Tribimaximal Mixing From Small Groups

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The Standard Model

Gauge group:

$$SU(3)_c \times SU(2)_L \times U(1)_Y$$

Particle content:

<table>
<thead>
<tr>
<th>$Q$</th>
<th>$(3, 2)_{1/3}$</th>
<th>$L$</th>
<th>$(1, 2)_{-1}$</th>
<th>$H$</th>
<th>$(1, 2)_{1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{u}$</td>
<td>$(\bar{3}, 1)_{-4/3}$</td>
<td>$\bar{e}$</td>
<td>$(1, 1)_{2}$</td>
<td>$\bar{H}$</td>
<td>$(1, 2)_{-1}$</td>
</tr>
<tr>
<td>$\bar{d}$</td>
<td>$(\bar{3}, 1)_{2/3}$</td>
<td>$\bar{\nu}$</td>
<td>$(1, 1)_{0}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Why Are We Not Happy With the Standard Model?

(i) Too many free parameters

Gauge sector: 3 couplings $g'$, $g$, $g_3$  
Quark sector: 6 masses, 3 mixing angles, 1 CP phase  
Lepton sector: 6 masses, 3 mixing angles and 1-3 phases  
Higgs sector: Quartic coupling $\lambda$ and vev $v$  
$\theta$ parameter of QCD  

3 + 10 + 10 + 2 + 1 = 26
Why Are We Not Happy With the Standard Model?

(ii) Structure of gauge symmetry

\[ \text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_Y \quad \subset \quad \text{SU}(5) \quad \subset \quad \text{SO}(10) \quad \subset \quad \text{E}_6 \quad \subset \quad \text{E}_8 \]

Why 3 different coupling constants \( g', g, g_3 \)?

(iii) Structure of family multiplets

\[
(3,2)_{1/3} + (\bar{3},1)_{-4/3} + (1,1)_{-2} + (\bar{3},1)_{2/3} + (1,2)_{-1} + (1,1)_{0} \overset{?}{=} 16
\]

\[
Q \quad \bar{u} \quad \bar{e} \quad \bar{d} \quad L \quad \bar{\nu}
\]
Why Are We Not Happy With the Standard Model?

(iv) Repetition of Families

Why is the pattern for 1 generation replicated 3 times?
(v) Mass Hierarchies and Yukawa Textures

up-quark mass $\sim 2 \times 10^{-3}$ GeV $\leftrightarrow$ top-quark mass $\sim 172.3$ GeV
Yukawa coupling of top $\sim 1$, but why are the other quarks so light?

Minimal mixing in quark sector

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \approx \begin{pmatrix} 0.97 & 0.22 & 0.00 \\ 0.22 & 0.97 & 0.04 \\ 0.00 & 0.04 & 0.99 \end{pmatrix}$$
(vi) Light neutrinos and texture of Yukawa couplings

Why are neutrinos so light?

\[ \Delta m^2_\nu \sim 10^{-2} - 10^{-5} \text{ eV}, \quad \sum m_\nu < 2 \text{ eV} \]

Maximal mixing in lepton sector

\[
U_{\text{PMNS}} = \begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} \\
U_{\tau1} & U_{\tau2} & U_{\tau3}
\end{pmatrix} 
\cong \begin{pmatrix}
0.8 & 0.5 & 0.0 \\
-0.4 & 0.6 & 0.7 \\
0.4 & -0.6 & 0.7
\end{pmatrix}
\]
Why Are We Not Happy With the Standard Model?

➢ And many other problems/shortcomings:

Hierarchy problem, dark matter, dark energy, quantum gravity, baryon asymmetry, charge quantization, ... 

➢ Our work addresses the questions:

- Number of parameters in the SM $\rightarrow$ (i)
- Repetition of families $\rightarrow$ (iv)
- Light neutrinos and form of $U_{\text{PMNS}}$ $\rightarrow$ (vi)

➢ There are cross-connections to:

- Mass hierarchies and form of $U_{\text{CKM}}$ $\rightarrow$ (v)
- Grand Unification (see-saw scale) $\rightarrow$ (ii) and (iii)
- Baryon asymmetry (leptogenesis)
Horizontal Symmetries

➢ Introduce relations between families of quarks and leptons
Neutrino Mixing Matrix

What we know about the mixing angles . . .

\[
U_{\text{PMNS}} = \begin{pmatrix}
  c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-c_{23}s_{12} - s_{13}s_{23}c_{12}e^{i\delta} & c_{23}c_{12} - s_{13}s_{23}s_{12}e^{i\delta} & s_{23}c_{13} \\
 s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta} & -s_{23}s_{12} - s_{13}c_{23}s_{12}e^{i\delta} & c_{23}c_{13}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  1 & 0 & 0 \\
 0 & c_{23} & s_{23} \\
 0 & -s_{23} & c_{23}
\end{pmatrix} \begin{pmatrix}
  c_{13} & 0 & e^{-i\delta}s_{13} \\
 0 & 1 & 0 \\
-e^{i\delta}s_{13} & 0 & c_{13}
\end{pmatrix} \begin{pmatrix}
  c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]


<table>
<thead>
<tr>
<th>Angle</th>
<th>1σ</th>
<th>2σ</th>
<th>3σ</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\theta_{12})</td>
<td>32.46° – 34.82°</td>
<td>31.31° – 36.27°</td>
<td>30.00° – 37.47°</td>
</tr>
<tr>
<td>(\theta_{23})</td>
<td>41.55° – 49.02°</td>
<td>38.65° – 52.54°</td>
<td>36.87° – 54.94°</td>
</tr>
<tr>
<td>(\theta_{13})</td>
<td>0.00° – 9.28°</td>
<td>0.00° – 11.54°</td>
<td>0.00° – 13.69°</td>
</tr>
</tbody>
</table>
Presently our best guess . . .


\[ U_{\text{HPS}} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \]

Suggestive of an underlying symmetry . . .

