Helpful tools in finite group theory

Patrick Ludl

Faculty of Physics, University of Vienna

October 12th 2010

Outline of the talk

- Basic definitions: Finite groups and their representations
- Principal series and their applications
- Finite groups with faithful 3-dimensional irreducible representations
- An inequality for characters

Basic definitions I: Order, conjugacy classes

The number of elements of a group is called **order of the group**.

Finite groups can be divided into **conjugacy classes**.

Conjugacy classes

Two elements a and b of a finite group G are *conjugate*, if

$$\exists c \in G : \quad b = c^{-1}ac.$$

To a given element a of a group one can assert the equivalence class of all elements which are equivalent to a. This equivalence class is called **conjugacy class**.

Basic definitions II: Representations

Homomorphisms and Representations

$\phi : G \to G'$ is called **group homomorphism**, if

$$\phi(ab) = \phi(a)\phi(b) \quad \forall a, b \in G.$$

Representation: Homomorphism $D : G \to D(G)$.

$D(G)$ ... Linear operators over a vectorspace V_D. $\dim D := \dim V_D$

Equivalent representations

$$D' \sim D \Leftrightarrow \exists C : \; D' = C^{-1} D C.$$

Theorem

Every representation of a finite group is equivalent to a unitary representation.

Basic definitions III: Irreducible representations (irreps)

Invariant subspaces and irreducible representations

Let D be a representation of G on a vectorspace V_D. A subspace $W \subset V_D$ is called **invariant**, if

$$D(a)W = W \quad \forall a \in G.$$

A representation is called **irreducible**, if there is no nontrivial invariant subspace of V_D.

Basic definitions III: Irreducible representations (irreps)

D not irreducible $\Rightarrow$ **completely reducible** (for finite groups).

$$\Rightarrow D = D_1 \oplus D_2 \oplus ... \oplus D_k$$

Example: Matrix representations:

$D = D_1 \oplus D_2 \Rightarrow \exists M$:

$$M^{-1} D(a) M = \begin{pmatrix} D_1(a) & \mathbf{0} \\ \mathbf{0} & D_2(a) \end{pmatrix} \quad \forall a \in G. \quad \text{"\textit{block-diagonal}"}$$

In the reduced block-diagonal form one can see the invariant subspaces.

Basic definitions IV: Characters

Characters

Let $a \in G$:

$$\chi_D(a) := Tr(D(a))$$

is called **character** of the representation D.

Equivalent representations have the same characters!

Equivalent group elements have the same characters!

Characters of all non-equivalent irreps of a finite group

$\Rightarrow$ **Character table**

Basic definitions IV: Characters

Example for a character table ($\omega = e^{\frac{2\pi i}{3}}$):

A_4	$C_1(1)$	$C_2(3)$	$C_3(4)$	$C_4(4)$
$\underline{\mathbf{1}}$	1	1	1	1
$\underline{\mathbf{1}}'$	1	1	ω	ω^2
$\underline{\mathbf{1}}''$	1	1	ω^2	ω
$\underline{\mathbf{3}}$	3	-1	0	0

The character table of the group A_4.

Theorem

$\#$ conjugacy classes $= \#$ non-equivalent irreps

Basic definitions IV: Orthogonality of characters

Scalar product for characters

$$(\chi_{D_1}, \chi_{D_2}) := \frac{1}{\mathrm{ord}(G)} \sum_{a \in G} \chi^*_{D_1}(a) \chi_{D_2}(a)$$

Orthogonality relation for characters

Characters of non-equivalent irreps are orthonormal.

$$(\chi_{D_i}, \chi_{D_j}) = \delta_{ij}.$$

Application: Reduction of tensor products using the character table

$$\chi_{D_1 \otimes D_2} = \chi_{D_1} \cdot \chi_{D_2}$$

$$D_1 \otimes D_2 = \bigoplus_{\lambda} b_\lambda D^\lambda \Rightarrow b_\lambda = (\chi_{D_1} \cdot \chi_{D_2}, \chi_{D^\lambda}).$$

Example: A_4

$$\underline{\mathbf{3}} \otimes \underline{\mathbf{3}} = \underline{\mathbf{1}} \oplus \underline{\mathbf{1}}\mathbf{'} \oplus \underline{\mathbf{1}}\mathbf{''} \oplus \underline{\mathbf{3}} \oplus \underline{\mathbf{3}}.$$

Principal series and their applications

Principal series of a group

Definition

$H \lhd G$: $\quad H \subsetneq G$ is an invariant subgroup (normal subgroup) of G.

