The Path Integral Formalism

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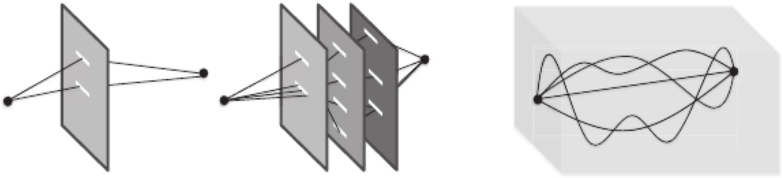
Overview

Idea



- Thought Experiment in Quantum Mechanics:
  - Double slit experiment: $\mathcal{M}_{\text{Slit1+Slit2}} = \mathcal{M}_{\text{Slit1}} + \mathcal{M}_{\text{Slit2}}$
  - More slits and/or more screens $\to$ Addition of all amplitudes with possible paths through slits.
  - What happens in the continuum limit of infinitely many screens and slits? Total amplitude: Sum over all possible paths. $\to$ Path Integral.

Motivation: Some great Things about the Path Integral in QFT

- No operators, just (anti-)commuting functions.
- Very simple quantization.
- Useful to quantify non-perturbative effects (e.g. Lattice QCD).
- Intrinsic Lorentz invariance due to Lagrangian formalism.
- Difficulty of calculation depends on specific problem.
- Hard to see that Hamiltonian is positive and hermitian.

Path Integrals in Quantum Mechanics - Definition

- Consider transition matrix with 1 d.o.f.:

$$\langle x', t' | x, t \rangle = \left\langle x' \left| \exp\left(-i\hat{H}(t'-t)\right) \right| x \right\rangle$$

with $\hat{x}_H(t) |x,t\rangle = x |x,t\rangle$, $\hat{x}_S |x\rangle = x |x\rangle$.

- To write down our thought experiment in a mathematical form, we divide time interval $(t'-t)$ into $n+1$ equal parts of length ε:

$$t' = t + (n+1)\varepsilon, \qquad t_j := t + j\varepsilon$$

Path Integrals in Quantum Mechanics - Definition

- Using completeness relation $\int \mathrm{d}x_j \; |x_j, t_j\rangle \langle x_j, t_j| = 1$:

$$\langle x', t' | x, t \rangle = \prod_{\substack{j=0 \\ i=1}}^{n} \int \mathrm{d}x_i \, \langle x_{j+1}, t_{j+1} | x_j, t_j \rangle$$

- Using $\int \frac{\mathrm{d}p_j}{2\pi} \; |p_j\rangle \langle p_j| = 1$ and assuming $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$ for a short-time matrix element:

$$\begin{aligned}
\langle x_j, t_j | x_{j-1}, t_{j-1} \rangle &= \left\langle x_j \left| \exp\left(-i\varepsilon \hat{H}\right) \right| x_{j-1} \right\rangle \\
&= \left\langle x_j \left| \left(1 - i\varepsilon \hat{H}\right) \right| x_{j-1} \right\rangle + \mathcal{O}(\varepsilon^2) \\
&= \int \frac{\mathrm{d}p_j}{2\pi} \; \mathrm{e}^{ip_j(x_j - x_{j-1})} \left(1 - i\varepsilon H(p_j, x_{j-1})\right) + \mathcal{O}(\varepsilon^2) \\
&= \int \frac{\mathrm{d}p_j}{2\pi} \; \mathrm{e}^{ip_j(x_j - x_{j-1}) - i\varepsilon H(p_j, x_{j-1})} + \mathcal{O}(\varepsilon^2)
\end{aligned}$$

Path Integrals in Quantum Mechanics - Definition

- Using all that for the full transition matrix element we get

$$\langle x', t' | x, t \rangle = \lim_{n\to\infty} \int \left(\prod_{j=1}^{n} \mathrm{d}x_j\right) \int \left(\prod_{j=1}^{n+1} \frac{\mathrm{d}p_j}{2\pi}\right) \times \exp\left(i \sum_{j=1}^{n+1} (p_j(x_j - x_{j-1}) - H(p_j, x_{j-1})\varepsilon)\right)$$

$$= \mathcal{N} \lim_{n\to\infty} \int \left(\prod_{j=1}^{n} \mathrm{d}x_j\right) \exp\left(i \sum_{j=1}^{n+1} \left(\frac{(x_j - x_{j-1})^2 m}{2\varepsilon} - V(x_{j-1})\right)\right)$$

$$=: \mathcal{N} \int \mathcal{D}x \exp\left(i \int_t^{t'} \mathrm{d}\tau \underbrace{\left(\frac{\dot{x}^2 m}{2} - V(x)\right)}_{=L}\right) = \mathcal{N} \int \mathcal{D}x \exp(iS[x])$$

- Boundaries: $x(t) = x$, $x(t') = x'$.
- We used $\int \frac{\mathrm{d}p_j}{2\pi} \exp\left(i(p_j \Delta x - \frac{\varepsilon p_j^2}{2m})\right) = \sqrt{2\pi \frac{m}{\varepsilon}} \exp\left(\frac{\Delta x^2 m}{2\varepsilon}\right)$.

