The Frobenius group T_7 as a symmetry in the lepton sector

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06.10.2011

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- Yukawa terms and mass matrices
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- Interactions
- The scalar potential of Φ
- Conclusions

Introduction

Neutrinos are puzzling:

- experimentally found properties (i.e. nonzero mass differences) can not be included in the SM without righthanded neutrinos
- mixing angles (one vanishingly small, others large) can not be explained
- absolute masses still unknown
- smallness of masses can not be explained
- mass spectrum can not be explained
- Dirac or Majorana nature is unsolved
- extremely hard to measure due to small interaction rates
- ...

Introduction

Favoured:
Tribimaximal neutrino mixing matrix (good agreement with experimental data)

$$U_{PMNS} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{1}$$

$\Rightarrow$ underlying family symmetry: discrete subgroups of $SU(3)$

Smallest group with 3-dim irreps: A_4
Smallest group with two nonequivalent 3-dim irreps: T_7, suggested by Cao, Khalil, Ma, Okada (2010)

Elements of T_7

21 elements, two generators:

$$a = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{2}$$

$$\rho = e^{\frac{2i\pi}{7}}$$

Important identities:

$$a^7 = e, \; b^3 = e, \; ba = a^2 b \tag{3}$$

Elements of T_7

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad b^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$a = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^4 \end{pmatrix} \quad ab = \begin{pmatrix} 0 & \rho & 0 \\ 0 & 0 & \rho^2 \\ \rho^4 & 0 & 0 \end{pmatrix} \quad ab^2 = \begin{pmatrix} 0 & 0 & \rho \\ \rho^2 & 0 & 0 \\ 0 & \rho^4 & 0 \end{pmatrix}$$

$$a^2 = \begin{pmatrix} \rho^2 & 0 & 0 \\ 0 & \rho^4 & 0 \\ 0 & 0 & \rho \end{pmatrix} \quad a^2b = \begin{pmatrix} 0 & \rho^2 & 0 \\ 0 & 0 & \rho^4 \\ \rho & 0 & 0 \end{pmatrix} \quad a^2b^2 = \begin{pmatrix} 0 & 0 & \rho^2 \\ \rho^4 & 0 & 0 \\ 0 & \rho & 0 \end{pmatrix}$$

$$a^3 = \begin{pmatrix} \rho^3 & 0 & 0 \\ 0 & \rho^6 & 0 \\ 0 & 0 & \rho^5 \end{pmatrix} \quad a^3b = \begin{pmatrix} 0 & \rho^3 & 0 \\ 0 & 0 & \rho^6 \\ \rho^5 & 0 & 0 \end{pmatrix} \quad a^3b^2 = \begin{pmatrix} 0 & 0 & \rho^3 \\ \rho^6 & 0 & 0 \\ 0 & \rho^5 & 0 \end{pmatrix}$$

$$a^4 = \begin{pmatrix} \rho^4 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{pmatrix} \quad a^4b = \begin{pmatrix} 0 & \rho^4 & 0 \\ 0 & 0 & \rho \\ \rho^2 & 0 & 0 \end{pmatrix} \quad a^4b^2 = \begin{pmatrix} 0 & 0 & \rho^4 \\ \rho & 0 & 0 \\ 0 & \rho^2 & 0 \end{pmatrix}$$

$$a^5 = \begin{pmatrix} \rho^5 & 0 & 0 \\ 0 & \rho^3 & 0 \\ 0 & 0 & \rho^6 \end{pmatrix} \quad a^5b = \begin{pmatrix} 0 & \rho^5 & 0 \\ 0 & 0 & \rho^3 \\ \rho^6 & 0 & 0 \end{pmatrix} \quad a^5b^2 = \begin{pmatrix} 0 & 0 & \rho^5 \\ \rho^3 & 0 & 0 \\ 0 & \rho^6 & 0 \end{pmatrix}$$

$$a^6 = \begin{pmatrix} \rho^6 & 0 & 0 \\ 0 & \rho^5 & 0 \\ 0 & 0 & \rho^3 \end{pmatrix} \quad a^6b = \begin{pmatrix} 0 & \rho^6 & 0 \\ 0 & 0 & \rho^5 \\ \rho^3 & 0 & 0 \end{pmatrix} \quad a^6b^2 = \begin{pmatrix} 0 & 0 & \rho^6 \\ \rho^5 & 0 & 0 \\ 0 & \rho^3 & 0 \end{pmatrix}$$

Irreducible representations of T_7

$$\sum_i d_i^2 = |G| \tag{4}$$

dim.	name	a	b
1	$\underline{1}_1$	1	1
1	$\underline{1}_2$	1	ω
1	$\underline{1}_3$	1	ω^2
3	$\underline{3}$	$\begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^4 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
3	$\underline{3}^*$	$\begin{pmatrix} \rho^6 & 0 & 0 \\ 0 & \rho^5 & 0 \\ 0 & 0 & \rho^3 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$$\omega = e^{\frac{2i\pi}{3}}$$

