Bottom and charm quark masses from quarkonium at N^3LO

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In collaboration with P.G. Ortega

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Outline

- Motivation: Heavy quark masses
- Quarkonium, the Cornell and static QCD potential
- Master formula and the MSR mass
- Massive lighter quarks in quarkonium and MSR mass
- Determination of charm and bottom mass (and α_s)
- Calibration of the Cornell model
- Conclusions

Motivation

Heavy quark masses

Fundamental parameters of the Standard Model, need to be known with high precision

In this talk we focus only on **Bottom and Charm**

Play a fundamental role in flavor physics:

- Unitarity triangle
- Rare kaon decays
- Test the Standard Model at the precision frontier

Also play a role in Higgs physics (branching ratios)

Heavy quark masses

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Play a fundamental role in flavor physics:

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Also play a role in Higgs physics (branching ratios)

But… quarks are confined particles, therefore their mass is not observable!

The mass of a heavy quark needs to be defined within perturbation theory…

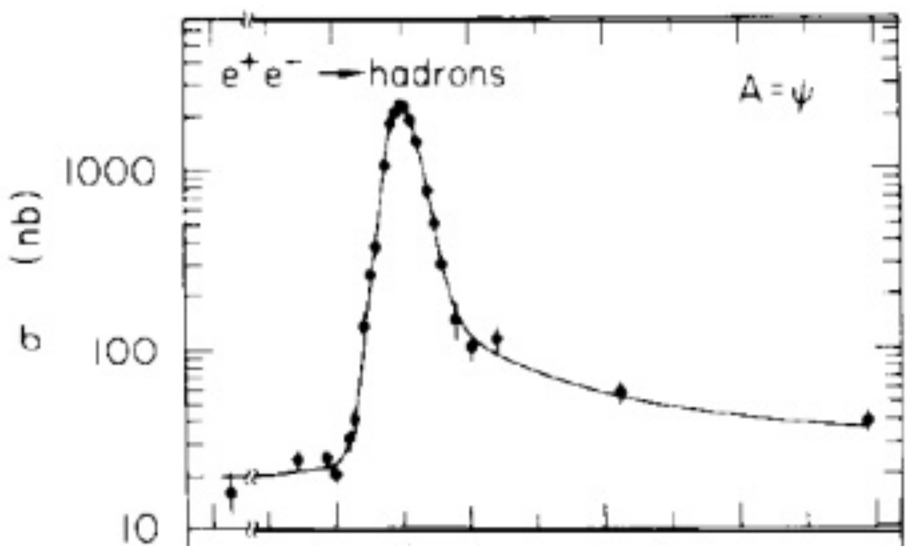
… as any other parameter in the QCD Lagrangian (renormalization, μ-dependence)

Only indirect measurements of quark masses possible.

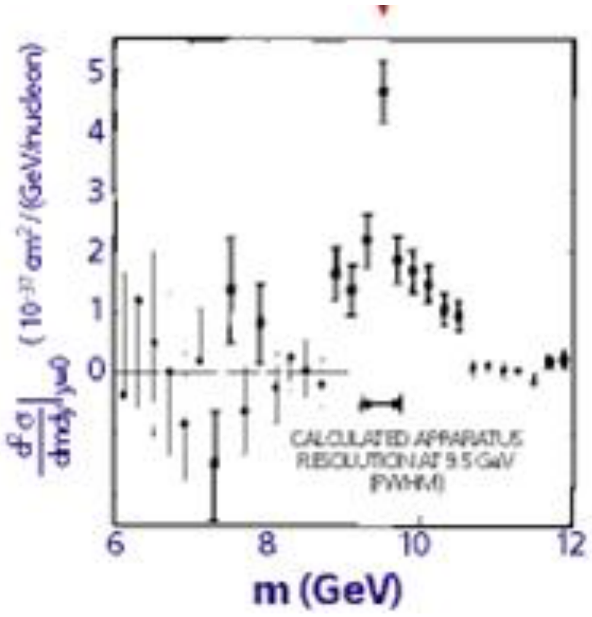
Quarkonium

Charm and Bottom quarks discovered as $Q\bar{Q}$ bound states

SLAC and BNL (1974), J/Ψ bound state



Fermilab (1977), Υ bound state



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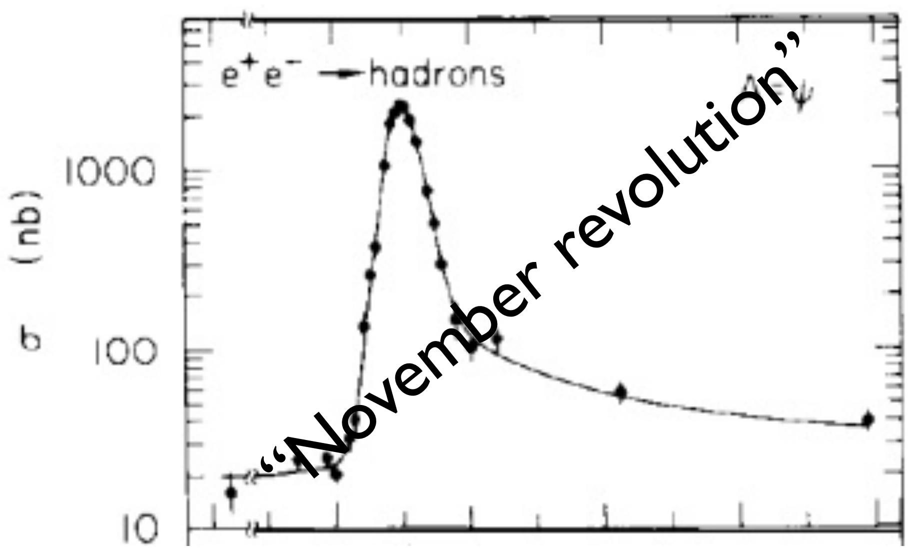
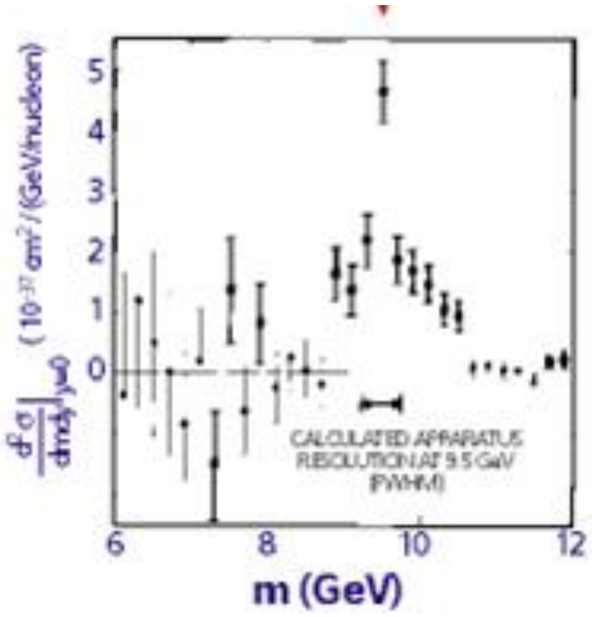


Fermilab (1977), ϒ bound state



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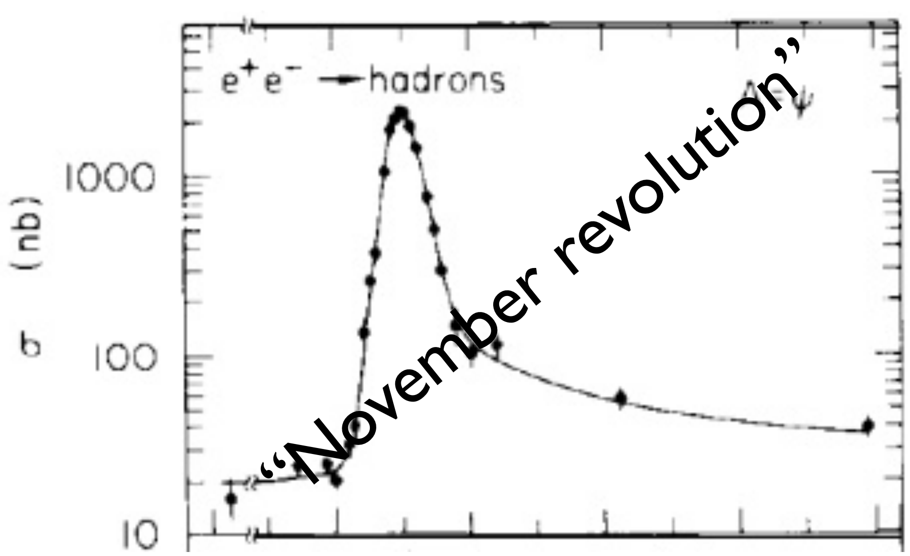