Path Integrals in field theory

- Generalization to more degrees of freedom.
- $x(t) \to \phi(x,t)$: D.o.f. labeled by "index" $x \to$ infinite d.o.f.
- Various mathematical issues in defining the path integral in field theory (e.g. discrete $\to$ continuous).
- Much more ambiguous than in quantum mechanics, nevertheless of great value.

Path Integrals in field theory

- Heuristically, the previous derivation can be done also for fields.
- Intermediate states much more complicated.
- Completeness relation:

$$\int \mathcal{D}\phi \; |\phi\rangle \langle\phi| = 1$$

- Analogous to quantum mechanics:

$$\left\langle \phi_b(\vec{x}) \left| \mathrm{e}^{-i\hat{H}T} \right| \phi_a(\vec{x}) \right\rangle = \mathcal{N} \int \mathcal{D}\phi \; \exp\left(i \int_0^T \mathrm{d}^4x \, \mathcal{L} \right)$$

with $\phi(0, \vec{x}) = \phi_a(\vec{x})$, $\phi(T, \vec{x}) = \phi_b(\vec{x})$.

Basics - The Two-Point Function

- We want to do something useful with our new formulation.
- What's the equivalent to $\langle\Omega\,|\mathcal{T}\phi_H(x_1)\phi_H(x_2)|\,\Omega\rangle$?
- Consider

$$\mathcal{I} := \int \mathcal{D}\phi\,\phi(x_1)\phi(x_2)\exp\left(i\int_{-T}^{T}\mathrm{d}^4x\,\mathcal{L}\right)$$

with $\phi(-T,\vec{x}) = \phi_a(\vec{x})$, $\phi(T,\vec{x}) = \phi_b(\vec{x})$.

Basics - The Two-Point Function

- We want to use our expression for matrix elements from above, so we break up the integral:

$$\int \mathcal{D}\phi(x) = \int \mathcal{D}\phi_1(\vec{x})\, \mathcal{D}\phi_2(\vec{x}) \int_{\phi(x_1^0,\vec{x})=\phi_1(\vec{x})}^{\phi(x_2^0,\vec{x})=\phi_2(\vec{x})} \mathcal{D}\phi(x)$$

- Assuming $x_1^0 < x_2^0$ we get

$$\begin{aligned}
\mathcal{I} = & \int \mathcal{D}\phi_1(\vec{x})\, \mathcal{D}\phi_2(\vec{x})\, \phi_1(\vec{x}_1)\phi_2(\vec{x}_2) \\
& \times \left\langle \phi_b \middle| \mathrm{e}^{-iH(T-x_2^0)} \middle| \phi_2 \right\rangle \left\langle \phi_2 \middle| \mathrm{e}^{-iH(x_2^0-x_1^0)} \middle| \phi_1 \right\rangle \\
& \times \left\langle \phi_1 \middle| \mathrm{e}^{-iH(x_1^0+T)} \middle| \phi_a \right\rangle \\
= & \left\langle \phi_b \middle| \mathrm{e}^{-iH(T-x_2^0)} \phi_S(\vec{x}_2)\, \mathrm{e}^{-iH(x_2^0-x_1^0)} \phi_S(\vec{x}_1)\, \mathrm{e}^{-iH(x_1^0+T)} \middle| \phi_a \right\rangle \\
= & \left\langle \phi_b \middle| \mathrm{e}^{-iHT} \mathcal{T}\left[\phi_H(x_1)\phi_H(x_2)\right] \mathrm{e}^{-iHT} \middle| \phi_a \right\rangle
\end{aligned}$$

Basics - The Two-Point Function

- Something we know from particle physics 2! Take limit $T \to \infty(1 - i\epsilon)$ to get correlation function.
- Conditions:
  - $\exists$ overlap of $|\phi_a\rangle$ and $|\Omega\rangle$,
  - there is a finite step between the ground energy E_0 and the next energy state.

$$\mathrm{e}^{-iHT}\,|\phi_a\rangle = \sum_n \mathrm{e}^{-iE_nT}\,|n\rangle\,\langle n\,|\phi_a\,\rangle$$

$$\overset{T\to\infty(1-i\epsilon)}{\longrightarrow}\;\mathrm{e}^{-iE_0\infty(1-i\epsilon)}\,\langle\Omega\,|\phi_a\,\rangle\,|\Omega\rangle$$

Basics - The Two-Point Function

- Like in particle physics 2, the overlap factor vanishes, if we normalize the expression.
- Analogous to the operator formalism, the normalization factor corresponds to vacuum bubbles.

$$\langle \Omega | \mathcal{T} \phi_H(x_1)\phi_H(x_2) | \Omega \rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi \, \phi(x_1)\phi(x_2) \exp\left(i \int_{-T}^{T} \mathrm{d}^4x \, \mathcal{L} \right)}{\int \mathcal{D}\phi \, \exp\left(i \int_{-T}^{T} \mathrm{d}^4x \, \mathcal{L} \right)}$$

Basics - The Generating Functional

- Generating functional: Great way to calculate correlation functions in the p.i. formalism.
- Consider an action in presence of an ext. source $J(x)$. In this case $J(x)$ is just an auxiliary field without physical meaning.
- The vacuum amplitude is then the generating functional:

$$Z[J] = \int \mathcal{D}\phi \, \exp\left(iS[\phi] + i \int \mathrm{d}^4x \, J(x)\phi(x) \right)$$