Some groups that have been considered in the literature:


\( S_3, \ D_4, \ D_7, \ A_4, \ A_5, \ \tilde{T}, \ S_4, \ (C_3 \times C_3) \rtimes \varphi \ C_3, \ C_7 \rtimes \varphi \ C_3, \ \text{PSL}_2(7) \)

\( \sim \) As a paradigm, we will consider a model with \( A_4 \times C_3 \) symmetry and then generalize it to other symmetry groups
Altarelli-Feruglio Model Revisited


1 Symmetries of the model

\[ \text{SU}(2)_L \times \text{U}(1)_Y \times \text{U}(1)_R \times A_4 \times C_3 \]

2 Particle content and charges

<table>
<thead>
<tr>
<th>Field</th>
<th>SU(2)_L × U(1)_Y</th>
<th>U(1)_R</th>
<th>A_4</th>
<th>C_3</th>
<th>A_4 × C_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L )</td>
<td>(2, -1)</td>
<td>1</td>
<td>3</td>
<td>( \omega )</td>
<td>3'</td>
</tr>
<tr>
<td>( e )</td>
<td>(1, 2)</td>
<td>1</td>
<td>1</td>
<td>( \omega^2 )</td>
<td>1'</td>
</tr>
<tr>
<td>( \mu )</td>
<td>(1, 2)</td>
<td>1</td>
<td>1''</td>
<td>( \omega^2 )</td>
<td>1^{(8)}</td>
</tr>
<tr>
<td>( \tau )</td>
<td>(1, 2)</td>
<td>1</td>
<td>1'</td>
<td>( \omega^2 )</td>
<td>1^{(5)}</td>
</tr>
<tr>
<td>( h_u )</td>
<td>(2, 1)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( h_d )</td>
<td>(2, -1)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \varphi_T )</td>
<td>(1, 0)</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>( \varphi_S )</td>
<td>(1, 0)</td>
<td>0</td>
<td>3</td>
<td>( \omega )</td>
<td>3'</td>
</tr>
<tr>
<td>( \xi )</td>
<td>(1, 0)</td>
<td>0</td>
<td>1</td>
<td>( \omega )</td>
<td>1''</td>
</tr>
</tbody>
</table>

3 Breaking the family symmetry

\[ \varphi_T = (v_T, v_T, v_T), \ \varphi_S = (v_S, 0, 0), \ \xi = v_\xi, \]

“GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory.”

group := SmallGroup(36,11);;
Display(StructureDescription(group));
chartab := Irr(group);;
Display(chartab);
SizesConjugacyClasses(CharacterTable(group));
LoadPackage("repsn");;
for i in [1..Size(chartab)] do
  rep := IrreducibleAffordingRepresentation(chartab[i]);
  for el in Elements(group) do
    Display(el^rep);
  od;
od;

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    od;
od;

➢ Specify the group that we will work with
Group Information from GAP


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  od;
od;

➤ The “human readable” name of the group

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   od;
od;

➤ The character table
Group Information from GAP


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    Display(el^rep);
  od;
od;
```

➤ Dimensions of the conjugacy classes
Group Information from GAP


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```gap
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chartab := Irr(group);;
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SizesConjugacyClasses(CharacterTable(group));
LoadPackage("repsn");;
for i in [1..Size(chartab)] do
  rep := IrreducibleAffordingRepresentation(chartab[i]);
  for el in Elements(group) do
    Display(el^rep);
  od;
od;

➢ The matrices for the representations
```
### The Character Table of $A_4 \times C_3$

<table>
<thead>
<tr>
<th></th>
<th>$K_1$</th>
<th>$K_2$</th>
<th>$K_3$</th>
<th>$K_4$</th>
<th>$K_5$</th>
<th>$K_6$</th>
<th>$K_7$</th>
<th>$K_8$</th>
<th>$K_9$</th>
<th>$K_{10}$</th>
<th>$K_{11}$</th>
<th>$K_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$1'$</td>
<td>1</td>
<td>1</td>
<td>$\omega^2$</td>
<td>1</td>
<td>1</td>
<td>$\omega^2$</td>
<td>$\omega$</td>
<td>$\omega^2$</td>
<td>$\omega$</td>
<td>$\omega$</td>
<td>$\omega$</td>
<td>1</td>
</tr>
<tr>
<td>$1''$</td>
<td>1</td>
<td>1</td>
<td>$\omega$</td>
<td>1</td>
<td>1</td>
<td>$\omega$</td>
<td>$\omega^2$</td>
<td>$\omega$</td>
<td>$\omega^2$</td>
<td>$\omega^2$</td>
<td>$\omega^2$</td>
<td>1</td>
</tr>
<tr>
<td>$1'''$</td>
<td>1</td>
<td>$\omega^2$</td>
<td>1</td>
<td>1</td>
<td>$\omega$</td>
<td>$\omega^2$</td>
<td>1</td>
<td>1</td>
<td>$\omega$</td>
<td>$\omega^2$</td>
<td>1</td>
<td>$\omega$</td>
</tr>
<tr>
<td>$1(4)$</td>
<td>1</td>
<td>$\omega$</td>
<td>1</td>
<td>1</td>
<td>$\omega^2$</td>
<td>$\omega$</td>
<td>1</td>
<td>1</td>
<td>$\omega^2$</td>
<td>$\omega$</td>
<td>1</td>
<td>$\omega^2$</td>
</tr>
<tr>
<td>$1(5)$</td>
<td>1</td>
<td>$\omega^2$</td>
<td>$\omega^2$</td>
<td>1</td>
<td>$\omega$</td>
<td>$\omega$</td>
<td>$\omega$</td>
<td>$\omega^2$</td>
<td>1</td>
<td>1</td>
<td>$\omega$</td>
<td>$\omega^2$</td>
</tr>
<tr>
<td>$1(6)$</td>
<td>1</td>
<td>$\omega$</td>
<td>$\omega$</td>
<td>1</td>
<td>$\omega^2$</td>
<td>$\omega^2$</td>
<td>$\omega^2$</td>
<td>$\omega$</td>
<td>1</td>
<td>1</td>
<td>$\omega^2$</td>
<td>$\omega$</td>
</tr>
<tr>
<td>$1(7)$</td>
<td>1</td>
<td>$\omega^2$</td>
<td>$\omega$</td>
<td>1</td>
<td>$\omega$</td>
<td>1</td>
<td>$\omega^2$</td>
<td>$\omega$</td>
<td>$\omega^2$</td>
<td>$\omega$</td>
<td>1</td>
<td>$\omega^2$</td>
</tr>
<tr>
<td>$1(8)$</td>
<td>1</td>
<td>$\omega$</td>
<td>$\omega^2$</td>
<td>1</td>
<td>$\omega^2$</td>
<td>1</td>
<td>$\omega$</td>
<td>$\omega^2$</td>
<td>$\omega$</td>
<td>$\omega^2$</td>
<td>$\omega$</td>
<td>1</td>
</tr>
<tr>
<td>$3$</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$3'$</td>
<td>3</td>
<td>0</td>
<td>$3\omega$</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>$3\omega^2$</td>
<td>$\omega$</td>
<td>0</td>
<td>0</td>
<td>1 + $\omega$</td>
<td>0</td>
</tr>
<tr>
<td>$3''$</td>
<td>3</td>
<td>0</td>
<td>$3\omega^2$</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>$3\omega$</td>
<td>1 + $\omega$</td>
<td>0</td>
<td>0</td>
<td>$\omega$</td>
<td>0</td>
</tr>
</tbody>
</table>