Principal series of G:

$$\{e\} \lhd G_1 \lhd \cdots \lhd G_{k-1} \lhd G_k \equiv G$$

such that

- $G_i \lhd G_j \quad \forall i < j$, i.e. G_i **is an invariant subgroup of all groups to the right of it**.
- G_j/G_{j-1} **is simple** (has no nontrivial invariant subgroup) $\forall j = 1, ..., k$. $\Rightarrow$ The principal series is maximal.

Why can the principal series be useful?

If the principal series

$$\{e\} \lhd G_1 \lhd \cdots \lhd G_{k-1} \lhd G_k \equiv G$$

has a reasonable length k it may be a useful concept

- to understand the structure of the group,
- to find the conjugacy classes, and
- to construct the irreps of G.

Example: $\Delta(27)$ and $\Delta(54)$

$\Delta(27)$: conjugacy classes $\rightarrow$ normal subgroups $\rightarrow$ principal series

Generators:

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{where } \omega = e^{2\pi i/3}.$$

Normal subgroups:

$$\begin{aligned} &\langle\langle \omega \mathbb{1} \rangle\rangle \cong \mathbb{Z}_3, \\ &\langle\langle \omega \mathbb{1}, C \rangle\rangle \cong \langle\langle \omega \mathbb{1}, E \rangle\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \end{aligned}$$

$\Rightarrow$ two principal series

$$\{\mathbb{1}\} \lhd \langle\langle \omega \mathbb{1} \rangle\rangle \lhd \langle\langle \omega \mathbb{1}, C \rangle\rangle \lhd \Delta(27)$$
$$\{\mathbb{1}\} \lhd \langle\langle \omega \mathbb{1} \rangle\rangle \lhd \langle\langle \omega \mathbb{1}, E \rangle\rangle \lhd \Delta(27)$$

These two principal series are **isomorphic** (Jordan-Hölder theorem)

$$\Rightarrow \quad \{e\} \lhd \mathbb{Z}_3 \lhd \mathbb{Z}_3 \times \mathbb{Z}_3 \lhd \Delta(27)$$

$\Delta(27)$: Construction of irreps from the principal series

$$\{e\} \lhd G_1 \lhd \cdots \lhd G_{k-1} \lhd G_k \equiv G$$

The principal series is a series of normal subgroups $\Rightarrow$

Every irrep of G/G_i is an irrep of G.

Moreover, for $i < j$:

Every irrep of G/G_j is an irrep of G/G_i.

Consequences for $\{e\} \lhd \mathbb{Z}_3 \lhd \mathbb{Z}_3 \times \mathbb{Z}_3 \lhd \Delta(27)$:

- Irreps of $\Delta(27)/(\mathbb{Z}_3 \times \mathbb{Z}_3) \cong \mathbb{Z}_3$ and $\Delta(27)/\mathbb{Z}_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ are irreps of $\Delta(27)$.
- Irreps of $\Delta(27)/(\mathbb{Z}_3 \times \mathbb{Z}_3) \cong \mathbb{Z}_3$ are irreps of $\Delta(27)/\mathbb{Z}_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Irreps of $\Delta(27)$

Irreps of $\Delta(27)/\mathbb{Z}_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ are irreps of $\Delta(27)$

$$\Rightarrow \quad \underline{\mathbf{1}}_{ij}: \;\; C \mapsto \omega^i, \;\; E \mapsto \omega^j, \quad i, j = 0, 1, 2.$$

Remaining irreps: defining representation $\underline{\mathbf{3}}$ and its complex conjugate $\underline{\mathbf{3}}^*$.

Now: $\Delta(54)$:

Generators: C, E and $B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

From relations of generators:

$$\{e\} \lhd \mathbb{Z}_3 \lhd \mathbb{Z}_3 \times \mathbb{Z}_3 \lhd \Delta(27) \lhd \Delta(54)$$

Conjugacy classes of $\Delta(54)$

From the principal series $\{e\} \lhd \mathbb{Z}_3 \lhd \mathbb{Z}_3 \times \mathbb{Z}_3 \lhd \Delta(27) \lhd \Delta(54)$ we know

$$\Delta(54)/\Delta(27) = \langle\langle \Delta(27)B \rangle\rangle \cong \mathbb{Z}_2$$

Theorem

Let

- H be a proper normal subgroup of G such that $G/H \cong \mathbb{Z}_r$, $r \geq 2$, and
- Hb be a generator of G/H.