Tensor products of the irreps of T_7

$\underline{1}_p \otimes \underline{1}_q \cong \underline{1}_{(p+q) mod(3)}$
$\underline{1}_p \otimes \underline{3} \cong \underline{3}$
$\underline{1}_p \otimes \underline{3}^* \cong \underline{3}^*$
$\underline{3} \otimes \underline{3} \cong \underline{3} \oplus \underline{3}^* \oplus \underline{3}^*$
$\underline{3}^* \otimes \underline{3}^* \cong \underline{3}^* \oplus \underline{3} \oplus \underline{3}$
$\underline{3} \otimes \underline{3}^* \cong \underline{3}^* \oplus \underline{3} \oplus \underline{1}_1 \oplus \underline{1}_2 \oplus \underline{1}_3$

Conjugacy classes of T_7

name	# elements	elements
C_1	1	e
C_2	3	$a,\ a^2,\ a^4$
C_3	3	$a^3,\ a^5,\ a^6$
C_4	7	$b,\ ab,\ a^2b,\ a^3b,\ a^4b,\ a^5b,\ a^6b$
C_5	7	$b^2,\ ab^2,\ a^2b^2,\ a^3b^2,\ a^4b^2,\ a^5b^2,\ a^6b^2$

Character table of T_7

	C_1	C_2	C_3	C_4	C_5
n	1	3	3	7	7
ord(g)	1	7	7	3	3
$\underline{1}_1$	1	1	1	1	1
$\underline{1}_2$	1	1	1	ω	ω^2
$\underline{1}_3$	1	1	1	ω^2	ω
$\underline{3}$	3	η	η^*	0	0
$\underline{3}^*$	3	η^*	η	0	0

$$\omega = e^{\frac{2i\pi}{3}}$$

$$\eta = -\frac{1}{2} + i\frac{\sqrt{7}}{2}$$

Frobenius groups: Definitions

Definition

Let G be a finite group that contains a nontrivial subgroup H with the property that $H \cap H^g = \{e\}$, where e is the neutral element of G and $\forall g \in G \backslash H$, H^g is defined as $H^g := \{g^{-1}hg | h \in H\}$. Then G is a **Frobenius group** and the subgroup H is called a **Frobenius complement** of G.

Definition

$K = \left(G \backslash \bigcup_{g \in G} H^g\right) \cup \{e\}$ is called the **Frobenius kernel** of G with respect to H.

Frobenius groups: Theorems

Theorem

Theorem of Frobenius
$K \leq G$.

Theorem

$G \cong K \rtimes H$ *as a semidirect product.*

T_7 as a Frobenius group

$H = \{e, b, b^2\}$ is a Frobenius complement of T_7.

The Frobenius kernel of T_7 with respect to H is

$$K = (T_7 \backslash \bigcup_{g \in T_7} H^g) \cup \{e\} = \{e, a, a^2, a^3, a^4, a^5, a^6\}. \qquad (5)$$

Every element $g \in T_7$ can be written as kh, $T_7 \cong K \rtimes H$.

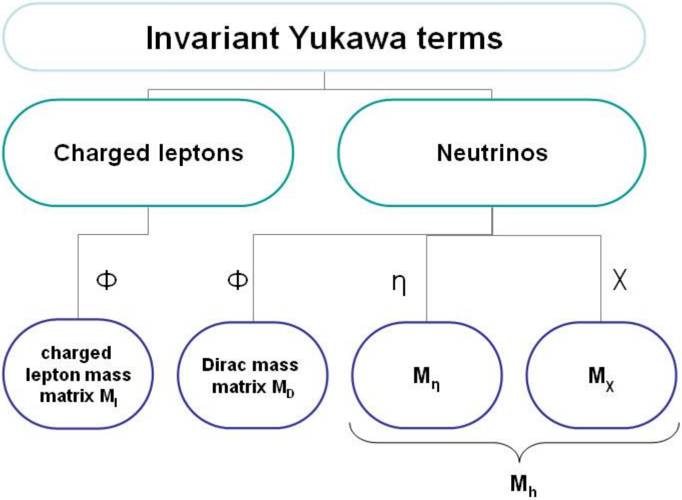
Transformation properties of the particles

Model suggested by Cao, Khalil, Ma, Okada (2010)

field	$L_{L,i}$	$\bar{\ell}_{R,i}$	$\nu_{R,i}$	Φ_i	$\tilde{\Phi}_i$	χ_i	η_i
$SU(2)_L$	$\underline{2}$	$\underline{1}$	$\underline{1}$	$\underline{2}$	$\underline{2}$	$\underline{1}$	$\underline{1}$
Y	-1	+2	0	+1	-1	0	0
T_7	$\underline{3}$	$\underline{1}_i$	$\underline{3}$	$\underline{3}$	$\underline{3}^*$	$\underline{3}$	$\underline{3}^*$
Y_{B-L}	-1	1	-1	0	0	-2	-2

$$\Phi_i = \begin{pmatrix} \Phi_i^+ \\ \Phi_i^0 \end{pmatrix}$$

Goal: Mass matrices



Yukawa terms for the charged leptons

Yukawa terms $\bar{L}_L \Phi \ell_R$ and $\bar{\ell}_R \Phi^\dagger L_L$
$\Rightarrow$ mass terms $\bar{\ell}_L M_\ell \ell_R$ and $\bar{\ell}_R M_\ell^\dagger \ell_L$