$\omega = e^{2\pi i/3}$ is the primitive third root of unity
### Decomposition of Tensor Products

From the character table and the dimensions of the conjugacy classes:

<table>
<thead>
<tr>
<th>Tensor Product</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \otimes 1$</td>
<td>1</td>
</tr>
<tr>
<td>$1 \otimes 1^{(5)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1 \otimes 3'$</td>
<td>3</td>
</tr>
<tr>
<td>$1' \otimes 1^{(4)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1' \otimes 3 = 3''$</td>
<td>3</td>
</tr>
<tr>
<td>$1'' \otimes 1^{(4)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1'' \otimes 3 = 3'$</td>
<td>3</td>
</tr>
<tr>
<td>$1''' \otimes 1^{(5)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1'''' \otimes 1^{(5)} = 1^{(8)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1^{(4)} \otimes 1^{(7)} = 1''$</td>
<td>1</td>
</tr>
<tr>
<td>$1^{(5)} \otimes 1^{(5)} = 1^{(6)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1^{(5)} \otimes 3' = 3$</td>
<td>1</td>
</tr>
<tr>
<td>$1^{(6)} \otimes 3' = 3'$</td>
<td>2</td>
</tr>
<tr>
<td>$1^{(7)} \otimes 3' = 3'$</td>
<td>2</td>
</tr>
<tr>
<td>$1^{(8)} \otimes 3' = 3$</td>
<td>2</td>
</tr>
<tr>
<td>$3 \otimes 3 = 1 + 1'''' + 1^{(4)} + 2 \otimes 3$</td>
<td>2</td>
</tr>
<tr>
<td>$3 \otimes 3'' = 1' + 1^{(5)} + 1^{(8)} + 2 \otimes 3''$</td>
<td>2</td>
</tr>
<tr>
<td>$3' \otimes 3'' = 1' + 1'''' + 1^{(4)} + 2 \otimes 3$</td>
<td>2</td>
</tr>
<tr>
<td>$3' \otimes 3'' = 1' + 1^{(5)} + 1^{(8)} + 2 \otimes 3''$</td>
<td>2</td>
</tr>
<tr>
<td>$3'' \otimes 3'' = 1' + 1^{(6)} + 1^{(7)} + 2 \otimes 3'$</td>
<td>2</td>
</tr>
<tr>
<td>$3'' \otimes 3'' = 1' + 1^{(5)} + 1^{(8)} + 2 \otimes 3''$</td>
<td>2</td>
</tr>
<tr>
<td>$1 \otimes 1^{(4)} = 1^{(4)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1 \otimes 3 = 3$</td>
<td>1</td>
</tr>
<tr>
<td>$1' \otimes 1'''' = 1^{(5)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1' \otimes 1'' = 1$</td>
<td>1</td>
</tr>
<tr>
<td>$1' \otimes 1^{(7)} = 1^{(4)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1' \otimes 1^{(7)} = 1'''$</td>
<td>1</td>
</tr>
<tr>
<td>$1' \otimes 1^{(8)} = 1^{(7)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1' \otimes 1^{(8)} = 1^{(4)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1'' \otimes 1'''' = 1^{(7)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1'' \otimes 1'' = 1'$</td>
<td>1</td>
</tr>
<tr>
<td>$1'' \otimes 1^{(7)} = 1^{(5)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1'' \otimes 1^{(8)} = 1^{(4)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1''' \otimes 1'''' = 1$</td>
<td>1</td>
</tr>
<tr>
<td>$1''' \otimes 1'' = 1$</td>
<td>1</td>
</tr>
<tr>
<td>$1''' \otimes 1^{(7)} = 1^{(4)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1''' \otimes 1^{(7)} = 1^{(5)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1''' \otimes 1^{(8)} = 1$</td>
<td>1</td>
</tr>
<tr>
<td>$1''' \otimes 1^{(8)} = 1^{(4)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1'''' \otimes 1^{(4)} = 1$</td>
<td>1</td>
</tr>
<tr>
<td>$1'''' \otimes 3 = 3$</td>
<td>1</td>
</tr>
<tr>
<td>$1^{(4)} \otimes 1^{(7)} = 1' + 1^{(4)} = 1''''$</td>
<td>1</td>
</tr>
<tr>
<td>$1^{(4)} \otimes 1^{(8)} = 1^{(7)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1^{(4)} \otimes 1^{(8)} = 1^{(5)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1^{(4)} \otimes 3' = 3'$</td>
<td>1</td>
</tr>
<tr>
<td>$1^{(5)} \otimes 1^{(7)} = 1^{(4)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1^{(5)} \otimes 1^{(8)} = 1^{(5)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1^{(6)} \otimes 1^{(6)} = 1^{(4)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1^{(6)} \otimes 1^{(7)} = 1^{(4)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1^{(6)} \otimes 1^{(7)} = 1^{(8)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1^{(7)} \otimes 1^{(7)} = 1^{(8)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1^{(8)} \otimes 1^{(8)} = 1^{(7)}$</td>
<td>1</td>
</tr>
<tr>
<td>$1^{(8)} \otimes 3 = 3'$</td>
<td>1</td>
</tr>
</tbody>
</table>