Then for a conjugacy class C_k of H there are two possibilities:

1. $bC_kb^{-1} = C_k \Rightarrow C_k$ is a conjugacy class of G.
2. $bC_kb^{-1} \cap C_k = \{\}$, then $C_k \cup bC_kb^{-1} \cup \ldots \cup b^{r-1}C_kb^{-(r-1)}$ is a conjugacy class of G.

Conjugacy classes of $\Delta(54)$

Now we can construct conjugacy classes of $\Delta(54)$ from those of $\Delta(27)$:

1. Classes which are invariant under $\mathcal{C} \mapsto B\mathcal{C}B^{-1}$:
   - $\mathcal{C}'_1 \equiv \mathcal{C}_1 = \{\mathbb{1}\}$,
   - $\mathcal{C}'_2 \equiv \mathcal{C}_2 = \{\omega\mathbb{1}\}$,
   - $\mathcal{C}'_3 \equiv \mathcal{C}_3 = \{\omega^2\mathbb{1}\}$.
2. Classes which are **not** invariant under $\mathcal{C} \mapsto B\mathcal{C}B^{-1}$:
   - $\mathcal{C}'_4 = \mathcal{C}_4 \cup B\mathcal{C}_4B^{-1} = \mathcal{C}_4 \cup \mathcal{C}_5$,
   - $\mathcal{C}'_5 = \mathcal{C}_6 \cup B\mathcal{C}_6B^{-1} = \mathcal{C}_6 \cup \mathcal{C}_7$,
   - $\mathcal{C}'_6 = \mathcal{C}_8 \cup B\mathcal{C}_8B^{-1} = \mathcal{C}_8 \cup \mathcal{C}_9$,
   - $\mathcal{C}'_7 = \mathcal{C}_{10} \cup B\mathcal{C}_{10}B^{-1} = \mathcal{C}_{10} \cup \mathcal{C}_{11}$,

11 conjugacy classes of $\Delta(27) \longmapsto$ 7 conjugacy classes of $\Delta(54)$.

Remaining conjugacy classes of $\Delta(54)$: $\mathcal{C}_B,\ \omega\mathcal{C}_B,\ \omega^2\mathcal{C}_B$.

Irreps of $\Delta(54)$

$$\{e\} \lhd \mathbb{Z}_3 \lhd \mathbb{Z}_3 \times \mathbb{Z}_3 \lhd \Delta(27) \lhd \Delta(54)$$

Factor groups:

- $\Delta(54)/\Delta(27) \cong \mathbb{Z}_2$,
- $\Delta(54)/(\mathbb{Z}_3 \times \mathbb{Z}_3) \cong S_3$,
- $\Delta(54)/\mathbb{Z}_3 \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

Thus:

- Irreps of $\mathbb{Z}_2$ are irreps of S_3,
- irreps of S_3 are irreps of $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ and
- irreps of $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ are irreps of $\Delta(54)$.

Together with $\underline{\mathbf{3}}$, $\underline{\mathbf{3}}^*$ and their products with the $\mathbb{Z}_2$-irreps we have found all irreps of $\Delta(54)$.

Application: $\Sigma(36 \times 3)$, $\Sigma(72 \times 3)$, $\Sigma(216 \times 3)$ [W. Grimus & PL, 2010]

Finite groups with faithful 3-dimensional irreducible representations

Motivation

Motivation for studying finite groups in particle physics:

- Flavour symmetries (lepton mixing, quark mixing)
- Spontaneous symmetry breaking of discrete symmetries does not give rise to Goldstone bosons

Why do we study finite groups with **faithful 3-dimensional irreducible representations**?

- Physical motivation: Three generations of fermions $\Rightarrow$ We study groups with three-dimensional faithful representations, i.e. subgroups of $U(3)$.
- Irreducibility: excludes subgroups of $U(2)$ and $U(1)$.

Investigation of $U(3) \rightarrow$ theorems which are easily generaliseable to $U(m)$.

The finite subgroups of $U(3)$

Finite subgroups of $U(3)$ $\rightarrow$ as far as we know not yet classified.

Finite subgroups of $SU(3)$ $\rightarrow$ classified at the beginning of the 20^{th} century by Miller, Dickson and Blichfeldt.[1]

Important question for flavour physics:

Is it (in terms of model building) enough to consider $SU(3)$ instead of $U(3)$?

$\rightarrow$ Not answered yet.