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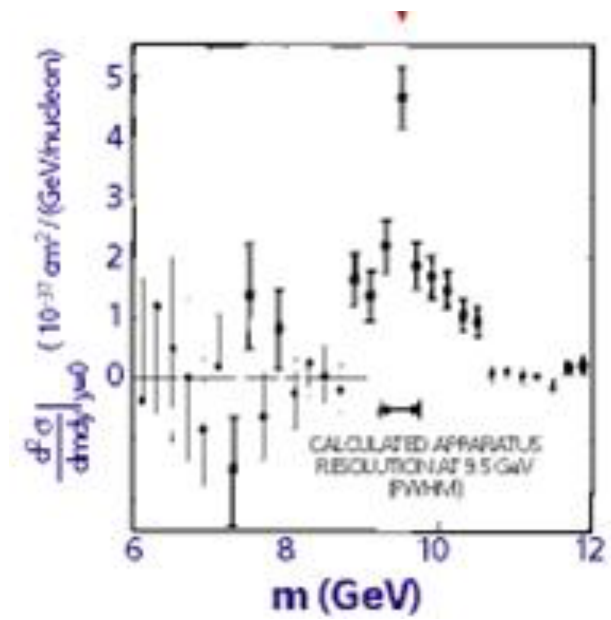


Theoretical description in early days in terms of a very simple non-relativistic description, the Cornell Model, with only three parameters: *[Eichten et. al. PRL 34:369–372 (1975)]*

- Quark mass: m_Q
- Coulomb-type interaction: α_s
- Linear raising potential: “string tension” σ.

confinement

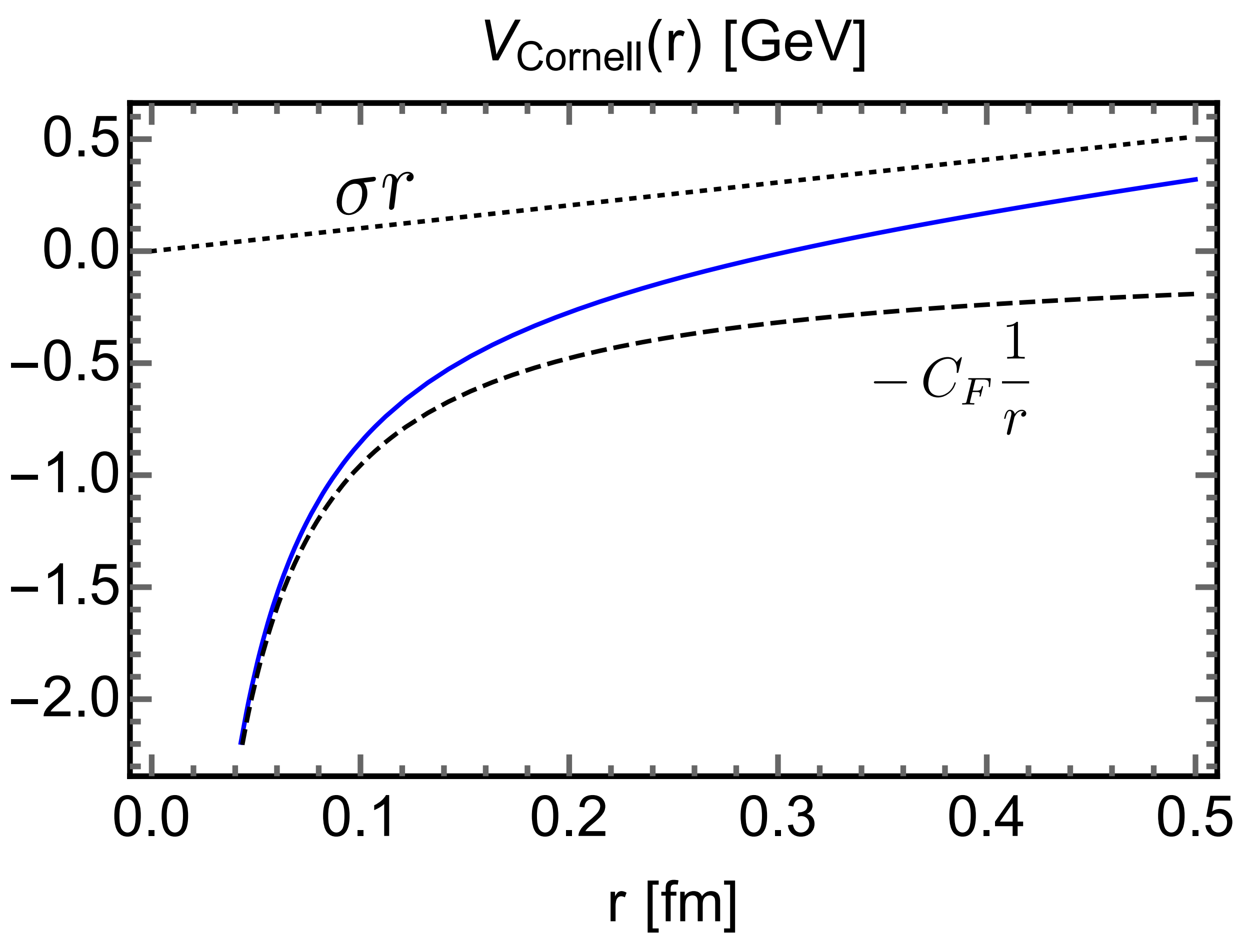
Interplay between perturbative and non-perturbative is crucial

Cornell model and the static QCD potential

Cornell Model

$$V_{\text{Cornell}}(r) = -C_F \frac{\alpha_s}{r} + \sigma r$$

"static potential"



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these terms yield dependence
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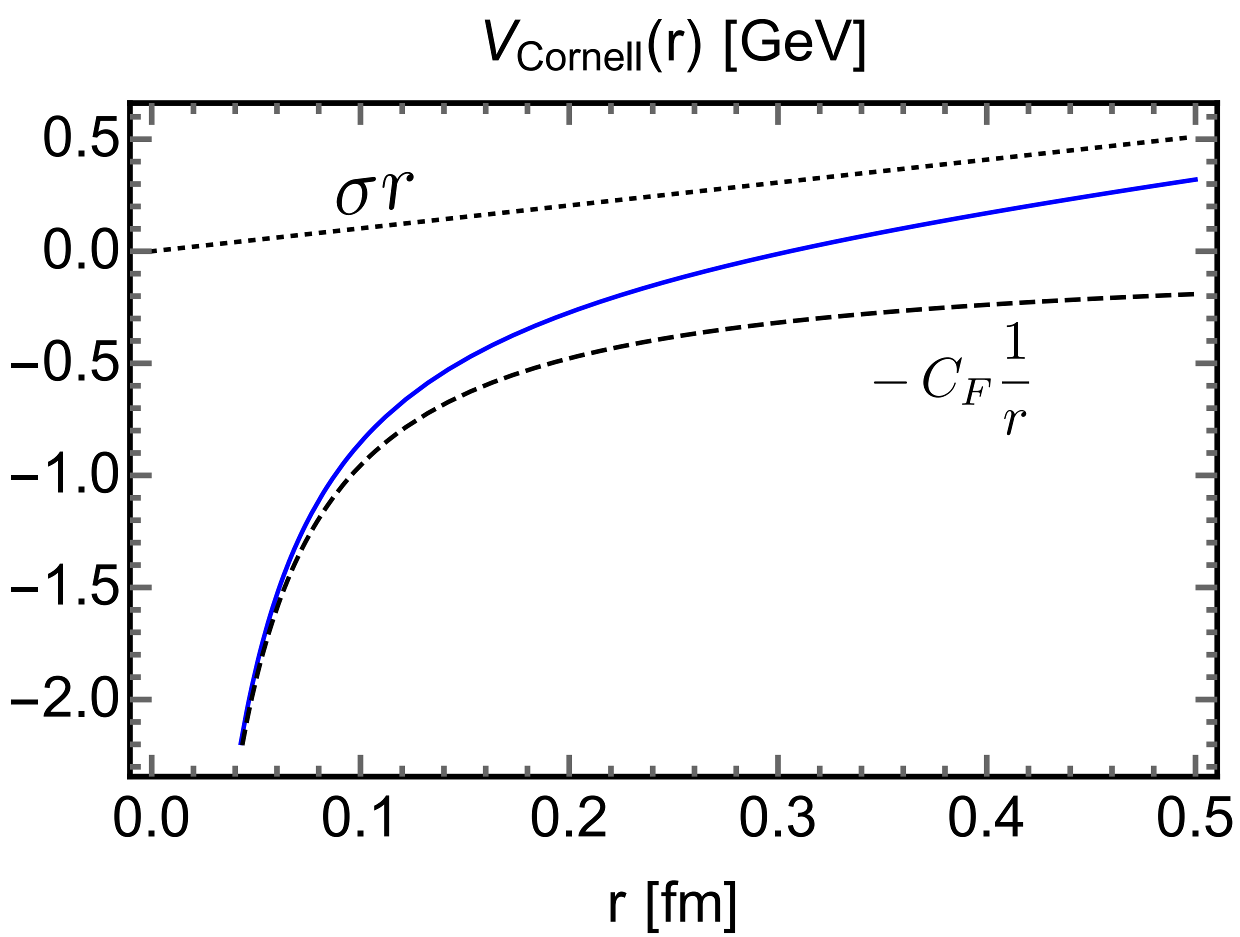
Solved numerically (Numerov)