Akin Wingerter, LPSC Grenoble
## Decomposition of Tensor Products

From the character table and the dimensions of the conjugacy classes:

<table>
<thead>
<tr>
<th>$1 \otimes 1 = 1$</th>
<th>$1 \otimes 1' = 1'$</th>
<th>$1 \otimes 1'' = 1''$</th>
<th>$1 \otimes 1''' = 1'''$</th>
<th>$1 \otimes 1^{(4)} = 1^{(4)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \otimes 1^{(5)} = 1^{(5)}$</td>
<td>$1 \otimes 1^{(6)} = 1^{(6)}$</td>
<td>$1 \otimes 1^{(7)} = 1^{(7)}$</td>
<td>$1 \otimes 1^{(8)} = 1^{(8)}$</td>
<td>$1 \otimes 1^{(9)} = 1^{(9)}$</td>
</tr>
<tr>
<td>$1 \otimes 3' = 3'$</td>
<td>$1 \otimes 3'' = 3''$</td>
<td>$1 \otimes 3''' = 3'''$</td>
<td>$1 \otimes 3^{(5)} = 3^{(5)}$</td>
<td>$1 \otimes 3^{(6)} = 3^{(6)}$</td>
</tr>
<tr>
<td>$1' \otimes 1^{(4)} = 1^{(8)}$</td>
<td>$1' \otimes 1^{(5)} = 1^{(7)}$</td>
<td>$1' \otimes 1^{(6)} = 1^{(4)}$</td>
<td>$1' \otimes 1^{(7)} = 1'''$</td>
<td>$1' \otimes 1^{(8)} = 1^{(6)}$</td>
</tr>
<tr>
<td>$1' \otimes 3' = 3$</td>
<td>$1' \otimes 3'' = 3''$</td>
<td>$1' \otimes 3''' = 3'''$</td>
<td>$1' \otimes 3^{(5)} = 3^{(5)}$</td>
<td>$1' \otimes 3^{(6)} = 3^{(6)}$</td>
</tr>
<tr>
<td>$1'' \otimes 1^{(4)} = 1^{(6)}$</td>
<td>$1'' \otimes 1^{(5)} = 1^{(5)}$</td>
<td>$1'' \otimes 1^{(6)} = 1^{(5)}$</td>
<td>$1'' \otimes 1^{(7)} = 1^{(8)}$</td>
<td>$1'' \otimes 1^{(8)} = 1^{(8)}$</td>
</tr>
<tr>
<td>$1'' \otimes 3' = 3$</td>
<td>$1'' \otimes 3'' = 3''$</td>
<td>$1'' \otimes 3''' = 3'''$</td>
<td>$1'' \otimes 3^{(5)} = 3^{(5)}$</td>
<td>$1'' \otimes 3^{(6)} = 3^{(6)}$</td>
</tr>
<tr>
<td>$1''' \otimes 1^{(4)} = 1^{(5)}$</td>
<td>$1''' \otimes 1^{(5)} = 1^{(7)}$</td>
<td>$1''' \otimes 1^{(6)} = 1^{(8)}$</td>
<td>$1''' \otimes 1^{(7)} = 1^{(8)}$</td>
<td>$1''' \otimes 1^{(8)} = 1^{(8)}$</td>
</tr>
<tr>
<td>$1''' \otimes 3' = 3$</td>
<td>$1''' \otimes 3'' = 3''$</td>
<td>$1''' \otimes 3''' = 3'''$</td>
<td>$1''' \otimes 3^{(5)} = 3^{(5)}$</td>
<td>$1''' \otimes 3^{(6)} = 3^{(6)}$</td>
</tr>
<tr>
<td>$1^{(4)} \otimes 1^{(5)} = 1^{(8)}$</td>
<td>$1^{(4)} \otimes 1^{(6)} = 1^{(5)}$</td>
<td>$1^{(4)} \otimes 1^{(7)} = 1^{(7)}$</td>
<td>$1^{(4)} \otimes 1^{(8)} = 1^{(8)}$</td>
<td>$1^{(4)} \otimes 1^{(9)} = 1^{(9)}$</td>
</tr>
<tr>
<td>$1^{(5)} \otimes 1^{(5)} = 1^{(6)}$</td>
<td>$1^{(5)} \otimes 1^{(6)} = 1^{(5)}$</td>
<td>$1^{(5)} \otimes 1^{(7)} = 1^{(8)}$</td>
<td>$1^{(5)} \otimes 1^{(8)} = 1^{(8)}$</td>
<td>$1^{(5)} \otimes 1^{(9)} = 1^{(9)}$</td>
</tr>
<tr>
<td>$1^{(5)} \otimes 3' = 3$</td>
<td>$1^{(5)} \otimes 3'' = 3''$</td>
<td>$1^{(5)} \otimes 3''' = 3'''$</td>
<td>$1^{(5)} \otimes 3^{(5)} = 3^{(5)}$</td>
<td>$1^{(5)} \otimes 3^{(6)} = 3^{(6)}$</td>
</tr>
<tr>
<td>$1^{(6)} \otimes 3' = 3'$</td>
<td>$1^{(6)} \otimes 3'' = 3''$</td>
<td>$1^{(6)} \otimes 3''' = 3'''$</td>
<td>$1^{(6)} \otimes 3^{(5)} = 3^{(5)}$</td>
<td>$1^{(6)} \otimes 3^{(6)} = 3^{(6)}$</td>
</tr>
<tr>
<td>$1^{(7)} \otimes 3' = 3'$</td>
<td>$1^{(7)} \otimes 3'' = 3''$</td>
<td>$1^{(7)} \otimes 3''' = 3'''$</td>
<td>$1^{(7)} \otimes 3^{(5)} = 3^{(5)}$</td>
<td>$1^{(7)} \otimes 3^{(6)} = 3^{(6)}$</td>
</tr>
<tr>
<td>$1^{(8)} \otimes 3' = 3'$</td>
<td>$1^{(8)} \otimes 3'' = 3''$</td>
<td>$1^{(8)} \otimes 3''' = 3'''$</td>
<td>$1^{(8)} \otimes 3^{(5)} = 3^{(5)}$</td>
<td>$1^{(8)} \otimes 3^{(6)} = 3^{(6)}$</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
3 \otimes 3 &= 1 + 1''' + 1^{(4)} + 2 \otimes 3 \\
3 \otimes 3'' &= 1' + 1^{(5)} + 1^{(8)} + 2 \otimes 3'' \\
3' \otimes 3'' &= 1 + 1''' + 1^{(4)} + 2 \otimes 3 \quad \text{(5)} \\
3' \otimes 3'' &= 1' + 1^{(5)} + 1^{(8)} + 2 \otimes 3'' \quad \text{(6)} \\
3'' \otimes 3'' &= 1 + 1^{(6)} + 1^{(7)} + 2 \otimes 3' \\
3'' \otimes 3'' &= 1' + 1^{(5)} + 1^{(8)} + 2 \otimes 3'' \quad \text{(8)} \\
3''' \otimes 3''' &= 1 + 1^{(6)} + 1^{(8)} + 2 \otimes 3' \quad \text{(7)} \\
3''' \otimes 3''' &= 1' + 1^{(5)} + 1^{(8)} + 2 \otimes 3'' \quad \text{(8)}
\end{align*}
\]
Invariant Lagrangian