Basics - The Generating Functional

$$Z[J] = \int \mathcal{D}\phi \, \exp\left(iS[\phi] + i \int \mathrm{d}^4x \, J(x)\phi(x) \right)$$

- Why is it called the generating functional? We see

$$- i \frac{\delta Z[J]}{\delta J(x_1)} = \int \mathcal{D}\phi \; \mathrm{e}^{iS[\phi]+i\int \mathrm{d}^4x \, J(x)\phi(x)} \phi(x_1)$$

$$\Rightarrow \quad - iZ[0]^{-1} \left.\frac{\delta Z[J]}{\delta J(x_1)}\right|_{J=0} = \langle \Omega \, | \phi_H(x_1) | \, \Omega \rangle$$

- Analogous:

$$(-i)^n Z[0]^{-1} \left.\frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)}\right|_{J=0} = \langle \Omega \, | \mathcal{T} \phi_H(x_1) \cdots \phi_H(x_n) | \, \Omega \rangle$$

- Comment: Possible do define special generating functionals for connected and 1PI Green's functions.

Example 1 - Free scalar

- Possible to get closed expression for generating functional if we assume that ∃ continuum version of

$$\int_{-\infty}^{\infty} \mathrm{d}^n x \ \mathrm{e}^{-\frac{i}{2}\vec{x}^T A \vec{x} + i\vec{J}\cdot\vec{x}} = \sqrt{\frac{(-2\pi i)^n}{\det A}} \ \mathrm{e}^{\frac{i}{2}\vec{J}^T A^{-1} \vec{J}}$$

- Our generating functional

$$Z_{\text{free}}[J] = \int \mathcal{D}\phi \ \mathrm{e}^{i\int \mathrm{d}^4x \left(-\frac{1}{2}\phi(x)\left(\Box + m^2 - i\varepsilon\right)\phi(x) + J(x)\phi(x)\right)}$$

  has this form with $A = \left(\Box + m^2 - i\varepsilon\right)$.
- $i\varepsilon$-term needed for convergence. Needed sign corresponds to Feynman-Stückelberg interpretation without additional thinking!
- Square root: Canceled by normalization constant.

Example 1 - Free scalar

- To use the integral formula, we need the inverse of A, a function $S(x-y)$, which satisfies

$$(\Box_x + m^2 - i\varepsilon)S(x-y) = -i\delta^{(4)}(x-y)$$

- This is the Feynman propagator:

$$S(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} \mathrm{e}^{-ip(x-y)}$$

$$\Rightarrow \quad Z_{\text{free}}[J] = \mathcal{N} \exp\left(- \int \mathrm{d}^4x\, \mathrm{d}^4y\, \frac{1}{2} J(x) S(x-y) J(y) \right)$$

- Easy to check:

$$\langle 0 | \mathcal{T} \phi_H(x_1) \phi_H(x_2) | 0 \rangle = (-i)^2 Z_{\text{free}}[0]^{-1} \left. \frac{\delta^2 Z_{\text{free}}[J]}{\delta J(x_1)\, \delta J(x_2)} \right|_0 = S(x_1 - x_2)$$

Example 2 - ϕ^4

- $\mathcal{L} = \mathcal{L}_{\text{free}} - \frac{\lambda}{4!}\phi^4$
- No analytical expression known. We use the free expression and get the known perturbative expansion:

$$
\begin{aligned}
Z[J] &= \int \mathcal{D}\phi \ \mathrm{e}^{i\left(S_{\text{free}}[\phi] + S_{\text{int}}[\phi] + \int \mathrm{d}^4x \ J(x)\phi(x)\right)} \\
&= \mathrm{e}^{iS_{\text{int}}[-i\frac{\delta}{\delta J}]} \int \mathcal{D}\phi \ \mathrm{e}^{S_{\text{free}}[\phi] + \int \mathrm{d}^4x \ J(x)\phi(x)} \\
&= \underbrace{\mathrm{e}^{iS_{\text{int}}[-i\frac{\delta}{\delta J}]}}_{\text{expandable in } \lambda \to \text{perturbative series}} Z_{\text{free}}[J]
\end{aligned}
$$

Example 2 - ϕ^4

- Check:

$$\langle \Omega | \mathcal{T} \phi_H(x_1) \phi_H(x_2) | \Omega \rangle = - Z[0]^{-1} \left. \frac{\delta^2 Z[J]}{\delta J(x_1)\, \delta J(x_2)} \right|_0$$

$$= - \left(Z_{\text{free}}[0] - \frac{i\lambda}{4!} \int d^4x \left. \frac{\delta^4 Z_{\text{free}}[J]}{\delta J(x)^4} \right|_0 \right)^{-1}$$

$$\times \left(\left. \frac{\delta^2 Z_{\text{free}}}{\delta J(x_1)\, \delta J(x_2)} \right|_0 - \frac{i\lambda}{4!} \int d^4x \left. \frac{\delta^6 Z_{\text{free}}[J]}{\delta J(x_1)\, \delta J(x_2)\, \delta J(x)^4} \right|_0 \right) + \mathcal{O}(\lambda^2)$$