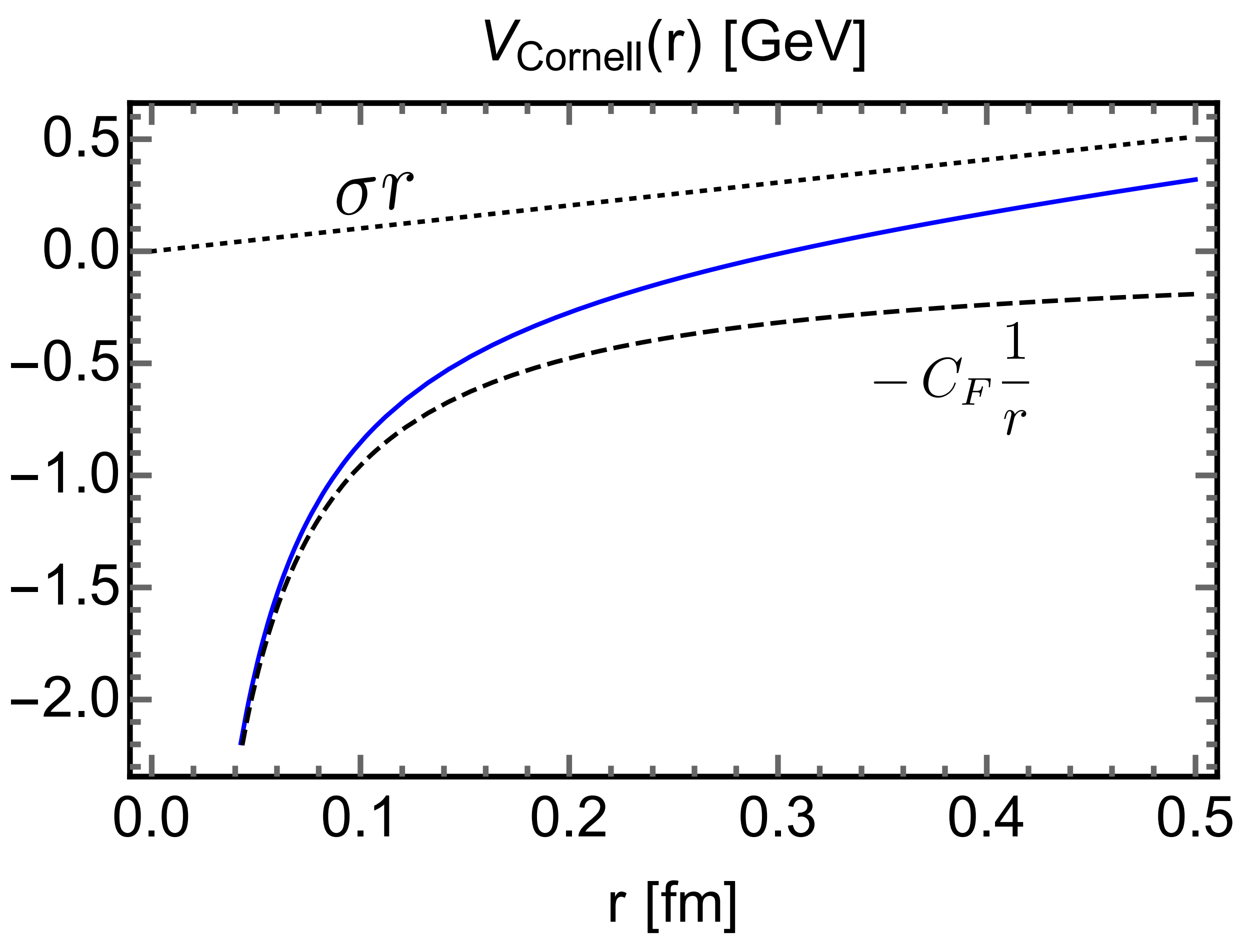
Cornell Model

$$V_{\mathrm{Cornell}}(r) = -C_F \frac{\alpha_s}{r} + \sigma\, r \quad + V_{LS} + V_{SS} + V_T \text{ + nothing}$$

"static potential" — spin-dependent $1/m^2$ corrections

these terms yield dependence on (n,l) quantum numbers — these terms give dependence on (s,j) quantum numbers

Solved numerically (Numerov) — Use perturbation theory



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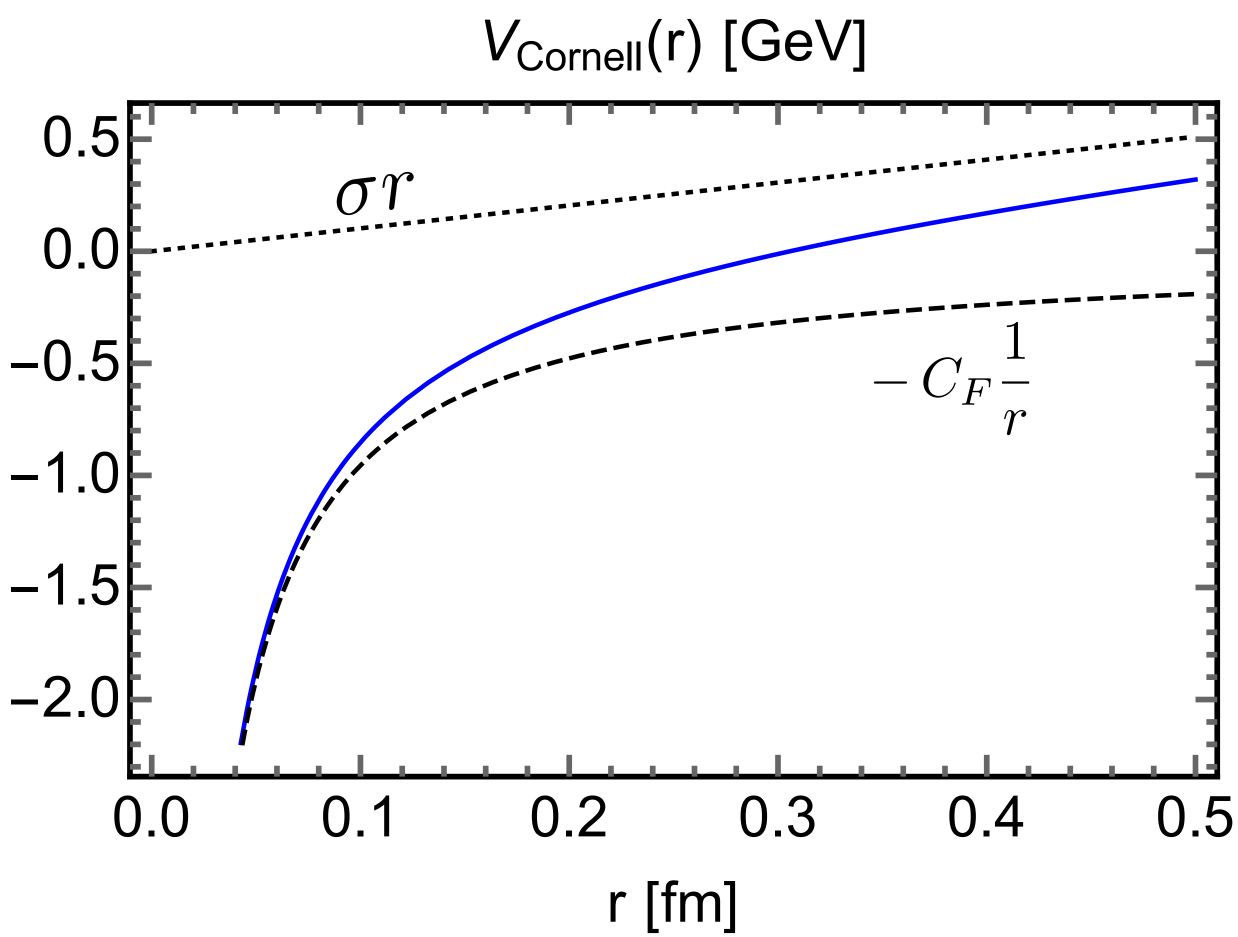