[1] *Theory And Applications of Finite Groups*; John Wiley & Sons, New York, 1916

The finite subgroups of $SU(3)$

type	subgroup	order of the subgroup
$\Sigma(n \times 3),\ n = 36, 72, 216, 360$	$\Sigma(36 \times 3)$	108
	$\Sigma(72 \times 3)$	216
	$\Sigma(216 \times 3)$	648
	$\Sigma(360 \times 3)$	1080
$\Sigma(m),\ m = 60, 168$	$\Sigma(60) \simeq A_5$	60
	$\Sigma(168)$	168
$\Delta(3n^2),\ n \in \mathbb{N}\backslash\{0,1\}$	$\Delta(3n^2)$; $\Delta(12) \simeq A_4$	$3n^2$
$\Delta(6n^2),\ n \in \mathbb{N}\backslash\{0,1\}$	$\Delta(6n^2)$; $\Delta(24) \simeq S_4$	$6n^2$
(C)-groups	$C(n, a, b)$	no general formula
(D)-groups	$D(n, a, b; d, r, s)$	no general formula

Finite subgroups of $U(3)$

Finite subgroups of $U(3)$ are not yet classified.

How can we get an idea of the finite subgroups of $U(3)$?

Two helpful tools:

- the **SmallGroups Library**,
- the computer algebra system **GAP** (Groups, algorithms and programming)[2]

- SmallGroups library contains information on all finite groups up to order 2000 (except 1024).
- GAP: read information from the library and calculate character tables, irreps,...

[2] www.gap-system.org

Finite subgroups of $U(3)$

We are interested in finite groups which

- have a faithful
- 3-dimensional
- irreducible representation and
- cannot be written as a direct product with a cyclic group.

Examples: A_4, S_4, $\Delta(54)$, ... but not S_3, $A_4 \times \mathbb{Z}_n$, ...

Why not direct products?

Let G be a finite group with a an m-dimensional faithful irrep. Let c be the order of the center of G.

Then $\mathbb{Z}_n \times G$ has a faithful m-dimensional irrep if and only if n and c have no common divisor.

In that case construction of the irreps of $\mathbb{Z}_n \times G$ from those of G is easy.

Extraction of groups from the SmallGroups library

How are finite groups listed in the SmallGroups library? $\rightarrow [g, j]$

g: order of the group, j: counter. Example: $A_4 \cong [12, 3]$

```
gap> NumberSmallGroups(12);
5
gap> g:=SmallGroup(12,3);
<pc group of size 12 with 3 generators>
gap> IsAbelian(g);
false
gap> ct:=CharacterTable(g);
CharacterTable( <pc group of size 12 with 3 generators> )
gap> Display(ct);
CT2

     2  2  .  2  .
     3  1  1  .  1

       1a 3a 2a 3b
    2P 1a 3b 1a 3a
    3P 1a 1a 2a 1a

X.1      1  1  1  1
X.2      1  A  1 /A
X.3      1 /A  1  A
X.4      3  . -1  .

A = E(3)^2
  = (-1-ER(-3))/2 = -1-b3
```

- There are five groups of order 12,
- SmallGroup number $[12, 3]$ is non-Abelian,
- character table shows a 3-dimensional faithful irrep.

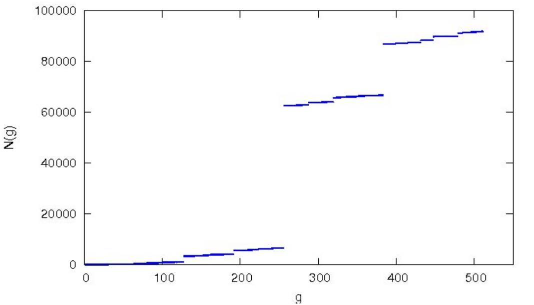
Extraction of groups from the SmallGroups library

GAP-command `StructureDescription` → direct products

```
gap>StructureDescription(SmallGroup([12,3]));
"A4"
gap>StructureDescription(SmallGroup([24,13]));
"C2 x A4"
```

⇒ We have tools to search the SmallGroups library for finite subgroups of $U(3)$.

How many non-Abelian groups are there?



→ finite subgroups of $U(3)$ of order smaller than 512.

The finite subgroups of $U(3)$ of order smaller than 512

List (including generators) of the finite subgroups of $U(3)$ of order smaller than 512 [PL, 2010]

Noteworthy results:

- $SU(3)$-subgroups: Classification scheme of Miller, Blichfeldt and Dickson confirmed (up to order 511)
- Smallest group of type (D): $[162, 14] \cong D(9, 1, 1; 2, 1, 1)$ cannot be interpreted as irrep of some $\Delta(6n^2)$.
- $U(3)$-subgroups: some series of finite subgroups of $U(3)$ found.