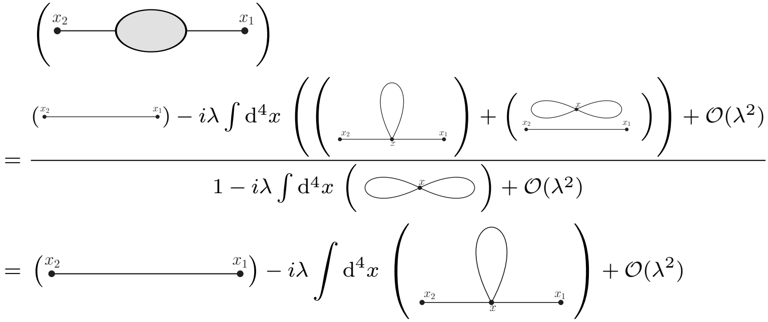
$$= S(x_1 - x_2) - i\lambda \int d^4x\, S(x_1 - x)S(x_2 - x)S(x - x) + \mathcal{O}(\lambda^2)$$

- Obviously the same as in operator formalism:

$$\langle 0 | \mathcal{T} \phi_I(x_1)\phi_I(x_2) | 0 \rangle - \frac{i\lambda}{4!} \int d^4x \left\langle 0 \left| \mathcal{T} \phi_I(x_1)\phi_I(x_2)\phi_I(x)^4 \right| 0 \right\rangle + \mathcal{O}(\lambda^2)$$

Example 2 - ϕ^4

- The known combinatorial factors arise from the multiple functional derivative.
- Graphical representation:



Basics - Path Integral for Fermions

- How to quantize fermionic fields?
- For $x_1 \neq x_2$:

$$\left\langle \Omega \middle| \mathcal{T} \psi_H(x_1) \bar{\psi}_H(x_2) \middle| \Omega \right\rangle = - \left\langle \Omega \middle| \mathcal{T} \bar{\psi}_H(x_2) \psi_H(x_1) \middle| \Omega \right\rangle$$

- "Classical" Fermi fields must be anticommuting numbers: These are called "Grassmann numbers".
- For Grassmann variables η and χ:

$$\{\eta, \chi\} = 0 \quad \Rightarrow \quad \eta^2 = 0$$

$$\Rightarrow \quad \text{For some function } f(\eta)\text{: } f(\eta) = f_0 + f_1 \eta$$

- Taylor expansion stops after the linear term.

Fermions - Grassmann Numbers

- Definition of left/right derivative:

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\eta = \eta\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\eta} = 1$$

- Integral over Grassmann variables: Has to maintain shift invariance

$$\Rightarrow \quad \int \mathrm{d}\eta\, 1 = 0 \text{ (from shift invariance)}$$
$$\int \mathrm{d}\eta\, \eta = 1 \text{ (normalization)}$$

- Check:

$$\int \mathrm{d}\eta\, f(\eta) = f_0 \int \mathrm{d}\eta\, 1 + f_1 \int \mathrm{d}\eta\, \eta \overset{\eta\to\eta+\chi}{=} (f_0 + \chi f_1) \int \mathrm{d}\eta\, 1 + f_1 \int \mathrm{d}\eta\, \eta = f_1$$

Fermions - Grassmann Numbers

- Remember the important integral

$$\int_{-\infty}^{\infty} \mathrm{d}^n x \; \mathrm{e}^{-\frac{i}{2}\vec{x}^T A\vec{x}+i\vec{J}\cdot\vec{x}} = \sqrt{\frac{(-2\pi i)^n}{\det A}}\,\mathrm{e}^{\frac{i}{2}\vec{J}^T A^{-1}\vec{J}}$$

- For Grassmann variables:

$$\int \mathrm{d}^n\bar{\eta}\,\mathrm{d}^n\eta \; \mathrm{e}^{i\vec{\bar{\eta}}^T A\vec{\eta}+i\vec{\bar{\beta}}\cdot\vec{\eta}+i\vec{\bar{\eta}}\cdot\vec{\beta}} = \det(-iA)\,\mathrm{e}^{-i\vec{\bar{\beta}}^T A^{-1}\vec{\beta}}$$

- Check in 1 dimension using $\eta \to \eta - a^{-1}\beta$, $\bar{\eta} \to \bar{\eta} - \bar{\beta}a^{-1}$:

$$\int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta \; \mathrm{e}^{i\bar{\eta}a\eta+i\bar{\beta}\eta+i\bar{\eta}\beta} = \mathrm{e}^{-i\bar{\beta}a^{-1}\beta}\int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\;\underbrace{\mathrm{e}^{i\bar{\eta}a\eta}}_{=1+i\bar{\eta}a\eta}$$

$$= -ia\,\mathrm{e}^{-i\bar{\beta}a^{-1}\beta}$$

Fermions - Dirac Propagator

- Dirac propagator from the generating functional:

$$Z[\alpha,\bar{\alpha}] = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\; \mathrm{e}^{i\int \mathrm{d}^4x\,\left(\bar{\psi}(x)(i\partial\!\!\!/-m+i\varepsilon)\psi(x)+\bar{\alpha}(x)\psi(x)+\bar{\psi}(x)\alpha(x)\right)}$$

$$= \mathcal{N}\,\mathrm{e}^{-i\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{\alpha}(x)(i\partial\!\!\!/-m+i\varepsilon)^{-1}\alpha(y)}$$

$$= \mathcal{N}\,\mathrm{e}^{-\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{\alpha}(x)S(x-y)\alpha(y)}$$

$$(i\partial\!\!\!/_{(x)}-m+i\varepsilon)S(x-y) = i\delta^{(4)}(x-y),\quad S(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{i\,\mathrm{e}^{-ip(x-y)}}{p\!\!\!/-m+i\varepsilon}$$