Terms that are invariant, have 2 leptons and mass dimension $\leq 6$:

\[ L L h_u h_u \phi_S, \quad L L h_u h_u \xi, \quad L e h_d \phi_T, \quad L \mu h_d \phi_T, \quad L \tau h_d \phi_T \]
Invariant Lagrangian

Terms that are invariant, have 2 leptons and mass dimension $\leq 6$:

$$LLh_u h_u \varphi_S, \quad LLh_u h_u \xi, \quad Le h_d \varphi_T, \quad L\mu h_d \varphi_T, \quad L\tau h_d \varphi_T$$

$$3' \otimes 3' \otimes 1 \otimes 1 \otimes 3' = (1' + 1^{(5)} + 1^{(8)} + 2 \times 3'') \otimes 3' = 2 \times 1 + 2 \times 1'' + 2 \times 1^{(4)} + 7 \times 3$$
Invariant Lagrangian

Terms that are invariant, have 2 leptons and mass dimension $\leq 6$:

\[ L L h_u h_u \varphi_S, \quad L L h_u h_u \xi, \quad L e h_d \varphi_T, \quad L \mu h_d \varphi_T, \quad L \tau h_d \varphi_T \]

\[ 3' \otimes 3' \otimes 1 \otimes 1 \otimes 3' = (1' + 1^{(5)} + 1^{(8)} + 2 \times 3'') \otimes 3' = 2 \times 1 + 2 \times 1''' + 2 \times 1^{(4)} + 7 \times 3 \]

Contract family indices:

\[ \frac{1}{\sqrt{3}} L_2 L_3 h_u h_u \varphi_{S,1} + \frac{1}{\sqrt{3}} L_3 L_1 h_u h_u \varphi_{S,2} + \frac{1}{\sqrt{3}} L_1 L_2 h_u h_u \varphi_{S,3} + \frac{1}{\sqrt{3}} L_1 L_1 h_u h_u \xi \]

\[ + \frac{1}{\sqrt{3}} L_2 L_2 h_u h_u \xi + \frac{1}{\sqrt{3}} L_3 L_3 h_u h_u \xi \]
Invariant Lagrangian

- Terms that are invariant, have 2 leptons and mass dimension \( \leq 6 \):

\[
L L h_u h_u \varphi_S, \quad L L h_u h_u \xi, \quad L e h_d \varphi_T, \quad L \mu h_d \varphi_T, \quad L \tau h_d \varphi_T
\]

\[
3' \otimes 3' \otimes 1 \otimes 1 \otimes 3' = (1' + 1(5) + 1(8) + 2 \times 3'' \otimes 3' = 2 \times 1 + 2 \times 1'' + 2 \times 1(4) + 7 \times 3
\]