Tensor products:

$$\begin{aligned} & f_1(\bar{L}_{L,1}\Phi_1 + \bar{L}_{L,2}\Phi_2 + \bar{L}_{L,3}\Phi_3)\ell_{R,1} + \\ & f_2(\bar{L}_{L,1}\Phi_1 + \omega^2\bar{L}_{L,2}\Phi_2 + \omega\bar{L}_{L,3}\Phi_3)\ell_{R,2} + \\ & f_3(\bar{L}_{L,1}\Phi_1 + \omega\bar{L}_{L,2}\Phi_2 + \omega^2\bar{L}_{L,3}\Phi_3)\ell_{R,3} \end{aligned} \tag{6}$$

Invariance under T_7

Sums inside brackets transform as 1-dimensional representations:

$$\begin{aligned}(\bar{L}_{L,1}\Phi_1 + \bar{L}_{L,2}\Phi_2 + \bar{L}_{L,3}\Phi_3) &\sim \underline{1}_1,\\ (\bar{L}_{L,1}\Phi_1 + \omega^2\bar{L}_{L,2}\Phi_2 + \omega\bar{L}_{L,3}\Phi_3) &\sim \underline{1}_2,\\ (\bar{L}_{L,1}\Phi_1 + \omega\bar{L}_{L,2}\Phi_2 + \omega^2\bar{L}_{L,3}\Phi_3) &\sim \underline{1}_3.\end{aligned} \tag{7}$$

$\ell_{R,i}$ outside brackets: $\ell_{R,1} \sim \underline{1}_1$, $\ell_{R,2} \sim \underline{1}_3$, $\ell_{R,3} \sim \underline{1}_2$
$\Rightarrow$ invariance under the group T_7

Mass matrix for the charged leptons

Read off charged lepton mass matrix:

$$M_\ell = \begin{pmatrix} f_1 v_1 & f_2 v_1 & f_3 v_1 \\ f_1 v_2 & f_2 \omega^2 v_2 & f_3 \omega v_2 \\ f_1 v_3 & f_2 \omega v_3 & f_3 \omega^2 v_3 \end{pmatrix} \tag{8}$$

v_i are VEV of Φ_i^0

Hermitian conjugate:

$$M_\ell^\dagger = \begin{pmatrix} f_1^* v_1^* & f_1^* v_2^* & f_1^* v_3^* \\ f_2^* v_1^* & f_2^* \omega v_2^* & f_2^* \omega^2 v_3^* \\ f_3^* v_1^* & f_3^* \omega^2 v_2^* & f_3^* \omega v_3^* \end{pmatrix} \tag{9}$$

Yukawa terms for the neutrinos

Goal: Find seesaw mass matrix

$$M_{light} = -M_D^T (M_h^*)^{-1} M_D \tag{10}$$

Yukawa terms $-\bar{\nu}_R \tilde{\Phi}^\dagger L_L$ and $-\overline{L}_L \tilde{\Phi} \nu_R$
$\Rightarrow$ Dirac mass terms $\bar{\nu}_R M_D \nu_L$ and $\bar{\nu}_L M_D^\dagger \nu_R$

Symmetry $\rightarrow$ all coupling constants need to be equal,
$f_{D,1} = f_{D,2} = f_{D,3} =: f_D$

Dirac mass matrix for the neutrinos

Dirac neutrino mass matrix:

$$M_D = f_D^* \begin{pmatrix} 0 & 0 & v_3 \\ v_1 & 0 & 0 \\ 0 & v_2 & 0 \end{pmatrix} \tag{11}$$

Hermitian conjugate:

$$M_D^\dagger = f_D \begin{pmatrix} 0 & v_1^* & 0 \\ 0 & 0 & v_2^* \\ v_3^* & 0 & 0 \end{pmatrix} \tag{12}$$

Additional Yukawa terms for the neutrinos

Two more invariant Yukawa terms:

$$\chi^\dagger \nu_R^T C^{-1} \nu_R \quad \text{and} \quad \eta^\dagger \nu_R^T C^{-1} \nu_R \tag{13}$$

and their hermitian conjugates

$$\nu_R^\dagger C \nu_R^* \chi \quad \text{and} \quad \nu_R^\dagger C \nu_R^* \eta \tag{14}$$

$\Rightarrow$ heavy Majorana mass matrix $M_h = M_\chi + M_\eta$

$B - L$ quantum number crucial here: guarantees that χ and η are independent fields

Majorana mass matrix for the neutrinos, part 1

$$M_\chi = h \begin{pmatrix} u_2^* & 0 & 0 \\ 0 & u_3^* & 0 \\ 0 & 0 & u_1^* \end{pmatrix} \tag{15}$$

u_i...VEV of χ_i, h...coupling constant of $\nu_{R,i}$ (all equal!)