- Check:

$$Z[0]^{-1}\,(-i)^2\,\frac{\delta}{\delta\bar{\alpha}(x_1)}Z[\alpha,\bar{\alpha}]\,\frac{\overleftarrow{\delta}}{\delta\alpha(x_2)}\Bigg|_0 = S(x_1-x_2)$$

- Attention!

$$(-i)^2\,\frac{\delta}{\delta\bar{\alpha}(x_1)}Z\frac{\overleftarrow{\delta}}{\delta\alpha(x_2)}\Bigg|_0 = -(-i)^2\,\frac{\delta}{\delta\bar{\alpha}(x_1)}\frac{\delta}{\delta\alpha(x_2)}Z\Bigg|_0$$

Fermions - Fermion Loops

- For fermion loops one has something like

$$\begin{aligned}
&\left\langle 0 \left| \mathcal{T} \bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y) \right| 0 \right\rangle \\
&= Z[0]^{-1} \left(\frac{i^2}{i^2} \right) \left. \frac{\delta^4 Z[\alpha, \bar{\alpha}]}{\delta\alpha(x)\, \delta\bar{\alpha}(x)\, \delta\alpha(y)\, \delta\bar{\alpha}(y)} \right|_0 \\
&= -Z[0]^{-1} \left. \frac{\delta^4 Z[\alpha, \bar{\alpha}]}{\delta\bar{\alpha}(y)\, \delta\alpha(x)\, \delta\bar{\alpha}(x)\, \delta\alpha(y)} \right|_0 \\
&= -S(y-x)S(x-y)
\end{aligned}$$

$\rightarrow$ minus sign for fermion loops.

Quantization of the Electromagnetic Field - Faddeev-Popov

- Consider

$$\int \mathcal{D}A \; \mathrm{e}^{iS[A]}$$

$$S[A] = \int \mathrm{d}^4x \left(-\frac{1}{4}\left(F_{\mu\nu}\right)^2\right) = \frac{1}{2}\int \mathrm{d}^4x \, A_\mu(x)\left(\Box g^{\mu\nu} - \partial^\mu\partial^\nu\right)A_\nu(x)$$

$$= \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\tilde{A}_\mu(k)\left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right)\tilde{A}_\nu(-k)$$

- Redundant integration over continuous infinity of physically equivalent field configurations due to gauge invariance.
- Very bad: $A^\mu = k^\mu a$ is gauge equivalent to 0!
- This related to the problem that

$$(-k^2 g^{\mu\nu} + k^\mu k^\nu)\tilde{S}^{(\gamma)}_{\nu\rho}(k) = i\delta^\mu_\rho$$

has no solution.

Quantization of the Electromagnetic Field - Faddeev-Popov

- We want to count every physical configuration just once!
- Let $G(A)$ be a function we want to be zero as a gauge fixing condition (e.g. $G(A) = \partial_\mu A^\mu$).
- Idea: Insert "functional delta function" $\delta(G(A))$. We insert this by using

$$1 = \int \mathcal{D}\alpha \, \delta(G(A^\alpha)) \det \left(\frac{\delta G(A^\alpha)}{\delta \alpha} \right)$$

  where $A^\alpha_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$.
- Continuum limit of $1 = \left(\prod_i \int \mathrm{d}a_i \right) \delta^{(n)}(\vec{g}(\vec{a})) \det \left(\frac{\partial g_i}{\partial a_j} \right)$
- For non-Abelian theories $(A^\alpha)^a_\mu = A^a_\mu + \frac{1}{g} D_\mu \alpha^a$

Quantization of the Electromagnetic Field - Faddeev-Popov

- After insertion of 1 we have

$$\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\int \mathcal{D}\alpha\,\mathcal{D}A\ \mathrm{e}^{iS[A]}\delta(G(A^\alpha))$$

- Next step: Change of variables from A to A^α. Shift and gauge invariance $\Rightarrow$ $\mathcal{D}A = \mathcal{D}A^\alpha$, $S[A] = S[A^\alpha]$.
- Now A^α is just a dummy variable $\rightarrow$ Rename back to A

$$\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A\ \mathrm{e}^{iS[A]}\delta(G(A))$$

Quantization of the Electromagnetic Field - Faddeev-Popov

$$\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\left(\int\mathcal{D}\alpha\right)\int\mathcal{D}A\ \mathrm{e}^{iS[A]}\delta(G(A))$$

- Specify gauge-fixing function $G(A)$: $G(A)=\partial^\mu A_\mu(x)-\omega(x)$ $\Rightarrow \det\left(\frac{\delta G}{\delta\alpha}\right)=\det\left(\frac{\Box}{e}\right)$, independent of any field $\rightarrow$ can be treated as normalization constant.

$$\det\left(\frac{\Box}{e}\right)\left(\int\mathcal{D}\alpha\right)\int\mathcal{D}A\ \mathrm{e}^{iS[A]}\delta(\partial^\mu A_\mu-\omega(x))$$

- This is not valid in a non-Abelian gauge theory, because $\det\left(\frac{\partial_\mu D^\mu}{g}\right)$ is not field independent.