- Contract family indices:

\[
\frac{1}{\sqrt{3}} L_2 L_3 h_u h_u \varphi_{S,1} + \frac{1}{\sqrt{3}} L_3 L_1 h_u h_u \varphi_{S,2} + \frac{1}{\sqrt{3}} L_1 L_2 h_u h_u \varphi_{S,3} + \frac{1}{\sqrt{3}} L_1 L_1 h_u h_u \xi
\]

\[
+ \frac{1}{\sqrt{3}} L_2 L_2 h_u h_u \xi + \frac{1}{\sqrt{3}} L_3 L_3 h_u h_u \xi
\]

- Contract SU(2) indices and substitute vevs \( \langle \varphi_S \rangle = (\nu_S, 0, 0) \), etc:

\[
\frac{1}{\sqrt{3}} L_2^{(1)} L_3^{(1)} \nu_u \nu_u \nu_S + \frac{1}{\sqrt{3}} L_1^{(1)} L_1^{(1)} \nu_u \nu_u \nu_\xi + \frac{1}{\sqrt{3}} L_2^{(1)} L_2^{(1)} \nu_u \nu_u \nu_\xi + \frac{1}{\sqrt{3}} L_3^{(1)} L_3^{(1)} \nu_u \nu_u \nu_\xi
\]
Finally, the Mixing Angles

Mass matrices

\[
M_{\ell+} = \begin{pmatrix} L_1^{(2)} & L_2^{(2)} & L_3^{(2)} \end{pmatrix}
\begin{pmatrix}
-\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\
-\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}}
\end{pmatrix},
M_\nu = \begin{pmatrix} L_1^{(1)} & L_2^{(1)} & L_3^{(1)} \end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{3}} & 0 & 0 \\
0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}}
\end{pmatrix}
\]
Finally, the Mixing Angles

▷ Mass matrices

\[
M_{\ell^+} = \begin{pmatrix} L_1^{(2)} & L_2^{(2)} & L_3^{(2)} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{pmatrix}, \quad M_\nu = \begin{pmatrix} L_1^{(1)} & L_2^{(1)} & L_3^{(1)} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}
\]

▷ Singular value decomposition: \( \hat{M}_{\ell^+} = D_L M_{\ell^+} D_R^\dagger \), \( \hat{M}_\nu = U_L M_\nu U_R^\dagger \)

\[
D_L = \begin{pmatrix} -0.5774+i 0.0000 & -0.5774+i 0.0000 & -0.5774+i 0.0000 \\ 0.5738-i 0.0636 & -0.2319+i 0.5287 & -0.3420-i 0.4652 \\ 0.5731-i 0.0702 & -0.3474-i 0.4612 & -0.2257+i 0.5314 \end{pmatrix}, \quad U_L = \begin{pmatrix} 0.0000 & -0.7071 & -0.7071 \\ -1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.7071 & -0.7071 \end{pmatrix}
\]
Finally, the Mixing Angles

> Mass matrices

\[
M_{\ell^+} =
\begin{pmatrix}
L_1^{(2)} & e & \mu & \tau \\
L_2^{(2)} & -1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\
L_3^{(2)} & -1/\sqrt{3} & 1/2\sqrt{3} & 1/2\sqrt{3}
\end{pmatrix},
\]

\[
M_\nu =
\begin{pmatrix}
L_1^{(1)} & L_2^{(1)} & L_3^{(1)} \\
L_1^{(1)} & 1/\sqrt{3} & 0 & 0 \\
L_2^{(1)} & 0 & 1/\sqrt{3} & 1/2\sqrt{3} \\
L_3^{(1)} & 0 & 1/2\sqrt{3} & 1/3
\end{pmatrix}
\]

> Singular value decomposition: \( \hat{M}_{\ell^+} = D_L M_{\ell^+} D_R^\dagger \), \( \hat{M}_\nu = U_L M_\nu U_R^\dagger \)

\[
D_L =
\begin{pmatrix}
0.5774+i0.0000 & -0.5774+i0.0000 & -0.5774+i0.0000 \\
0.5738-i0.0636 & -0.2319+i0.5287 & -0.3420-i0.4652 \\
0.5731-i0.0702 & -0.3474-i0.4612 & -0.2257+i0.5314
\end{pmatrix},
\]

\[
U_L =
\begin{pmatrix}
0.0000 & -0.7071 & -0.7071 \\
-1.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.7071 & -0.7071
\end{pmatrix}
\]

> Neutrino mixing matrix: \( U_{PMNS} = D_L U_L^\dagger \) (needs rephasing)

\[
U_{PMNS} =
\begin{pmatrix}
0.8165 + i0.0000 & 0.5774 + i0.0000 & 0.0000 + i0.0000 \\
0.4058 - i0.0449 & -0.5738 + i0.0636 & 0.0778 + i0.7028 \\
0.4052 - i0.0497 & -0.5731 + i0.0702 & -0.0860 - i0.7019
\end{pmatrix}
\]

> Mixing angles: \( \theta_{12} = 35.26 \), \( \theta_{23} = 45.00 \), \( \theta_{13} = 0.00 \) Tribimaximal ✔️
Finally, the Mixing Angles

>We Mass matrices

\[ M_{\ell^+} = \begin{pmatrix} L_1^{(2)} & L_2^{(2)} & L_3^{(2)} \end{pmatrix} \begin{pmatrix} e & \mu & \tau \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{pmatrix}, \quad M_\nu = \begin{pmatrix} L_1^{(1)} & L_2^{(1)} & L_3^{(1)} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{pmatrix} \]

>Singular value decomposition: \( \hat{M}_{\ell^+} = D_L M_{\ell^+} D_R^\dagger, \quad \hat{M}_\nu = U_L M_\nu U_R^\dagger \)

\[ D_L = \begin{pmatrix} -0.5774+i 0.0000 & -0.5774+i 0.0000 & -0.5774+i 0.0000 \\ 0.5738-i 0.0636 & -0.2319+i 0.5287 & -0.3420-i 0.4652 \\ 0.5731-i 0.0702 & -0.3474+i 0.4612 & -0.2257+i 0.5314 \end{pmatrix}, \quad U_L = \begin{pmatrix} 0.0000 & -0.7071 & -0.7071 \\ -1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.7071 & -0.7071 \end{pmatrix} \]