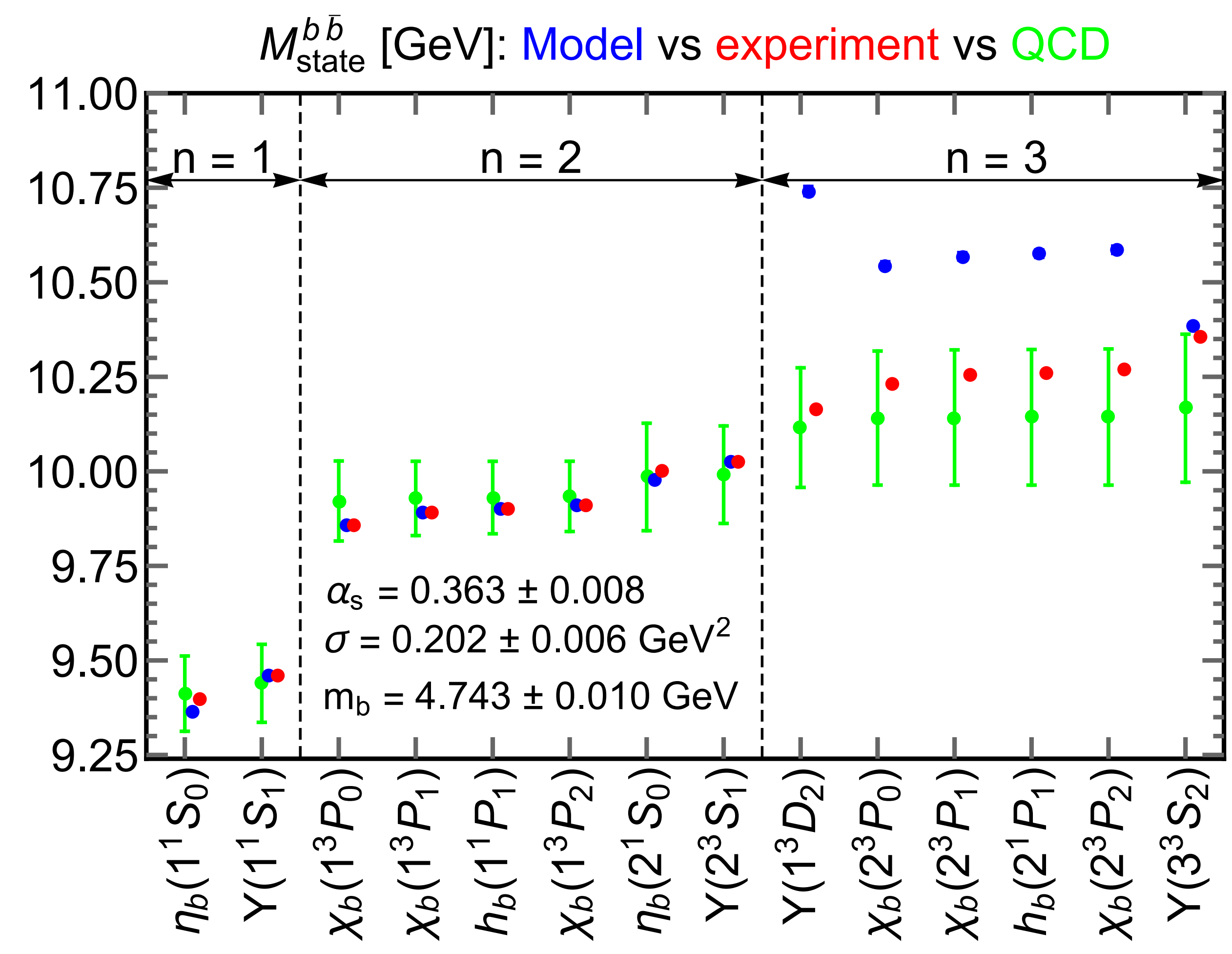

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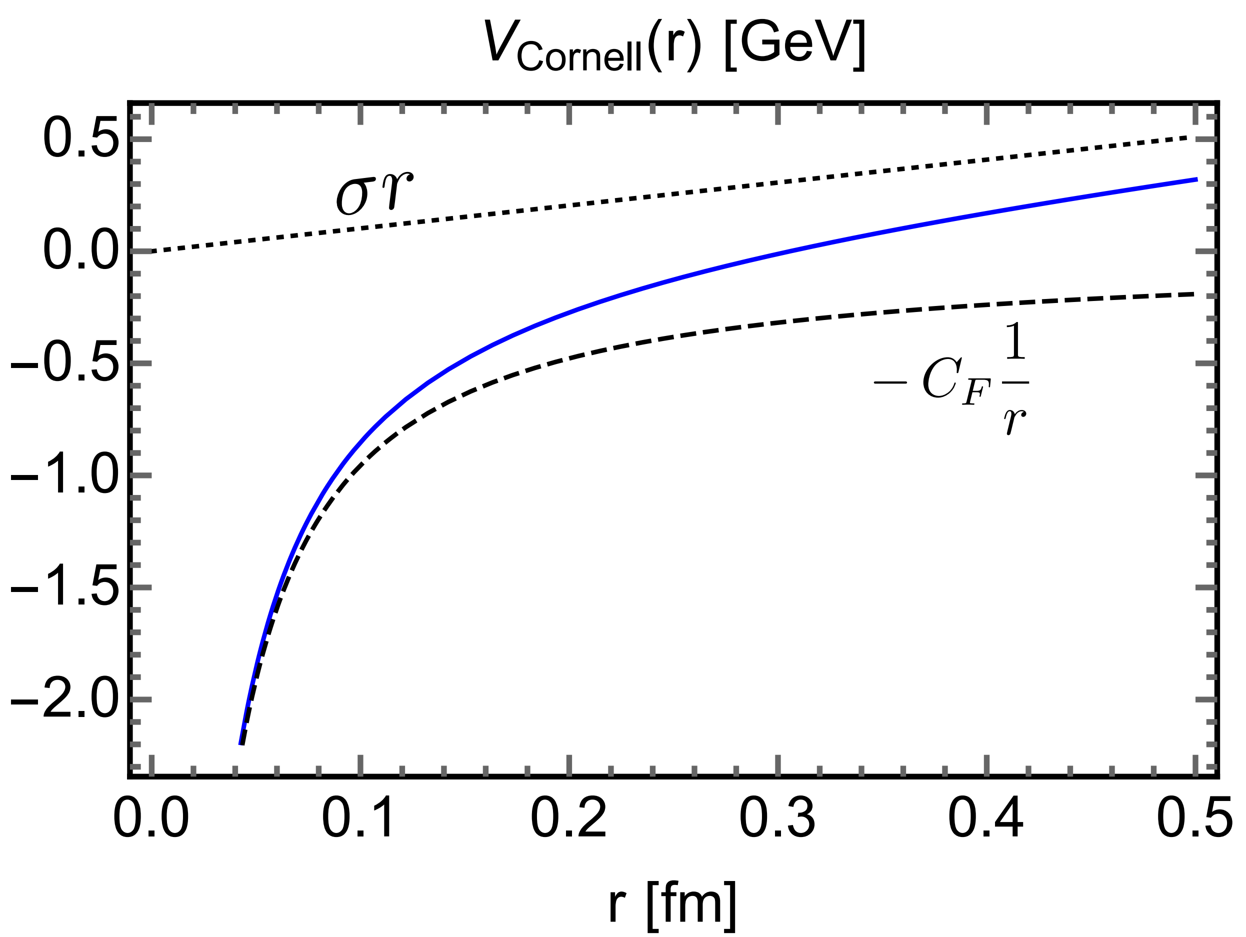
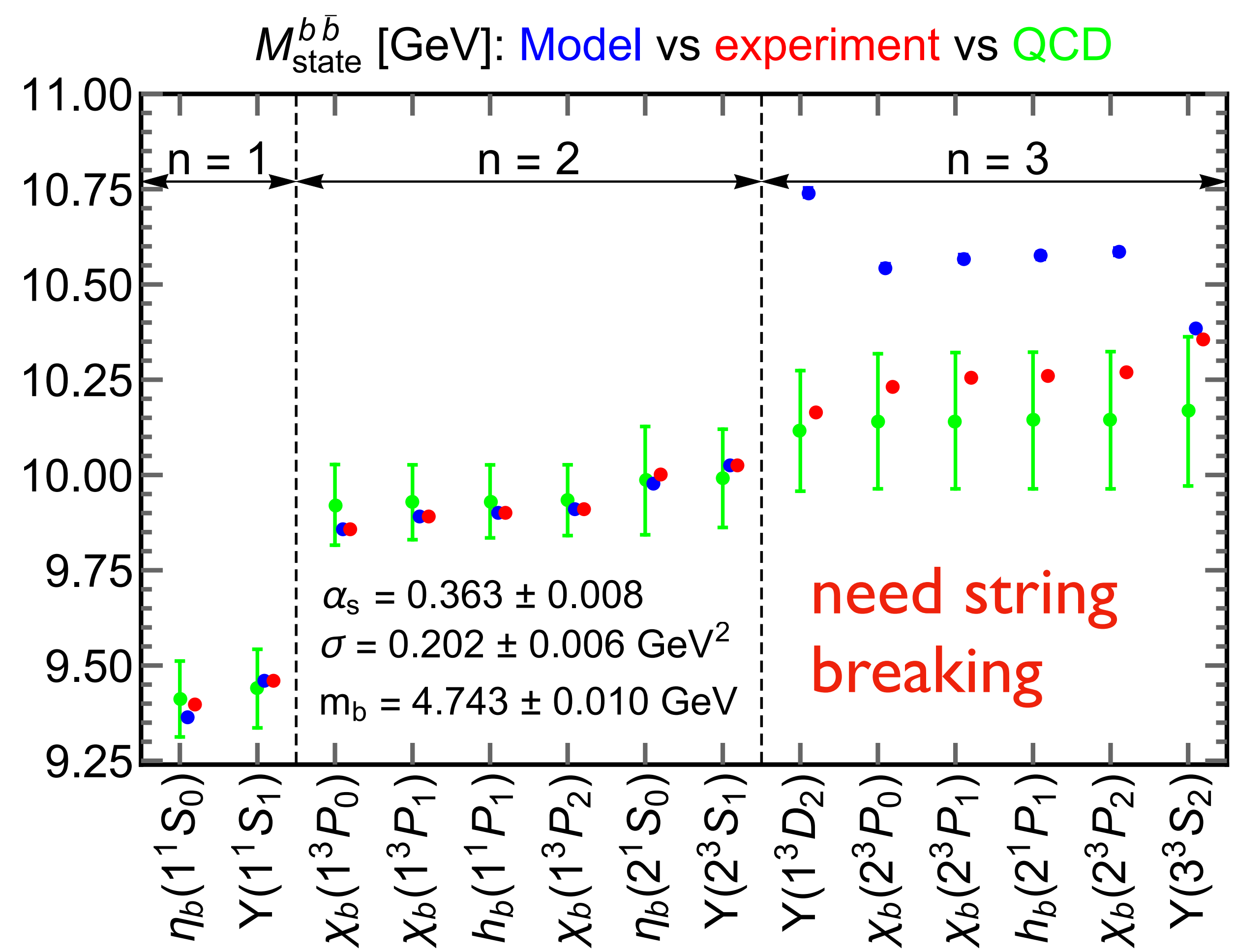