Hermitian conjugate:

$$M_\chi^\dagger = h^* \begin{pmatrix} u_2 & 0 & 0 \\ 0 & u_3 & 0 \\ 0 & 0 & u_1 \end{pmatrix} \tag{16}$$

Majorana mass matrix for the neutrinos, part 2

$$M_\eta = h' \begin{pmatrix} 0 & u'^*_3 & u'^*_2 \\ u'^*_3 & 0 & u'^*_1 \\ u'^*_2 & u'^*_1 & 0 \end{pmatrix} \qquad (17)$$

Hermitian conjugate:

$$M_\eta^\dagger = h'^* \begin{pmatrix} 0 & u'_3 & u'_2 \\ u'_3 & 0 & u'_1 \\ u'_2 & u'_1 & 0 \end{pmatrix} \qquad (18)$$

u'_i...VEV of η_i
Assumption: $u_i \sim u'_i$.

Note: M_χ and M_η are symmetric

Heavy Majorana mass matrix for the neutrinos

$$M_h = M_\chi + M_\eta = h \begin{pmatrix} u_2^* & 0 & 0 \\ 0 & u_3^* & 0 \\ 0 & 0 & u_1^* \end{pmatrix} + h' \begin{pmatrix} 0 & {u'}_3^* & {u'}_2^* \\ {u'}_3^* & 0 & {u'}_1^* \\ {u'}_2^* & {u'}_1^* & 0 \end{pmatrix} \tag{19}$$

Hermitian conjugate:

$$M_h^\dagger = M_\chi^\dagger + M_\eta^\dagger = h^* \begin{pmatrix} u_2 & 0 & 0 \\ 0 & u_3 & 0 \\ 0 & 0 & u_1 \end{pmatrix} + {h'}^* \begin{pmatrix} 0 & {u'}_3 & {u'}_2 \\ {u'}_3 & 0 & {u'}_1 \\ {u'}_2 & {u'}_1 & 0 \end{pmatrix} \tag{20}$$

Assumptions on the seesaw mass matrix

$$M_{light} = -M_D^T (M_h^*)^{-1} M_D \tag{21}$$

- $u := u_1 = u_2 = u_3 \neq 0 \rightarrow \chi$ breaks in the $(1,1,1)$ direction
- $u_1' = u_2' = 0$ but $u_3' \neq 0 \rightarrow \eta$ breaks in the $(0,0,1)$ direction
- $v_1 = v_2 = v_3 =: v$

$\Rightarrow Z_3 - Z_2$ misalignment

Simplifying the seesaw mass matrix

Define $A := h^* u$ and $B := h'^* u'_3$

$$M_h = \begin{pmatrix} h^* u & h'^* u'_3 & 0 \\ h'^* u'_3 & h^* u & 0 \\ 0 & 0 & h^* u \end{pmatrix} = \begin{pmatrix} A & B & 0 \\ B & A & 0 \\ 0 & 0 & A \end{pmatrix} \tag{22}$$

Inverse:

$$M_h^{-1} = \frac{1}{A^3 - AB^2} \begin{pmatrix} A^2 & -AB & 0 \\ -AB & A^2 & 0 \\ 0 & 0 & A^2 - B^2 \end{pmatrix} \tag{23}$$

Light neutrino mass matrix

$$M_{light} = -M_D^T M_h^{-1} M_D = -\frac{(f_D^*)^2 v^2}{A^3 - AB^2} \begin{pmatrix} A^2 & 0 & -AB \\ 0 & A^2 - B^2 & 0 \\ -AB & 0 & A^2 \end{pmatrix} \tag{24}$$

Assumption on scales already made in the context of $B - L$ gauging:

$$M_D \sim v \text{ and } M_h \sim u \sim u_3'$$
$$v << u \rightarrow M_D << M_h$$

The seesaw mass matrix in terms of three independent parameters

Define $C := f_D^{*2} v^2$, $p_1^2 := \frac{CA^2}{A^3 - AB^2}$ and $p_2^2 := \frac{CB^2}{A^3 - AB^2}$, absorb overall phase: $p_1 = e^{i\alpha_1}|p_1|$

$$M_{light} = e^{2i\alpha_1} \begin{pmatrix} -|p_1|^2 & 0 & e^{-i\alpha_1}|p_1|p_2 \\ 0 & -|p_1|^2 + e^{-2i\alpha_1} p_2^2 & 0 \\ e^{-i\alpha_1}|p_1|p_2 & 0 & -|p_1|^2 \end{pmatrix} \tag{25}$$

Three real parameters $|p_1|$, $\Im(e^{-i\alpha_1}p_2)$ and $\Re(e^{-i\alpha_1}p_2)$

PMNS neutrino mixing matrix

Pontecorvo-Maki-Nakagawa-Sakata (PMNS) neutrino mixing matrix: $V = U_L^{\ell\dagger} U_\nu$

Diagonalize mass matrices:

$$U_L^{\ell\dagger} M_\ell U_R^\ell = \hat{M}_\ell \qquad (26)$$

and

$$U_\nu^T M_{light} U_\nu = \hat{M}_{light} \qquad (27)$$

Bidiagonalization of the charged lepton mass matrix

U_L^ℓ: bidiagonalize the charged lepton mass matrix:

$$M_\ell = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \sqrt{3} v \begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & f_3 \end{pmatrix} \tag{28}$$

so

$$U_L^\ell = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \tag{29}$$

and

$$U_L^{\ell\dagger} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \tag{30}$$