>Neutrino mixing matrix: \( U_{\text{PMNS}} = D_L U_L^\dagger \) (needs rephasing)

\[ |U_{\text{PMNS}}| = \begin{pmatrix} 0.8165 & 0.5774 & 0.0000 \\ 0.4082 & 0.5774 & 0.7071 \\ 0.4082 & 0.5774 & 0.7071 \end{pmatrix} \]

>Mixing angles: \( \theta_{12} = 35.26, \quad \theta_{23} = 45.00, \quad \theta_{13} = 0.00 \) Tribimaximal ✔
Finally, the Mixing Angles

> Mass matrices

\[
M_{\ell^+} = \begin{pmatrix}
L_1^{(2)} & L_2^{(2)} & L_3^{(2)} \\
-\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\
-\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\
-\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\
\end{pmatrix},
\]

\[
M_\nu = \begin{pmatrix}
L_1^{(1)} & L_2^{(1)} & L_3^{(1)} \\
\frac{1}{\sqrt{3}} & 0 & 0 \\
0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \\
0 & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\end{pmatrix}
\]

> Singular value decomposition: \( \hat{M}_{\ell^+} = D_L M_{\ell^+} D_R^\dagger, \quad \hat{M}_\nu = U_L M_\nu U_R^\dagger \)

\[
D_L = \begin{pmatrix}
-0.5774+i 0.0000 & -0.5774+i 0.0000 & -0.5774+i 0.0000 \\
0.5738-i 0.0636 & 0.5287-i 0.3420 & 0.4652-i 0.0000 \\
0.5731-i 0.0702 & 0.4612-i 0.2257 & 0.5314-i 0.0000 \\
\end{pmatrix}, \quad U_L = \begin{pmatrix}
0.0000 & -0.7071 & -0.7071 \\
-1.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.7071 & -0.7071 \\
\end{pmatrix}
\]

> Neutrino mixing matrix: \( U_{\text{PMNS}} = D_L U_L^\dagger \) (needs rephasing)

\[
|U_{\text{PMNS}}| = \begin{pmatrix}
\sqrt{2/3} & 1/\sqrt{3} & 0 \\
1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\
1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\
\end{pmatrix}
\]

> Mixing angles: \( \theta_{12} = 35.26, \quad \theta_{23} = 45.00, \quad \theta_{13} = 0.00 \) Tribimaximal ✅
What is the Bottom Line?

- We used the Altarelli-Feruglio model only as a paradigm.
- The analysis is completely independent of the family symmetry.
- GAP gives us all the relevant information about the group.
- Complexity of group is hereby irrelevant.
- We use Python to interact w/GAP and do symbolic manipulations.
- From symmetry to lagrangian to mixing angles takes less than 1 second per model (checking all vacua!)
Where Do We Go From Here?

1. Generalize family symmetry:
   \[ A_4 \times C_3 \rightarrow 1048 \text{ groups of order } \leq 100 \rightarrow 90 \text{ w/3-dim irreps} \]

2. Keep same particle content:
   \[ L, e, \mu, \tau, h_u, h_d, \varphi_T, \varphi_S, \xi \]

3. Generalize family charge assignments:
   \[ (L, e, \mu, \tau, h_u, h_d, \varphi_T, \varphi_S, \xi) \rightarrow (3', 1', 1^{(8)}, 1^{(5)}, 1, 1, 3, 3', 1'') \rightarrow (*, *, *, *, *, *, *, *, *) \]

4. Generalize symmetry breaking patterns:
   \[ \langle \varphi_T \rangle = (*, *, *), \quad \langle \varphi_S \rangle = (*, *, *), \quad \langle \xi \rangle = (*, *, *) \]
Scanning for the Models

- Consider $A_4 \times C_3$
- 14,594,580 different family charge assignments (particles w/same gauge/R-symmetry charges are considered identical)

Computer takes 17 hours (3 GHz Intel Xeon) $\rightsquigarrow$ Distribute job to 17 machines and get results in 1 hour

- 39,900 different Lagrangians

- Plus results for 75 more groups!
Results for $\mathbf{A}_4 \times \mathbf{C}_3$

- We consider 2 models equivalent, if their Lagrangians are the same after contracting the family indices, but before the vevs are substituted.
- In this sense, we have 39,900 inequivalent models/Lagrangians.
- 22,932 models have non-singular charged lepton and neutrino mass matrices:

\[
\hat{M}_\ell^+ = D_L M_\ell + D_R^\dagger, \quad \hat{M}_\nu = U_L M_\nu U_R^\dagger, \quad U_{PMNS} \equiv D_L U_L^\dagger
\]

- 4,481 consistent w/experiment at 3$\sigma$ level (19.5%).
- 4,233 are tribimaximal (18.5%).
- Probably largest set of viable neutrino models ever constructed!
Most Likely Mixing Angles

(a) Number of models that give $\theta_{ij}$ with no constraints on the other 2 angles. Each histogram has 15,992,118 entries.

(b) Number of models that give $\theta_{ij}$ with the other 2 angles restricted to their $3\sigma$ interval. The histograms have 838,289, 148,886 and 225,844 entries, respectively.
Correlation Between Pairs of Mixing Angles

(c) Number of models that give $\theta_{ij}$ and $\theta_{mn}$ with no constraint on the remaining angle. Each histogram has 15,768,810 entries.

(d) Number of models that give $\theta_{ij}$ and $\theta_{mn}$ with the remaining angle restricted to its $3\sigma$ interval. The histograms have 2,591,752, 4,060,640 and 1,214,874 entries, respectively.
Correlation Between All Mixing Angles

(e) The 12,230 bins that are $\geq 1$.  
(f) The 1,586 bins that are $\geq 1000$. 

