17. Due to its critical point at zero temperature, the critical properties of the $1d$- Ising model are a little special. It is advantageous to introduce $\tilde{f} = (f + J)/T$. Thermodynamic critical exponents $\alpha_x, \beta_x, \gamma_x, \delta_x$ are defined with $t = \frac{T - T_c}{T_c}$ being replaced by $x = exp(-\frac{4J}{T})$, e.g. $C(x) \propto \frac{\partial^2 \tilde{f}}{\partial x^2} \propto x^{-\alpha_x}$. Calculate the critical exponents $\alpha_x, \beta_x, \gamma_x, \delta_x$ of the $1d$- Ising model.

18. Prove the critical exponent inequality $\alpha + 2\beta + \gamma \geq 2$.
Hint: $\chi(C_H - C_M) = T \left(\frac{\partial M}{\partial T}\right)_H^2$

19. Calculate the critical exponents α, γ, δ in the case of the van der Waals gas.

20. Calculate the critical exponent β in the case of the van der Waals gas.
Hint: Apply the Maxwell construction

21. A generalized Ising model in d dimensions has the usual Ising model Hamiltonian, but each spin variable takes the values

$$S_i = -t, -t + 1, ..., t - 1, t$$

Here t may be either an integer or a half-odd integer. Using mean field theory find the critical temperature of this system and recover the critical temperature of the usual Ising model.

22. The quantum Ising model Hamiltonian is given by

$$\hat{H} = -J \sum_{<i,j>} \sigma_i^z \sigma_j^z - \Gamma \sum_i \sigma_i^x$$

Here $J > 0, \Gamma \geq 0$ and σ_i^x, σ_i^z are Pauli matrices which measure the x- and y-components of the spins residing on the lattice sites of a hypercubic lattice. Let us denote the eigenstates of the σ_i^z by $| + >$ and $| - >$, with eigenvalues ± 1.

(a) Show that for $\Gamma = 0$ and in the above basis spanned by the eigenstates of the σ_i^z, the Hamiltonian $\hat{H}$ reduces to the familiar classical Ising model.

(b) Introduce $g = \frac{\Gamma}{\gamma J}$, where $\gamma = 2d$ is the coordination number, to tune a quantum phase transition at $T = 0$. Show that for $g = 0 \quad (\Gamma = 0)$ and for $g \longrightarrow \infty \quad (J \longrightarrow \infty)$ you get two qualitatively different ground states, called ferromagnetic and paramagnetic ground states.

(c) Derive the mean field Hamiltonian $\hat{H}_{MF}$ of the quantum Ising model
Hint: Write $\sigma_i^z = m^z + \delta\sigma_i^z$, where $m^z =< \sigma_i^z >= \dfrac{Tr\left[\sigma_i^z exp(-\beta\hat{H})\right]}{Tr\left[exp(-\beta\hat{H})\right]}$

(d) Calculate the eigenvalues of the single site mean-field Hamiltonian and use these eigenvalues to evaluate the partititon function Z_{MF} and the Landau function L_{MF}. Minimize the Landau function with respect to m^z to obtain a selfconsistency condition.

(e) Solve the mean-field self consistency equation graphically, analyze regions in parameter space where $m^z \neq 0$.

(f) Draw a phase diagram, separating the ferromagnetic phase from the paramagnetic one. What kind of phase transitions can you identify?