Quantization of the Electromagnetic Field - Faddeev-Popov

- Valid for any ω $\Rightarrow$ can replace with normalized linear combination over all ω with Gaussian weight function centered at $\omega = 0$.

$$\det\left(\frac{\Box}{e}\right)\left(\int \mathcal{D}\alpha\right)\mathcal{N}(\xi)\int \mathcal{D}\omega\ \mathrm{e}^{-i\int \mathrm{d}^4x\ \frac{\omega^2}{2\xi}}\int \mathcal{D}A\ \mathrm{e}^{iS[A]}\delta(\partial^\mu A_\mu - \omega(x))$$

$$= \det\left(\frac{\Box}{e}\right)\left(\int \mathcal{D}\alpha\right)\mathcal{N}(\xi)\int \mathcal{D}A\ \mathrm{e}^{iS[A]}\ \mathrm{e}^{-i\int \mathrm{d}^4x\ \frac{1}{2\xi}(\partial^\mu A_\mu)^2}$$

where ξ can be any finite constant.

- Effectively we added the new term $-\frac{(\partial^\mu A_\mu)^2}{2\xi}$ to the Lagrangian!
- It seems like we introduced ξ as a new parameter, but we didn't!

Quantization of the Electromagnetic Field - Faddeev-Popov

- Is it now possible to derive the photon propagator? We define the generating functional

$$\int \mathcal{D}A \; \mathrm{e}^{\int \mathrm{d}^4x \left(\frac{i}{2}A_\mu(x)(\Box g^{\mu\nu}-\partial^\mu\partial^\nu)A_\nu(x)-\frac{i}{2\xi}(\partial^\mu A_\mu)^2+J_\mu A^\mu\right)}$$
$$= \int \mathcal{D}A \; \mathrm{e}^{i\int \mathrm{d}^4x \left(\frac{1}{2}A_\mu(x)\left(\Box g^{\mu\nu}-\partial^\mu\partial^\nu\left(1-\frac{1}{\xi}\right)\right)A_\nu(x)+J_\mu A^\mu\right)}$$
$$= \mathcal{N}\,\mathrm{e}^{-\frac{i}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\, J_\mu(x)\left(\Box g^{\mu\nu}-\partial^\mu\partial^\nu\left(1-\frac{1}{\xi}\right)\right)^{-1}J_\nu(y)}$$

- In momentum space wee need to solve

$$\left(-k^2 g_{\mu\nu} + \left(1-\frac{1}{\xi}\right)k_\mu k_\nu\right)\tilde{S}^{\nu\rho}_{(\gamma)}(k) = i\delta^\rho_\mu$$

$$\Rightarrow \quad \tilde{S}^{\nu\rho}_{(\gamma)}(k) = \frac{-i}{k^2+i\varepsilon}\left(g^{\nu\rho}-(1-\xi)\frac{k^\nu k^\rho}{k^2}\right).$$

Quantization of the Electromagnetic Field - Faddeev-Popov

- Correlation functions $\mathcal{N} \int \mathcal{D}A\, \mathcal{O}(A)\, \mathrm{e}^{iS[A]}$ of gauge invariant operators $\mathcal{O}(A)$ is clearly independent of ξ.
- In practice: Insert specific value for ξ. Popular choices:
  - Feynman gauge: $\xi = 1$,
  - Landau gauge: $\xi = 0$.

Faddeev-Popov: Comments on Non-Abelian Gauge Theories

- In non-Abelian theory: Problem of functional determinant solved by using

$$\det\left(\frac{\partial_\mu D^\mu}{g}\right) = \int \mathcal{D}\bar{c}\,\mathcal{D}c\ \mathrm{e}^{i\int \mathrm{d}^4x\, \bar{c}^a(-\partial^\mu D_\mu^{ac})c^c}$$

  where c^i and $\bar{c}^i$ are scalar Grassmann fields and $D_\mu^{ac} = \partial_\mu \delta^{ac} + g f^{abc} A_\mu^b$.
- This results in an additional term in the Lagrangian.
- c and $\bar{c}$ unphysical, but can be treated as additional excitations in diagrams → "Faddeev-Popov ghosts".
- QED: "ghosts" are propagating fields, but they do not interact:

$$\det\left(\frac{\Box}{e}\right) = \int \mathcal{D}\bar{c}\,\mathcal{D}c\ \mathrm{e}^{i\int \mathrm{d}^4x\, \bar{c}(-\Box)c}$$

The QED Ward-Takahashi Identities - General

- We will now derive the famous Ward-Takahashi Identities.
- Consider the generating functional of QED

$$Z[J,\alpha,\bar{\alpha}] = \int \mathcal{D}(A,\bar{\psi},\psi)\ \mathrm{e}^{i\int \mathrm{d}^4x\,\left(\mathcal{L}(A,\bar{\psi},\psi)+J_\mu A^\mu+\bar{\alpha}\psi+\bar{\psi}\alpha\right)}$$

  and an infinitesimal gauge transformation
  $\psi'(x) = (1-ig(x))\psi(x),\ A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu g(x)$
- Gauge dependent quantities in Z: Gauge-fixing and source terms.