Akın Wingerter, LPSC Grenoble
Tribimaximal Mixing From Small Groups
Are We Looking Under the Lamppost?
Small Groups

Arthur Cayley (1821-1895) is the first to systematically construct groups; in 1854, he determined all groups of order 4 and 6 . . .
The first few of the 1048 groups of order \( \leq 100 \)

\( \checkmark = U(n) \) and \( \checkmark = SU(n) \) for \( n = 2, 3 \)

<table>
<thead>
<tr>
<th>GAP ID</th>
<th>Group</th>
<th>3</th>
<th>U(3)</th>
<th>U(2)</th>
<th>U(2)×U(1)</th>
<th>A_4</th>
</tr>
</thead>
<tbody>
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<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>[2, 1]</td>
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<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>[3, 1]</td>
<td>C_3</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>[4, 1]</td>
<td>C_4</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>[4, 2]</td>
<td>C_2 × C_2</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>[5, 1]</td>
<td>C_5</td>
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<td>✓</td>
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<td>X</td>
</tr>
<tr>
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<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
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<tr>
<td>[8, 1]</td>
<td>C_8</td>
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<td>X</td>
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<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>
The first few of the 1048 groups of order \( \leq 100 \)

\[ \udden{=} U(n) \text{ and } \su{=} SU(n) \text{ for } n = 2, 3 \]

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<thead>
<tr>
<th>GAP ID</th>
<th>Group</th>
<th>3</th>
<th>U(3)</th>
<th>U(2)</th>
<th>U(2)×U(1)</th>
<th>A_4</th>
</tr>
</thead>
<tbody>
<tr>
<td>[8, 2]</td>
<td>( C_4 \times C_2 )</td>
<td>✗</td>
<td>✓</td>
<td>✗</td>
<td>✗</td>
<td>✗</td>
</tr>
<tr>
<td>[8, 3]</td>
<td>( D_4 )</td>
<td>✗</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
</tr>
<tr>
<td>[8, 4]</td>
<td>( Q_8 )</td>
<td>✗</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
</tr>
<tr>
<td>[8, 5]</td>
<td>( C_2 \times C_2 \times C_2 )</td>
<td>✗</td>
<td>✓</td>
<td>✗</td>
<td>✗</td>
<td>✗</td>
</tr>
<tr>
<td>[9, 1]</td>
<td>( C_9 )</td>
<td>✗</td>
<td>✗</td>
<td>✗</td>
<td>✗</td>
<td>✗</td>
</tr>
<tr>
<td>[9, 2]</td>
<td>( C_3 \times C_3 )</td>
<td>✗</td>
<td>✓</td>
<td>✗</td>
<td>✗</td>
<td>✗</td>
</tr>
<tr>
<td>[10, 1]</td>
<td>( D_5 )</td>
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<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
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<tr>
<td>[10, 2]</td>
<td>( C_{10} )</td>
<td>✗</td>
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<tr>
<td>[11, 1]</td>
<td>( C_{11} )</td>
<td>✗</td>
<td>✗</td>
<td>✗</td>
<td>✗</td>
<td>✗</td>
</tr>
<tr>
<td>[12, 1]</td>
<td>( C_3 \rtimes \varphi C_4 )</td>
<td>✗</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
</tr>
</tbody>
</table>
The first few of the 1048 groups of order $\leq 100$

$✓ = U(n)$ and $✓ = SU(n)$ for $n = 2, 3$

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<tr>
<th>GAP ID</th>
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<th>3</th>
<th>U(3)</th>
<th>U(2)</th>
<th>U(2)×U(1)</th>
<th>A_4</th>
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<tr>
<td>[12, 2]</td>
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<td>✓</td>
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<tr>
<td>[12, 4]</td>
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<td>C_6 × C_2</td>
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<tr>
<td>[14, 1]</td>
<td>D_7</td>
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<td>✓</td>
<td>✔</td>
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<tr>
<td>[14, 2]</td>
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<tr>
<td>[15, 1]</td>
<td>C_{15}</td>
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</tr>
<tr>
<td>[16, 1]</td>
<td>C_{16}</td>
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<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>[16, 2]</td>
<td>C_4 × C_4</td>
<td></td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
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Smallest group that can produce TBM: $\mathcal{G}(21, 1) = T_7$

Largest fraction of TBM models: $\mathcal{G}(39, 1) = T_{13}$. Special?
Conclusions

- Constructed thousands of new models of tribimaximal mixing
  - 18.5% of all $A_4 \times \mathbb{Z}_3$ models are TBM. Encouraging!
  - Prediction for $\theta_{13}$: If $A_4$ and $\theta_{13} \lesssim 12^\circ \Rightarrow \theta_{13} = 0^\circ$
  - Prediction for $\delta$: $0^\circ$
  - Correlations between mixing angles: Fix two, predict the third

- Constructed specific models
  - $\theta_{13} \neq 0$ possible: $\theta_{12} \simeq 34^\circ$, $\theta_{23} \simeq 41^\circ$ and $\theta_{13} \simeq 5^\circ$
  - Altarelli-Feruglio model works with $\mathbb{Z}_2$: $A_4 \times \mathbb{Z}_2 \simeq \Sigma(24)$
  - TBM possible for $T_7$, $\Sigma(24)$, $T_{13}$, $T_{14}$, $\Delta(48)$, $T_{19}$, ... 

- Is $A_4$ special? Are TBM and $A_4$ connected?
  - 50% of the 76 groups we scanned can accomodate TBM
  - Metacyclic group $T_{13}$ has larger fraction of TBM models
  - Smallest group w/TBM is $T_7$

- GAP as a new tool for model builders