Diagonalization of the neutrino mass matrix

Neutrino sector:

$$\tilde{U}_\nu = \begin{pmatrix} r & 0 & -r \\ 0 & 1 & 0 \\ r & 0 & r \end{pmatrix} \tag{31}$$

where $r := \frac{1}{\sqrt{2}}$

Diagonalizes M_{light}:

$$\hat{M}_{light} = \tilde{U}_\nu^T M_{light} \tilde{U}_\nu =$$
$$-\frac{(f_D^*)^2 v^2}{A^3 - AB^2} \begin{pmatrix} A^2 - AB & 0 & 0 \\ 0 & A^2 - B^2 & 0 \\ 0 & 0 & A^2 + AB \end{pmatrix} \tag{32}$$

$$U_\nu = \tilde{U}_\nu e^{i\hat{\beta}} \tag{33}$$

The tribimaximal neutrino mixing matrix

$$V = U_L^{\ell\dagger} U_\nu = \tag{34}$$

$$\frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & 1 & 0 \\ \frac{1+\omega^2}{\sqrt{2}} & \omega & \frac{-1+\omega^2}{\sqrt{2}} \\ \frac{1+\omega}{\sqrt{2}} & \omega^2 & \frac{-1+\omega}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{i\beta_3} \end{pmatrix} = \tag{35}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & -\omega^2 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{i\beta_3} \end{pmatrix} \tag{36}$$

The tribimaximal neutrino mixing matrix

Phases on both sides irrelevant for mixing $\Rightarrow$ tribimaximal mixing!
(Harrison, Perkins, Scott (1999,2002))

Define

$$\tilde{V} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{37}$$

the mixing matrix without the phases.

The tribimaximal neutrino mixing matrix

General form:

$$\tilde{V} = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{38}$$

$s_{ij} = \sin(\theta_{ij})$, $c_{ij} = \cos(\theta_{ij})$

Only parameters: the three mixing angles θ_{ij} and one phase δ

Summary of best fits from experimental data

parameter	$\pm 1\sigma$	$\pm 2\sigma$	$\pm 3\sigma$
$\sin^2(\theta_{12})$	$0.296 - 0.329$	$0.280 - 0.347$	$0.265 - 0.364$
$\sin^2(\theta_{23})$	$0.39 - 0.50$	$0.36 - 0.60$	$0.34 - 0.64$
$\sin^2(\theta_{13})$	$0.018 - 0.032$	$0.012 - 0.041$	$0.005 - 0.050$

Data from Fogli, Lisi, Marrone, Palazzo, Rotunno (2011)

Tribimaximal still possible, but not realized exactly (radiative corrections)

New decays of $\tau^{\pm}$ implied by the Z_3 symmetry

$$\tau^- \to \mu^- \mu^- e^+ \tag{39}$$

and

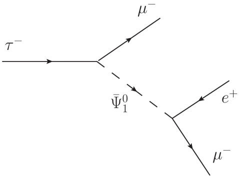
$$\tau^- \to e^- e^- \mu^+, \tag{40}$$

$$\tau^+ \to \mu^+ \mu^+ e^- \tag{41}$$

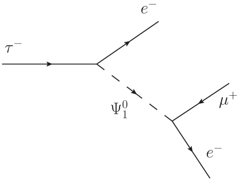
and

$$\tau^+ \to e^+ e^+ \mu^- \tag{42}$$

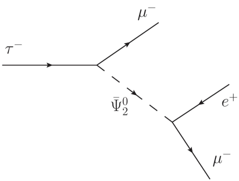
Diagrams of the new τ^- decays



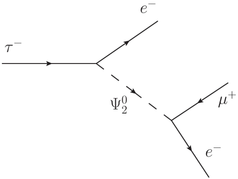
1.a First τ^- decay via Ψ_1^0



1.b Second τ^- decay via Ψ_1^0



1.c First τ^- decay via Ψ_2^0



1.d Second τ^- decay via Ψ_2^0

Experimental mass limits

Experimental upper limit:

$$B(\tau^+ \to \mu^+ \mu^+ e^-) \leq 2.3 \times 10^{-8} \tag{43}$$

$$B(\tau^+ \to \mu^+ \nu_\mu \bar{\nu}_\tau) = 0.1736 \tag{44}$$

$$B(\tau^+ \to \mu^+ \mu^+ e^-) = \frac{9 m_\tau^2 m_\mu^2 (m_1^2 + m_2^2)^2}{m_1^4 m_2^4} B(\tau^+ \to \mu^+ \nu_\mu \bar{\nu}_\tau) \tag{45}$$

Lower limit:

$$\frac{m_1^2 m_2^2}{\sqrt{m_1^2 + m_2^2}} \geq 39\ \mathrm{GeV} \tag{46}$$

Principles of an experimental test of this theory

Principle: observation of decay products of $B - L$ gauge boson Z'

$$Z' \to \Psi^0_{1,2}\bar{\Psi}^0_{1,2} \tag{47}$$

Final states:

1. $\tau^+\tau^-\mu^-\mu^+$
2. $\tau^+\tau^+\mu^-e^-$
3. $\tau^-\tau^-\mu^+e^+$
4. $\tau^-\tau^+e^+e^-$

The scalar potential of Φ

Only Φ_i-terms (possible if supersymmetrised)
Note: supersymmetrisation not consistent with Yukawa terms ($\bar{L}_L \Phi \ell_R$ and $-\bar{\nu}_R \tilde{\Phi}^\dagger L_L$)

$\Rightarrow$ consider all quadratic and quartic terms

Quadratic terms:

$$\sum_{i=1}^{3} \Phi_i^\dagger \Phi_i \tag{48}$$

Quartic terms of the scalar potential of Φ

Tensor product

$$(\underline{3}\otimes\underline{3}^*)\otimes(\underline{3}\otimes\underline{3}^*) = (\underline{1}_1\oplus\underline{1}_2\oplus\underline{1}_3\oplus\underline{3}\oplus\underline{3}^*)\otimes(\underline{1}_1\oplus\underline{1}_2\oplus\underline{1}_3\oplus\underline{3}\oplus\underline{3}^*) \quad (49)$$

Singlets:

1. $\underline{1}_1 \otimes \underline{1}_1$
2. $\underline{1}_2 \otimes \underline{1}_3 = \underline{1}_3 \otimes \underline{1}_2$
3. $\underline{3} \otimes \underline{3}^* = \underline{3}^* \otimes \underline{3}$

Quartic terms of the scalar potential of Φ

$\underline{1}_1 \otimes \underline{1}_1$ corresponds to

$$\left(\sum_{i=1}^{3} \Phi_i^\dagger \Phi_i \right)^2 \tag{50}$$

$\rightarrow$ sum of terms of the form

$$\Phi_i^\dagger \Phi_i \Phi_j^\dagger \Phi_j \tag{51}$$

Quartic terms of the scalar potential of Φ

$\underline{1}_2 \otimes \underline{1}_3$ corresponds to

$$(\Phi_1^\dagger \Phi_1 + \omega \Phi_2^\dagger \Phi_2 + \omega^2 \Phi_3^\dagger \Phi_3) \cdot (\Phi_1^\dagger \Phi_1 + \omega^2 \Phi_2^\dagger \Phi_2 + \omega \Phi_3^\dagger \Phi_3) \quad (52)$$

$\rightarrow$ sum of terms of the form

$$\Phi_i^\dagger \Phi_i \Phi_j^\dagger \Phi_j, \quad (53)$$

terms with $i = j$ have $+1$ as factor, $i \neq j$ have factor $-\frac{1}{2}$

The complete scalar potential of Φ

$\underline{3} \otimes \underline{3}^*$ corresponds to

$$\Phi_i^\dagger \Phi_j \Phi_j^\dagger \Phi_i \qquad (54)$$

with $i \neq j$

Potential:

$$\begin{aligned} V_\Phi &= \mu^2 \sum_{i=1}^{3} \Phi_i^\dagger \Phi_i \\ &+ (\lambda_1 + \lambda_2) \sum_{i=1}^{3} (\Phi_i^\dagger \Phi_i)^2 \\ &+ (\lambda_1 - \frac{\lambda_2}{2}) \sum_{i,j=1, i\neq j}^{3} \Phi_i^\dagger \Phi_i \Phi_j^\dagger \Phi_j \\ &+ \lambda_3 \sum_{i,j=1, i\neq j}^{3} \Phi_i^\dagger \Phi_j \Phi_j^\dagger \Phi_i \end{aligned} \qquad (55)$$

Minimum of the scalar potential of Φ

VEV: $v_1 = v_2 = v_3 = v$

Minimum of potential:

$$V_{\Phi,min} = 3\mu^2 v^2 + 3(3\lambda_1 + 2\lambda_3)v^4 \tag{56}$$

Minimization condition on v:

$$v = \sqrt{\frac{-\mu^2}{6\lambda_1 + 4\lambda_3}} \tag{57}$$

Constraints on the parameters

- μ^2 real and negative (condition for spontaneous symmetry breaking)
- λ_1, λ_2 and λ_3 real (potential hermitian, each term individually hermitian)
- lower bound on each λ_i (physical requirement that potential is bounded from below)
- lower bounds $\rightarrow$ expression under the square in (57) positive
- $\rightarrow$ v real and positive

General formalism to find mass matrices for the charged and the neutral scalars

General potential (Grimus, Lavoura (2002)):

$$V = \sum_{i,j=1}^{n_H} \mu_{ij}^2 \phi_i^\dagger \phi_j + \sum_{i,j,k,l=1}^{n_H} \lambda_{ijkl} \left(\phi_i^\dagger \phi_j\right) \left(\phi_k^\dagger \phi_l\right) \tag{58}$$

In our case: $n_H = 3$, nonvanishing coefficients:

- $\mu_{ii}^2 = \mu^2$
- $\lambda_{iiii} = (\lambda_1 + \lambda_2)$
- $\lambda_{iikk} = (\lambda_1 - \frac{\lambda_2}{2})$
- $\lambda_{ijji} = \lambda_3$