EFT treatment

Modern method to deal with problems widely separated scales

For quarkonium such theory is called NRQCD:
[Bodwin, Braaten, Lepage, PRD 51 (1995) 1125-1171]

RGE-improved versions of NRQCD are called

pNRQCD *[Pineda, Soto; Brambilla, Pineda, Soto, Vairo NPB 566 (2000) 275]*
vNRQCD *[Luke, Manohar, Rothstein, PRD 61 (2000) 074025]*

Can be also used for other processes, such as $t\text{-}\bar{t}$ production at threshold

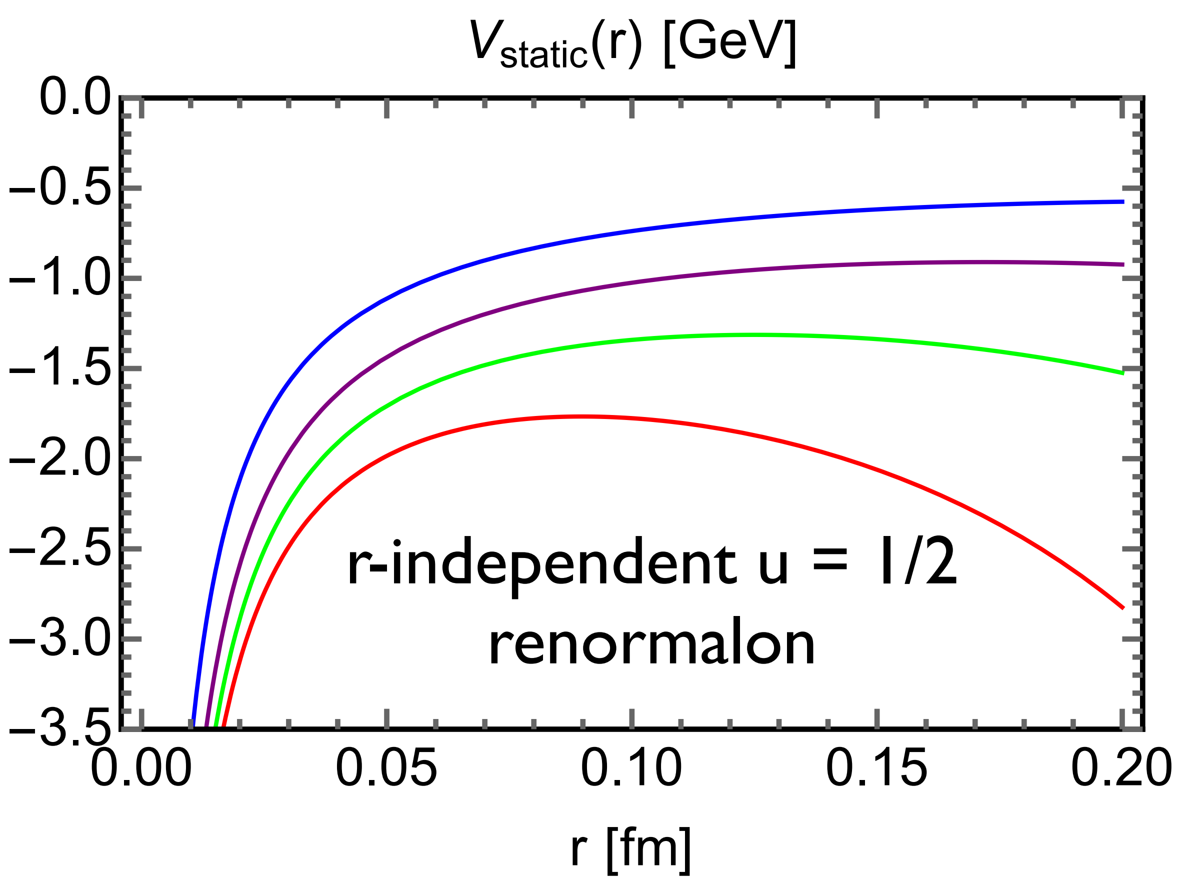
Static QCD potential

$$V_{\mathrm{QCD}}(r) = V_{\mathrm{static}}(r) + \frac{1}{m^n}\,\mathrm{corrections}$$

many more terms known,
also quantum corrections

$$V_{\mathrm{static}}(r) = -\,C_F\,\frac{\alpha_s(\mu)}{r}\left[1+\sum_{i=1}\left[\frac{\alpha_s(\mu)}{4\pi}\right]^i a_{ij}\,\log^j(r\,\mu\,e^{\gamma_E})\right]$$

known to $\mathcal{O}(\alpha_s^4)$



[Fischler, NPB 129 (1977) 157-174], [Anzai et al, PRL 104 (2010) 112004]
[Schröder, PLB 447 (1999) 321-326], [Brambilla et al, PRD60 (1999) 091502]
[Peter, PRL 78 (1997) 602-605] , [Lee et al, PRD 94 (2016) 054029]
[Smirnov et al, PLB 668 (2008) 293-298, PRL 104 (2010) 112003]