The QED Ward-Takahashi Identities - General

- After applying the transformation, we expand in $g(x)$ up to $\mathcal{O}(g)$:

$$\int \mathrm{d}^4x \int \mathcal{D}(A,\bar{\psi},\psi)\ \mathrm{e}^{i(S[A,\bar{\psi},\psi]+\text{Source})}$$
$$\times\left(1 - g(x)\left(-\frac{1}{e}\partial_\mu J^\mu(x) - i\bar{\alpha}(x)\psi(x) + i\bar{\psi}(x)\alpha(x) - \frac{1}{e}\partial_\mu \Box^{\mu\nu} A_\nu(x)\right)\right)$$

  with $\Box_{\mu\nu} = \Box g_{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu$.

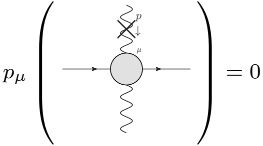
- We get rid of the 1 by subtracting the original expression ($\to \delta Z$). We now arrive at the Ward-Takahashi Identity:

$$0 = \left.\frac{\delta Z}{\delta g(y)}\right|_{g=0} = \frac{1}{e}\partial^\mu_{(y)} J_\mu(y) Z + i\bar{\alpha}(y)\frac{\delta Z}{\delta\bar{\alpha}(y)} + i\frac{\delta Z}{\delta\alpha(y)}\alpha(y) + \frac{1}{e}\partial^{(y)}_\mu \Box^{\mu\nu}_{(y)}\frac{\delta Z}{\delta J^\nu(y)}$$

The QED Ward Identity - Special Case

$$-\frac{1}{e}\partial_\mu^{(y)}\Box^{\mu\nu}_{(y)}\frac{\delta Z}{\delta J^\nu(y)} = \frac{1}{e}\partial^\mu_{(y)}J_\mu(y)Z + i\bar{\alpha}(y)\frac{\delta Z}{\delta\bar{\alpha}(y)} + i\frac{\delta Z}{\delta\alpha(y)}\alpha(y)$$

- Ward-identity: Important special case in a S-matrix element: $p_\mu\mathcal{M}^\mu(p) = 0$ with $\mathcal{M}(p) = \epsilon_\mu(p)\mathcal{M}^\mu(p)$.

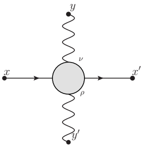


  - For a special case with two fermions and photons, we apply $\left.\frac{\delta}{\delta J^\rho(y')}\frac{\delta}{\delta\bar{\alpha}(x)}\frac{\delta}{\delta\alpha(x')}\right|_0$ and get

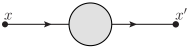
$$-\frac{1}{e}\partial_\mu^{(y)}\Box^{\mu\nu}_{(y)}\left.\frac{\delta^4 Z}{\delta J^\nu(y)\,\delta J^\rho(y')\,\delta\bar{\alpha}(x)\,\delta\alpha(x')}\right|_0 = \frac{1}{e}\partial_\rho^{(y)}\delta^{(4)}(y-y')\left.\frac{\delta^2 Z}{\delta\bar{\alpha}(x)\,\delta\alpha(x')}\right|_0$$
$$-i\delta^{(4)}(x-y)\left.\frac{\delta^3 Z}{\delta J^\rho(y')\,\delta\alpha(x')\,\delta\bar{\alpha}(y)}\right|_0 - i\delta^{(4)}(y-x')\left.\frac{\delta^3 Z}{\delta J^\rho(y')\,\delta\bar{\alpha}(x)\,\delta\alpha(y)}\right|_0$$

The QED Ward Identity - Special Case

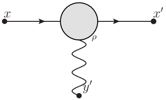
$$\partial_\mu^{(y)} \Box^{\mu\nu}_{(y)} \left(\vphantom{\int} \right.$$



$$\left. \vphantom{\int} \right) = -\partial_\rho^{(y)} \delta^{(4)}(y - y') \left(\vphantom{\int} \right.$$



$$\left. \vphantom{\int} \right) + e\left(\delta^{(4)}(x-y) - \delta^{(4)}(y-x')\right) \left(\vphantom{\int} \right.$$



$$\left. \vphantom{\int} \right)$$

- In terms of correlation functions:

$$\partial_\mu^{(y)} \Box^{\mu\nu}_{(y)} \left\langle A_\nu(y) A_\rho(y') \psi(x) \bar{\psi}(x') \right\rangle = -\partial_\rho^{(y)} \delta^{(4)}(y - y') \left\langle \psi(x) \bar{\psi}(x') \right\rangle$$
$$+e\left(\delta^{(4)}(x-y) - \delta^{(4)}(y-x')\right) \left\langle A_\rho(y') \psi(x) \bar{\psi}(x') \right\rangle$$

where we used the notation $\langle \Omega \,|\mathcal{T} \varphi(x_1) \cdots \varphi(x_n)|\, \Omega \rangle =: \langle \varphi(x_1) \cdots \varphi(x_n) \rangle$.