Mass matrix of the charged scalars $\phi_i^\pm$

$$M_{+ij}^2 = \mu_{ij}^2 + \Lambda_{ij}, \tag{59}$$

where

$$\Lambda_{ij} = \sum_{k,l=1}^{n_H} \lambda_{ijkl} v_k^* v_l \tag{60}$$

Here:

$$M_{+ij}^2 = \frac{\mu^2 \lambda_3}{3\lambda_1 + 2\lambda_3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{61}$$

Mass matrix of the neutral scalars $\phi_i^{\pm}$

$$M_0^2 = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \tag{62}$$

where

$$\begin{aligned} A_{ij} &= \Re(\mu_{ij}^2 + \Lambda_{ij} + K'_{ij}) + \Re(K_{ij}), & (63) \\ B_{ij} &= \Re(\mu_{ij}^2 + \Lambda_{ij} + K'_{ij}) - \Re(K_{ij}), & (64) \\ C_{ij} &= -\Im(\mu_{ij}^2 + \Lambda_{ij} + K'_{ij}) - \Im(K_{ij}) & (65) \end{aligned}$$

and

$$K_{ik} = \sum_{j,l=1}^{n_H} \lambda_{ijkl} v_j v_l \tag{66}$$

$$K'_{il} = \sum_{j,k=1}^{n_H} \lambda_{ijkl} v_j v_k^* \tag{67}$$

Mass matrix of the neutral scalars $\phi_i^\pm$ for our model

Here:

$C = 0$ as λ_i, μ^2 and v are real

$$A = \frac{\mu^2}{3\lambda_1 + 2\lambda_3} \begin{pmatrix} -2\lambda_1 - 2\lambda_2 & -2\lambda_1 + \lambda_2 - 4\lambda_3 & -2\lambda_1 + \lambda_2 - 4\lambda_3 \\ -2\lambda_1 + \lambda_2 - 4\lambda_3 & -2\lambda_1 - 2\lambda_2 & -2\lambda_1 + \lambda_2 - 4\lambda_3 \\ -2\lambda_1 + \lambda_2 - 4\lambda_3 & -2\lambda_1 + \lambda_2 - 4\lambda_3 & -2\lambda_1 - 2\lambda_2 \end{pmatrix}$$

$B = 0$

Eigenvalue equations

$$M_+^2 a = m_a^2 a \tag{68}$$

and

$$M_0^2 \begin{pmatrix} \Re(b) \\ \Im(b) \end{pmatrix} = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \begin{pmatrix} \Re(b) \\ \Im(b) \end{pmatrix} = m_b^2 \begin{pmatrix} \Re(b) \\ \Im(b) \end{pmatrix} \tag{69}$$

Rewrite (69) as

$$(\mu_{ij}^2 + \Lambda_{ij} + K'_{ij}) b_j + K_{ij} b_j^* = m_b^2 b_i \tag{70}$$

The existence of massless Goldstone bosons in this model

Potential (55) is invariant under $O(4)$:

$$\Phi_i = \frac{1}{\sqrt{2}} \begin{pmatrix} \Re(\Phi_i^+) + i\Im(\Phi_i^+) \\ \Re(\Phi_i^0) + i\Im(\Phi_i^0) \end{pmatrix} \tag{71}$$

four real parameters

$$\Phi_i^\dagger \Phi_i = \frac{1}{2}(\Re(\Phi_i^+)^2 + \Im(\Phi_i^+)^2 + \Re(\Phi_i^0)^2 + \Im(\Phi_i^0)^2) \tag{72}$$

Invariant $\Rightarrow$ $SO(4)$ sufficient, six parameters

The existence of massless Goldstone bosons in this model

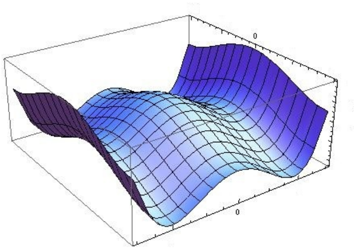
Additionally: Invariance under all phase transformations:

$$\Phi_i \to e^{i\alpha_i}\Phi_i, \tag{73}$$

$\Rightarrow$ Invariance under three independent $U(1)$ transformations
One transformation is weak hypercharge, already contained in $SO(4)$ symmetry

Independent parameters (group generators): 8
Three symmetry generators spontaneously broken $\Rightarrow$ Goldstone theorem $\Rightarrow$ number of massless Goldstone bosons: $8-3=5$

Schematic visualization of a simplified potential



Simplifications:

- neutral part of the potential only
- Φ_3^0 set to a fixed value (real but otherwise arbitrary)
- imaginary parts of Φ_1^0 and Φ_2^0 set to zero
- coefficients μ^2 and λ_i set to arbitrary real values

Pseudo Goldstone bosons

Goldstone bosons corresponding to longitudinal modes of the W and Z vector bosons:

$$a_W = \frac{1}{\sqrt{3}v}(v, v, v)^T \tag{74}$$

and

$$b_Z = \frac{i}{\sqrt{3}v}(v, v, v)^T \tag{75}$$

Masses are zero!