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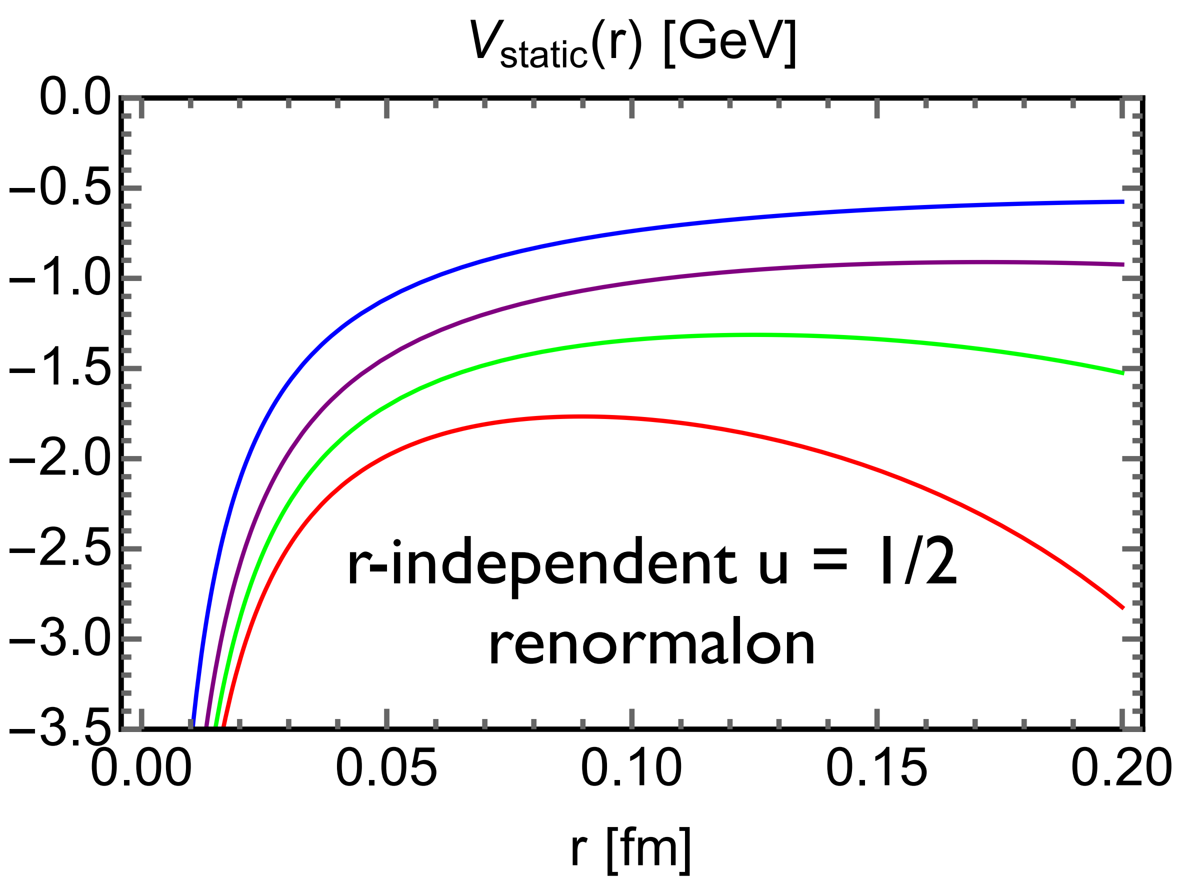

But a potential is not an observable!
Energy is an observable

$$E = 2\,m_Q^{\mathrm{pole}} + V_{\mathrm{static}}(r)$$

has a u = 1/2 mass-independent renormalon

[Pineda, PhD thesis]
[Hoang et al, PRD 59 (1999) 114014]
[Beneke et al, PLB 434 (1998) 115-125]

Static QCD potential

$$V_{\rm QCD}(r) = V_{\rm static}(r) + \frac{1}{m^n}\,\text{corrections}$$

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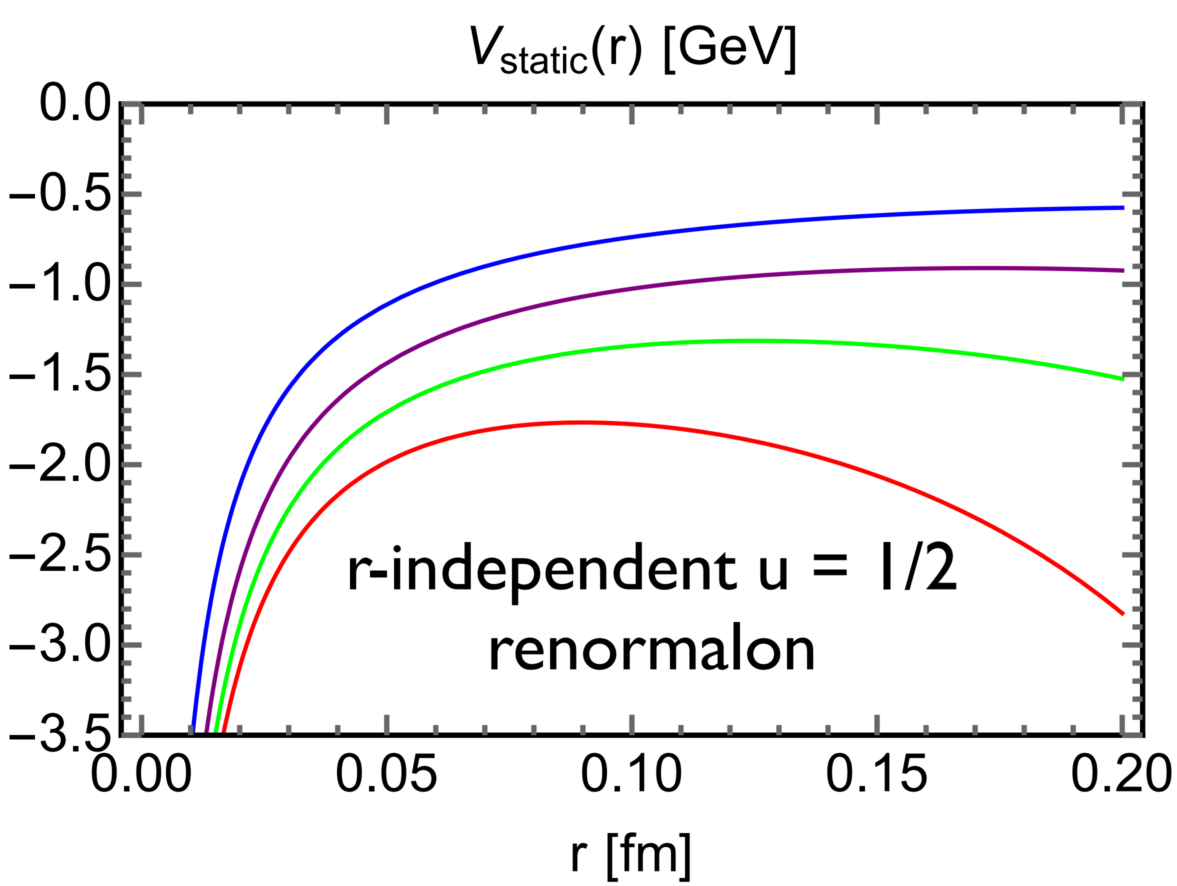

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Same renormalon! cancels in the difference

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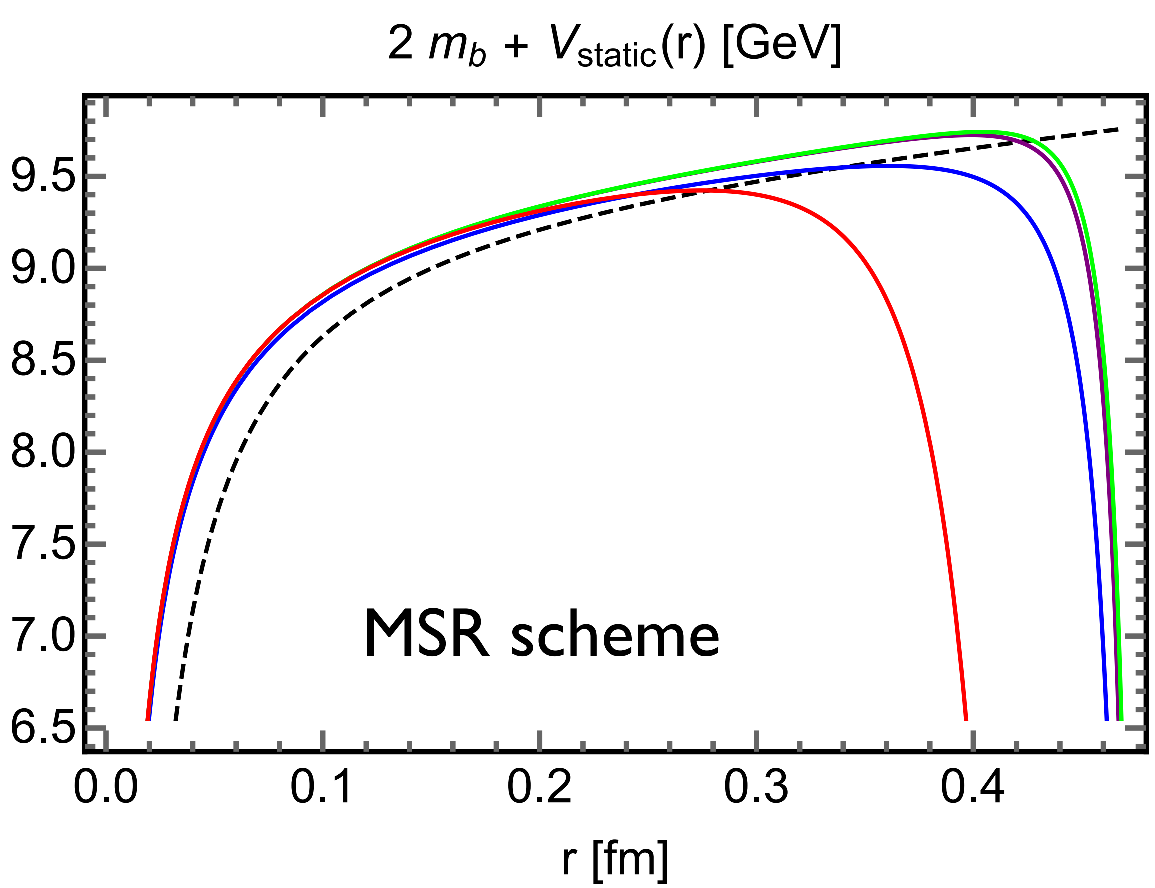