The QED Ward Identity - Special Case

- Note that $\Box^{\mu\nu}_{(y)}$ is $i \left(S^{-1}_{(\gamma)}\right)^{\mu\nu}(y)$!
- Fourier transform:

$$\int \mathrm{d}^4x\, \mathrm{d}^4y\, \mathrm{d}^4x'\, \mathrm{d}^4y'\; \mathrm{e}^{-iqx}\, \mathrm{e}^{-ipy}\, \mathrm{e}^{iq'x'}\, \mathrm{e}^{ip'y'}$$

$$\delta^{(4)}(x-y)\left\langle A_\rho(y')\psi(x)\bar{\psi}(x')\right\rangle \to \left\langle A_\rho(p')\psi(q')\middle|\, \psi(q+p)\right\rangle (2\pi)^4\delta^{(4)}(q+p-q'-p')$$

$$\delta^{(4)}(y-x')\left\langle A_\rho(y')\psi(x)\bar{\psi}(x')\right\rangle \to \left\langle A_\rho(p')\psi(q'-p)\middle|\, \psi(q)\right\rangle (2\pi)^4\delta^{(4)}(q+p-q'-p')$$

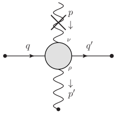
$$\partial^{(y)}_\rho\delta^{(4)}(y-y')\left\langle \psi(x)\bar{\psi}(x')\right\rangle \to ip_\rho\left\langle \psi(q')\middle|\, \psi(q)\right\rangle (2\pi)^8\delta^{(4)}(q-q')\delta^{(4)}(p-p')$$

$$\partial^{(y)}_\mu\left(S^{-1}_{(\gamma)}\right)^{\mu\nu}(y)\left\langle A_\rho(y')A_\nu(y)\psi(x)\bar{\psi}(x')\right\rangle$$

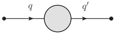
$$\to ip_\mu\left(\tilde{S}^{-1}_{(\gamma)}\right)^{\mu\nu}(p)\left\langle A_\rho(p')\psi(q')\middle|\, A_\nu(p)\psi(q)\right\rangle (2\pi)^4\delta^{(4)}(q+p-q'-p')$$

- $\langle \varphi(p_1)\cdots\varphi(p_n)|\,\varphi(q_1)\cdots\varphi(q_m)\rangle$ are momentum space Green's functions.

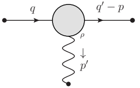
The QED Ward Identity - Special Case

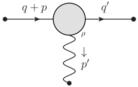
p_μ (


) $\delta^{(4)}(q+p-q'-p')$

$= ip_\rho$ (


) $(2\pi)^4\delta^{(4)}(q-q')\delta^{(4)}(p-p')$

$+ e$ ((


) $-$ (


)) $\delta^{(4)}(q+p-q'-p')$

- Our relation in momentum space:

$$p_\mu \left(\tilde{S}^{-1}_{(\gamma)}\right)^{\mu\nu}(p)\left\langle A_\rho(p')\psi(q')\middle| A_\nu(p)\psi(q)\right\rangle \delta^{(4)}(q+p-q'-p')$$
$$= ip_\rho \left\langle \psi(q')\middle|\psi(q)\right\rangle (2\pi)^4\delta^{(4)}(q-q')\delta^{(4)}(p-p')$$
$$+ e\left\langle A_\rho(p')\psi(q'-p)\middle|\psi(q)\right\rangle \delta^{(4)}(q+p-q'-p')$$
$$- e\left\langle A_\rho(p')\psi(q')\middle|\psi(q+p)\right\rangle \delta^{(4)}(q+p-q'-p')$$

The QED Ward Identity - Special Case

- Final step: The right hand side vanishes when applying the various inverse propagators for the amputation procedure and taking the on-shell limit for the left hand side:

$$p_\mu \left(\tilde{S}^{-1}_{(\gamma)}\right)^{\mu\nu}(p) \left\langle A_\rho(p')\psi(q') \middle| A_\nu(p)\psi(q) \right\rangle \delta^{(4)}(q+p-q'-p') \;\rightarrow\; p_\mu \mathcal{M}^\mu{}_\rho(p)\delta(\dots)$$

$$\begin{aligned} &= ip_\rho \left\langle \psi(q') \middle| \psi(q) \right\rangle (2\pi)^4 \delta^{(4)}(q-q')\delta^{(4)}(p-p') && \rightarrow \; 0 \;\; (p \neq p',\, q \neq q') \\ &\quad + e \left\langle A_\rho(p')\psi(q'-p) \middle| \psi(q) \right\rangle \delta^{(4)}(q+p-q'-p') && \rightarrow \; 0 \;\; (\text{no pole at } q'^2 = m_\psi^2) \\ &\quad - e \left\langle A_\rho(p')\psi(q') \middle| \psi(q+p) \right\rangle \delta^{(4)}(q+p-q'-p') && \rightarrow \; 0 \;\; (\text{no pole at } q^2 = m_\psi^2) \end{aligned}$$

- Note: For a physical scattering process $p \neq p'$, $q \neq q'$ and $p \neq 0$.

Conclusions

- Different approach to quantum theory.
- Explicit Lorentz Symmetry.
- Depending on problem, much easier or much more difficult calculation than in operator formalism.
  - Easy to derive Feynman rules directly from the Lagrangian.
  - Easy derivation of the Ward-identity from gauge symmetry.
  - Essential for quantizing gauge theories (especially non-Abelian).

References

- M. D. Schwartz: *Quantum Field Theory and the Standard Model.* Cambridge University Press, Cambridge (2014)
- S. Pokorski: *Gauge Field Theories, Second Edition.* Cambridge University Press, Cambridge (2000)
- M. E. Peskin, D. V. Schroeder: *An Introduction to Quantum Field Theory.* Perseus Books, New York (1995)