Pseudo Goldstone bosons

General expressions for physical charged and neutral scalars:

$$S_a^+ = \sum_{i=1}^{n_H} a_i^* \varphi_i^+ \tag{76}$$

and

$$S_b^0 = \sqrt{2} \sum_{i=1}^{n_H} \Re \left(b_i^* (\varphi_i^0)' \right) , \tag{77}$$

where $(\varphi_i^0)' = (\varphi_i^0) - v$ such that VEV of $(\varphi_i^0)'$ is zero.

Pseudo Goldstone bosons

In our case,

$$S^{+}_{a_W} = \frac{1}{\sqrt{3}} \sum_{i=1}^{3} \Phi_i^{+} \tag{78}$$

and

$$S^{0}_{b_Z} = \frac{\sqrt{2}}{\sqrt{3}} \sum_{i=1}^{3} \Im\left((\Phi_i^{0})'\right) \tag{79}$$

Two pseudo Goldstone bosons used to create the masses of the W and Z bosons

Two additional massless Goldstone bosons in this model

M_+^2: no other Goldstone boson fields exist

M_0^2:

$$M_0^2 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \tag{80}$$

Choose the vectors

$$b_1 = \frac{i}{\sqrt{2}v}(-v, v, 0)^T \tag{81}$$

and

$$b_2 = \frac{i}{\sqrt{6}v}(v, v, -2v)^T \tag{82}$$

Physical neutral and charged scalars

$$S^0_{b_1} = -\Im\left((\Phi^0_1)'\right) + \Im\left((\Phi^0_2)'\right) \tag{83}$$

and

$$S^0_{b_2} = \frac{1}{\sqrt{3}}\left(\Im\left((\Phi^0_1)'\right) + \Im\left((\Phi^0_2)'\right) - 2\Im\left((\Phi^0_3)'\right)\right) \tag{84}$$

Found two additional massless Goldstone bosons!

A is in general non singular, eigenvalues are

$$m^2_{a_1} = (4\lambda_3 - 3\lambda_2)\frac{\mu^2}{3\lambda_1 + 2\lambda_3}, \tag{85}$$

$$m^2_{a_2} = (4\lambda_3 - 3\lambda_2)\frac{\mu^2}{3\lambda_1 + 2\lambda_3}, \tag{86}$$

$$m^2_{a_3} = -2(4\lambda_3 + 3\lambda_1)\frac{\mu^2}{3\lambda_1 + 2\lambda_3} \tag{87}$$

Summary of massless Goldstone bosons

Charged sector: one charged complex massless Goldstone boson, remaining two charged complex particles are massive Higgs bosons.

Neutral sector: three massless real Goldstone bosons and three massive real Higgs bosons.

a_W and b_Z: used to create the masses of the W and the Z bosons
b_1 and b_2 remain massless particles that exist in this model

$\Rightarrow$ forces with infinite range

The existence of forces with infinite range?

Investigated experimentally: Mostepanenko, Sokolov (1993)

- measurements of the Casimir force
- gravitation experiments of both the Eotvos and the Cavendish type
- several other experiments

Upper limit for λ_n is (for $n = 1$) 10^{-47} in

$$\lambda_n (2z)^2 \frac{1}{r} \left(\frac{r_0}{r} \right)^{n-1} \tag{88}$$

(z...number of protons in the atom, $r_0 = 1\mathrm{F} = 10^{-15}\mathrm{m}$, r...distance, λ_n...dimensionless coupling constant)
$\Rightarrow$ existence of the predicted forces can be excluded!

The existence of forces with infinite range?

Relevant coupling constants f_i:

v is almost exactly 100 GeV, f_i fixed by the value of v and charged lepton masses:

$$f_1 \sim 10^{-6} \tag{89}$$
$$f_2 \sim 10^{-4} \tag{90}$$
$$f_3 \sim 10^{-2} \tag{91}$$

$\Rightarrow$ coupling constants many orders of magnitude larger than values in the experimentally allowed region!
(only tree-level approximation)

T_7 looks promising at first...

- has a set of irreps that can accommodate the multiplets, i.e. the particle content
- Yukawa terms lead to suitable mass matrix for the charged leptons
- Yukawa terms lead to suitable Dirac and Majorana mass matrices for the neutrinos
- plausible assumptions (some of which are made in the context of $B - L$ gauging) lead to a suitable seesaw neutrino mass matrix

T_7 looks promising at first...

- both mass hierarchy scenarios can be realized
- most importantly: the tribimaximal neutrino mixing matrix can be explained!
- new decays are predicted
- experimental test is possible by observing the decay of the new $B - L$ gauge boson Z'

But then...

- necessary supersymmetrization is in contradiction with the Yukawa terms
- scalar potential of only Φ leads to two massless Goldstone bosons that remain new massless particles
- these massless bosonic particles correspond to forces with infinite range
- such forces are experimentally excluded if the couplings are of the order of magnitude of the couplings in this model

By these arguments, it is excluded that this model describes the physical reality. This T_7 symmetry model is therefore ruled out as explanation of the symmetry in the lepton sector.