Construction of some series of finite subgroups of $U(3)$

Helpful theorem

Let $G = H \rtimes \mathbb{Z}_n$ be a finite group with the following properties:

1. G has a faithful m-dimensional irrep D.
2. n is prime.
3. The center of G is of order $c \neq n$ with c prime or $c = 1$.
4. G cannot be written as a direct product with a cyclic group.

Generators of $D(H)$: $A_1, \ldots, A_a$; generator of $D(\mathbb{Z}_n) : B$. Then

$$G' := \langle\langle A_1, \ldots, A_a, e^{2\pi i/b} B \rangle\rangle$$

(which has a faithful m-dim. irrep too) cannot be written as a direct product with a cyclic group if and only if

$$b = c^j n^k, \quad j, k \in \mathbb{N}.$$

Example: The group $S_4(m)$

$$S_4 = A_4 \rtimes \mathbb{Z}_2$$

Generators: A_4: $(14)(23)$, (123); $\mathbb{Z}_2$: (23)

- $S_4 = A_4 \rtimes \mathbb{Z}_2$ has a faithful 3-dimensional irrep $\underline{\mathbf{3}}$. $\Rightarrow n = 2$.
- $n = 2$ is prime.
- The center of S_4 is trivial. $\Rightarrow c = 1 \neq 2 = n$.
- S_4 cannot be written as a direct product with a cyclic group.

$$\underline{\mathbf{3}}: \quad (14)(23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} =: A, \quad (123) \mapsto \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} =: B, \; (23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} =: C.$$

$$S_4(m) := \langle\langle A, B, e^{2\pi i/2^m} C \rangle\rangle \cong A_4 \rtimes \mathbb{Z}_{2^m}, \quad m \in \mathbb{N}\backslash\{0\}.$$

$\rightarrow$ **Series of $U(3)$-subgroups.**

Series of finite subgroups of $U(3)$

group series	derived from
$T_n(m) = \mathbb{Z}_n \rtimes \mathbb{Z}_{3^m}$	$T_n = \mathbb{Z}_n \rtimes \mathbb{Z}_3$
$\Delta(3n^2, m) \cong (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_{3^m}$	$\Delta(3n^2) \cong (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3$
$S_4(m) \cong A_4 \rtimes \mathbb{Z}_{2^m}$	$S_4 \cong A_4 \rtimes \mathbb{Z}_2$
$\Delta(6n^2, m) \cong \Delta(3n^2) \rtimes \mathbb{Z}_{2^m},\ n \notin 3\mathbb{N}$	$\Delta(6n^2) \cong \Delta(3n^2) \rtimes \mathbb{Z}_2,\ n \notin 3\mathbb{N}$
$\Delta'(6n^2, j, k),\ n \in 3\mathbb{N}$	$\Delta(6n^2),\ n \in 3\mathbb{N}$

An inequality for characters

Construction of character tables

Especially in constructing character tables computer algebra systems are extremely helpful, however ...

... we must not rely on computer algebra systems only

Construction of some character tables by hand $\rightarrow$ "helpful tricks"

Here one such trick will be presented.

Characters of finite groups

Basic idea: $a \in G$ ... finite group:

$$a^n = e, \quad n := \mathrm{ord}(a) < \infty$$

$\Rightarrow$ in representation

$$D(a)^n = \mathbb{1}.$$

$\Rightarrow$ eigenvalues of $D(a)$ must be n-th roots of 1.

$\Rightarrow$ **character (trace) of $D(a)$ is a sum of n-th roots of 1.**

$\rightarrow$ What does this trivial insight tell us?