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Important to use MSR mass because neither $\log(r\mu)$ nor $\log(R/\mu)$ should be large

It also makes qualitative agreement with Cornell model better

Master formula for quarkonia masses

Master formula

[Kiyo, Sumino NP B889 (2014) 156-191]

$$E_X(\mu, n_\ell) = 2\, m_Q^{\text{pole}} \left[1 - \frac{C_F^2\, \alpha_s^{(n_\ell)}(\mu)^2}{8n^2} \sum_{i=0}^{\infty} \left(\frac{\alpha_s^{(n_\ell)}(\mu)}{4\pi} \right)^i \varepsilon^{i+1} P_i(L_{n_\ell}) \right]$$ + non-perturbative

[Penin, Steinhauser, PLB 538 (2002) 335-345]

[Beneke, Kiyo Schuller, PLB 714 (2005) 67-90]

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We assume $\mu_{\rm s} = m\,v \gg \Lambda_{\rm QCD}$ certainly true for n < 4

And either $\mu_{\rm us} = m\,v^2 \gg \Lambda_{\rm QCD}$ (true for n = 1) — local condensates

or $\mu_{us} \sim \Lambda_{\rm QCD}$ (possibly true for n = 2) — non-local condensates

For n = 3 one seems to have $\mu_{\rm us} < \Lambda_{\rm QCD}$ — perturbative and non-perturbative of the same order

We will estimate those by comparing fits with different datasets and by studies of perturbative stability

For a recent study of non-perturbative effects, see [T. Rauh 1803.05477 (2018)]

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$$L_{n_\ell} = \log\left(\frac{n\mu}{C_F \alpha_s^{(n_\ell)}(\mu) m_Q^{\text{pole}}} \right) + H_{n+\ell}$$

$$P_i(L) = \sum_{j=0}^{i} c_{i,j}\, L^j$$

First quantum correction is $\mathcal{O}(\alpha_s^2)$, but static potential starts at $\mathcal{O}(\alpha_s)$

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bookkeeping parameter that labels the various orders in the Υ-expansion

[Hoang, Ligeti, Manohar, PRL 82 (1999) 277-280; PRD 59 (1999) 074017]

Crucial when cancelling the static potential renormalon in the quarkonium mass

Important when figuring out alternative perturbative expansions

Sets up a counting for the MSR-mass parameter R

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Argument in logs non-trivial: explicit μ dependence as well as through $\alpha_s^{(n_\ell)}(\mu)$

Dependence gets even more complex when switching to the MSR mass

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Harmonic number $H_n \equiv \sum_{i=1}^{n}\frac{1}{i}$ grouped with the log for convenience

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$c_{i,j}$ depend on the bound-state quantum numbers: (n, l, j, s)

$c_{3,0}$ depends on $\log(\alpha_s)$: first hint of ultra-soft effects. Could be resumed within EFTs.

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This formula has an m_Q-independent renormalon equal to that of $-2\, m_Q^{\rm pole}$ inherited from the QCD static potential

Master formula in short-distance scheme and finite charm quark mass effects

Master formula in short-distance scheme

$$m_Q^{\text{pole}} = m_Q^{\text{SD}}\left[1 + \sum_{n=1} \varepsilon^n \, \delta_n^{\text{SD}} \left(\frac{\alpha_s^{(n_\ell)}(\mu)}{4\pi}\right)^n\right]$$

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Υ - counting scheme parameter

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Υ- counting scheme parameter

Depends on powers of $\log\left(\frac{\mu}{R}\right)$ and proportional to $\frac{R}{m_Q^{\text{SD}}}$

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$$E_X(\mu, n_\ell) = 2\, m_Q^{\text{SD}} \left\{ 1 + \sum_{i=1} \varepsilon^i \left(\frac{\alpha_s^{(n_\ell)}(\mu)}{4\pi} \right)^i \left[\delta_i^{\text{SD}} - F\, P_{i-1}^{\text{SD}} - (1 - \delta_{i,1})\, F \sum_{j=1}^{i-1} \delta_j^{\text{SD}} P_{i-j-1}^{\text{SD}} \right] \right\}$$

$$F = \frac{\pi\, C_F^2 \alpha_s^{(n_\ell)}(\mu)}{2n^2}$$

Master formula in short-distance scheme

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and $c_{i,j}$, $\delta_i^{\rm SD}$

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MSR mass

1-S mass

Effects from massive lighter quarks

$$E_X(\mu, n_\ell, m_Q^{\rm pole}, m_q^{\rm pole}) = E_X(\mu, n_\ell, m_Q^{\rm pole}) + \varepsilon^2 \delta E_X^{(1)} + \varepsilon^3 \delta E_X^{(2)} + \cdots$$

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m_Q heavy quark

m_q massive lighter quark

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higher order terms currently unknown, but …

Computed in *[Eiras, Soto PLB491 (2000), 101-110]* for any state

Computed in *[Hoang hep-ph/0008102]* for the ground state

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n_ℓ scheme: massless limit manifest, decoupling limit not well defined

$n_\ell - 1$ scheme: decoupling limit manifest, massless limit not well defined

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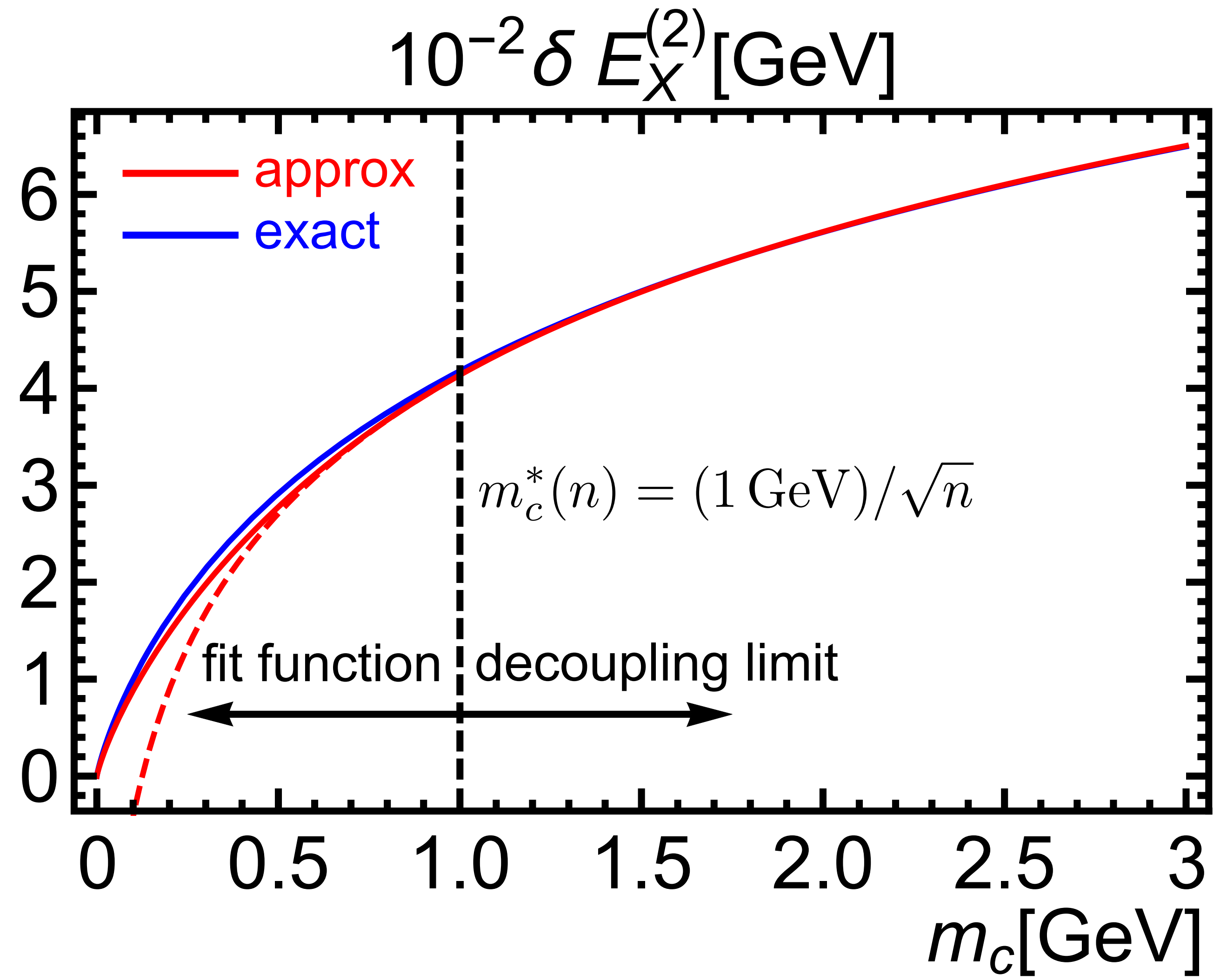
$n_\ell - 1$ scheme: decoupling limit manifest, massless limit not well defined

