - \frac{N}{6(1+N)}S_1^3 + \frac{1}{24}S_1^4$$
$$-\frac{(7+22N+10N^2)}{2(1+N)^2(2+N)}S_2 - \frac{19}{8}S_2^2 - \frac{1+4N+2N^2}{2(1+N)^2(2+N)}S_1^2 + \frac{9}{4}S_2 - \frac{(-9+4N)}{3(1+N)}S_3$$
$$-\frac{1}{4}S_4 - 2(-1)^N S_{-2,1} + \frac{(-1+6N)}{(1+N)}S_{2,1} + \frac{54+207N+246N^2+130N^3+32N^4+3N^5}{(1+N)^3(2+N)^2(3+N)^2}S_1$$
$$+4\zeta_3 S_1 - \frac{(-2+7N)}{2(1+N)}S_2S_1 + \frac{13}{3}S_3S_1 - 7S_{2,1}S_1 - 7S_{3,1} + 10S_{2,1,1}\Big\}$$

General Values of N: Higher Topologies



$$
\begin{aligned}
I(N) \quad = \quad & \frac{1}{(N+1)(N+2)}\Bigg\{\frac{2\left(1-13(-1)^N+(-1)^N 2^{3+N}+N-7(-1)^N N+3(-1)^N 2^{1+N}N\right)}{(1+N)(2+N)}\zeta_3 \\
& +\frac{1}{(2+N)}S_3+\frac{(-1)^N}{2(2+N)}S_1^3-\frac{(-1)^N(3+2N)}{2(1+N)^2(2+N)}S_2+\frac{5(-1)^N}{2}S_2^2 \\
& +\frac{(-1)^N(3+2N)}{2(1+N)^2(2+N)}S_1^2-\frac{(-1)^N}{2}S_2S_1^2+\frac{3(-1)^N(4+3N)}{(1+N)(2+N)}S_3+3(-1)^N S_4+\frac{2}{(2+N)}S_{-2,1} \\
& +2(-1)^N\zeta_3 S_1,(2)+\frac{2(-1)^N(3+N)}{(1+N)(2+N)}S_{2,1}-12(-1)^N S_1\zeta_3 \\
& +\frac{(-1)^N(5+7N)}{2(1+N)(2+N)}S_1S_2+3(-1)^N S_1S_3+4(-1)^N S_{2,1}S_1-4(-1)^N S_{3,1} \\
& -\frac{4\left((-1)^N 2^{2+N}-3(-2)^N N+3(-1)^N 2^{1+N}N\right)}{(1+N)(2+N)}S_{1,2}\left(\frac{1}{2},1\right)-5(-1)^N S_{2,1,1} \\
& +\frac{2\left(-(-1)^N 2^{2+N}-13(-2)^N N+5(-1)^N 2^{1+N}N\right)}{(1+N)(2+N)}S_{1,1,1}\left(\frac{1}{2},1,1\right) \\
& -2(-1)^N S_{1,1,2}\left(2,\frac{1}{2},1\right)-(-1)^N S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)\Bigg\}
\end{aligned}
$$

7. Conclusions

- The heavy flavor contributions to the structure function F_2 are rather large in the region of lower values of x.
- Competitive QCD precision analyzes therefore require the description of the heavy quark contributions to 3–loop order.
- The inclusive heavy flavor contributions to $F_{2,L}(x, Q^2)$ are known to NLO (in semi-analytic form with fast numeric implementations.)
- Complete analytic results are known in the region $Q^2 \gg m^2$ at NLO for $F_{2,L}(x, Q^2), g_{1,2}(x, Q^2)$. They are expressed in terms of massive operator matrix elements and the corresponding massless Wilson coefficients.
  Threshold resummations were performed.
- $F_L(x, Q^2)$ is known to NNLO for $Q^2 \gg m^2$.
- The calculation of fixed moments of the massive operator matrix elements at $O(a_s^3)$ has been performed for $N = 10, .., 14$.
- We also calculate the matrix elements necessary to transform from the FFNS to the VFNS.
- The calculation of the massive operator matrix elements at $O(a_s^3)$ for general values of N is underway. We reached Ladder and Benz topologies that far.