An inequality for characters

An inequality for characters

Let $a \in G$ ... finite group, and let D be a representation of G. If

$$n := \mathrm{ord}(D(a)) = 1, 2, 3, 4, 6$$

then

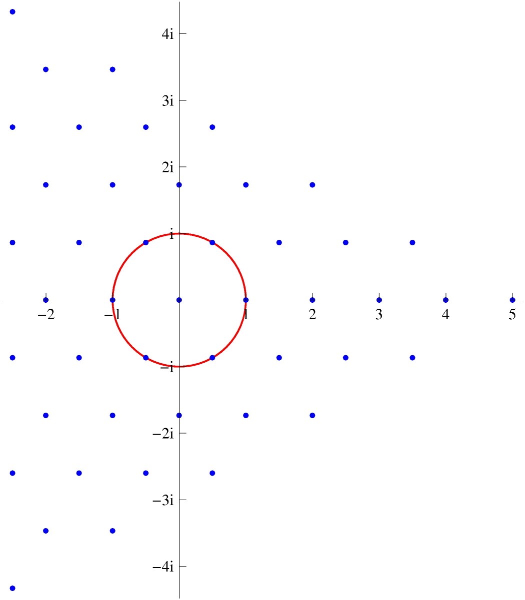
$$\boldsymbol{\chi_{\mathbf{D}}(\mathbf{a}) = 0 \quad \text{or} \quad |\chi_{\mathbf{D}}(\mathbf{a})| \geq 1.}$$

Proof: $n = 1, 2$: trivial ($\chi_D(a)$= sum of ± 1).

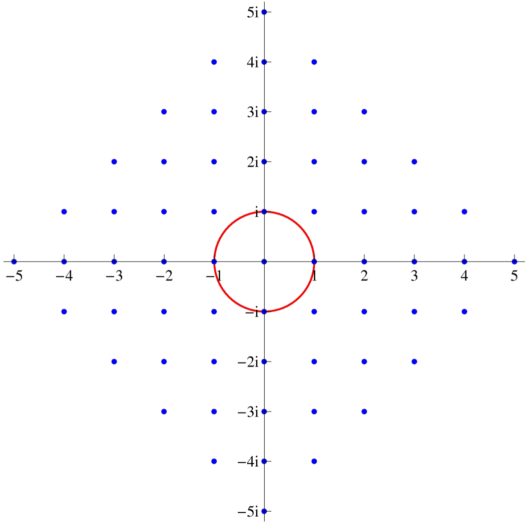
$n = 3, 4, 6$: **regular tilings of the plane.**

$\rightarrow$ **inequality for characters: trivial but useful**

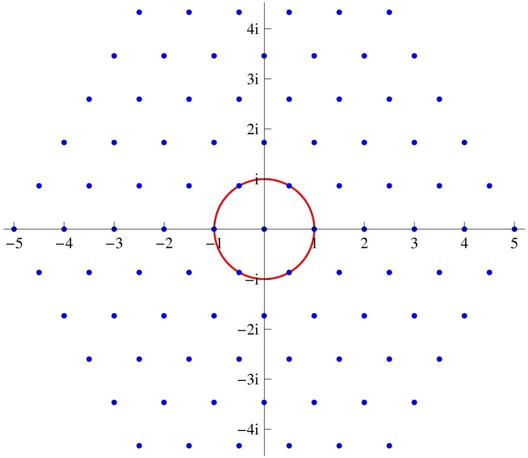
Sums of up to five third roots of 1



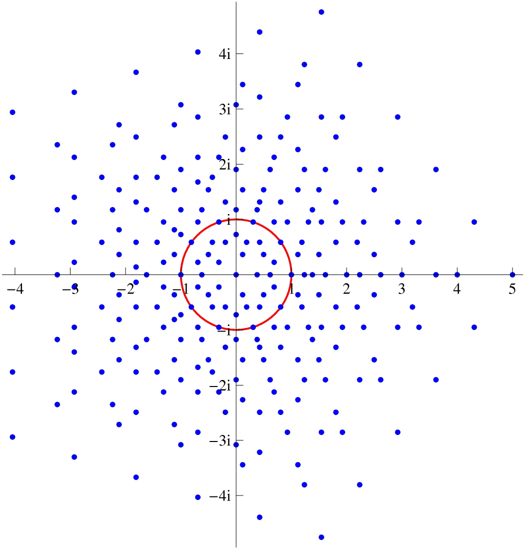
Sums of up to five fourth roots of 1



Sums of up to five sixth roots of 1



Sums of up to five fifth roots of 1



Summary

Principal series

- are helpful guides to understand the structure of a group
- can be helpful in constructing irreps.

The finite subgroups of $U(3)$ are not classified yet, but

- the finite subgroups of $SU(3)$ are.
- The SmallGroups library gives information on the finite groups of order ≤ 2000.
- The computer algebra system GAP is a powerful tool to work with finite groups.
- The finite subgroups of $U(3)$ of order < 512 have been listed.
- Several series of finite subgroups of $U(3)$ have been derived from this list.

Another helpful tool mentioned is

- the inequality for characters derived from the regular tilings of the plane.