Observation made in *[Brambilla, Sumino, Vairo PRD65 (2002) 043001]*: true answer very very close to decoupling limit:

Use decoupling limit plus trick to parametrize

Use $n_\ell - 1$ scheme plus corrections to incorporate $\mathcal{O}(\varepsilon^3)$ charm quark mass effects

Effects from massive lighter quarks



below m_c^* use fit function of the form $f(m_c) = m_c\,[\,a + b\,\log(m_c)\,]$

above use decoupling limit

demand smooth junction

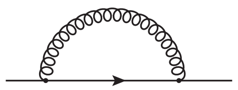
It appears clear we will be in need of massive lighter quarks effect on the short-distance mass as well

Schemes for quarks masses

The pole mass

$\Sigma(p, m_0)$

μ- independent
has divergences

$+ \quad \cdots \quad = \quad \dfrac{1}{\not{p} - m_0 - }$ 

m_0 = bare mass

quark mass defined in context of perturbation theory

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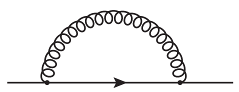

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quark mass defined in context of perturbation theory

Pole scheme: propagator has a pole for $\not{p} \to m_p$

$m_p = m_0 + \Sigma(m_p, m_0)$ pole mass is μ - independent

The whole diagram at $p^2 = m^2$ is absorbed into the mass definition

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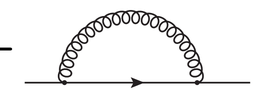
$+ \quad \cdots \quad = \quad \dfrac{1}{\not{p} - m_0 - }$ 

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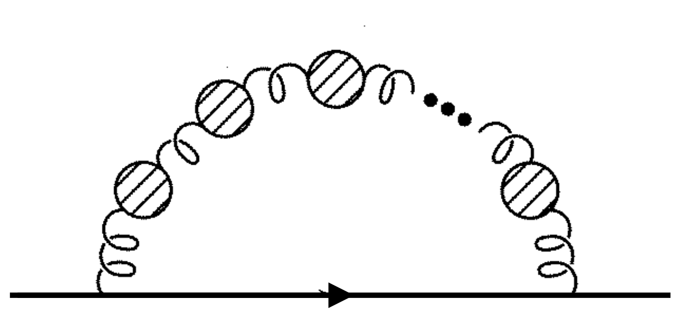
Absorbs into mass parameter UV fluctuations from scales > 0



$$\Sigma(m, m) \sim m \sum_n \alpha_s^{n+1} (2\beta_0)^n \, n!$$

Linear sensitivity to infrared momenta: factorially growing coefficients in perturbation theory

Sensitivity to non-perturbative regime

$\mathcal{O}(\Lambda_{\rm QCD})$ renormalon

The $\overline{\mathrm{MS}}$ mass

$\overline{\text{MS}}$ scheme: propagator is finite, subtract only $\frac{1}{\epsilon}$ in dimensional regularization

$\overline{m}(\mu) \,=\, m_0 \,+ \Sigma(m_p, m_0)|_{\frac{1}{\epsilon}}$ $\overline{\text{MS}}$ mass is μ-dependent

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$m_p - \overline{m}(\mu) = \Sigma(m_p, m_0)|_{\text{finite}} \equiv \delta m_{\overline{\text{MS}}}(\mu)$ no renormalon problem

μ - dependent

Absorbs into mass parameter UV fluctuations from scales > $\overline{m}(\overline{m})$

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$$\delta m_{\overline{\text{MS}}}(\mu) = \overline{m}(\mu) \sum_{n=1} \left[\frac{\alpha_s^{(n_\ell+n_h)}(\mu)}{4\pi} \right]^n \sum_{m=0}^{n} a_{n,m}(n_\ell, n_h) \log^m \left(\frac{\overline{m}(\mu)}{\mu} \right)$$

This equations encodes the μ-anomalous dimension of the $\overline{\text{MS}}$ mass

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Let us define $\overline{m} \equiv \overline{m}(\overline{m})$:

$$m_p - \overline{m} = \overline{m} \sum_{n=1} \left[\frac{\alpha_s^{(n_\ell+n_h)}(\mu)}{4\pi} \right]^n a_{n,0}(n_\ell, n_h)$$

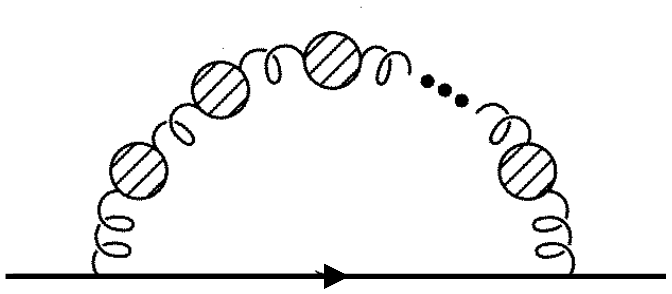
This series contains the renormalon

The $\overline{\text{MS}}$ mass is very far from a kinetic or threshold mass, resembles a coupling constant

Cannot be used in processes for which the quark mass is no longer a dynamical scale

Pole mass ambiguity

[Beneke Phys. Lett. B344 (1995) 341-347]



Based on the observation that the B-meson mass $M_B = m_b^{\text{pole}} + \overline{\Lambda}$ is renormalon free

ambiguity cancels in the sum of these two terms

Pole mass ambiguity

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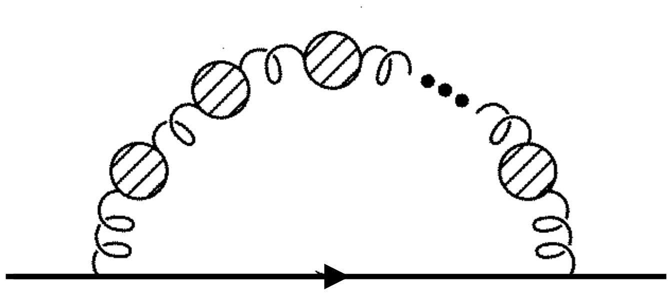

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m_b-independent

therefore the ambiguity in the pole mass is mass-independent

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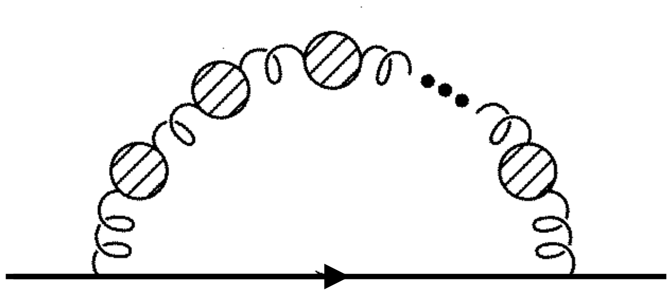

Based on the observation that the B-meson mass $M_B = m_b^{\text{pole}} + \overline{\Lambda}$ is renormalon free

Wilson coefficient is 1 to all orders
No anomalous dimension

therefore the ambiguity in the pole mass is scheme and scale independent

Status of perturbative coefficients

Considering all lighter quarks massless, coefficients known up to $\mathcal{O}(\alpha_s^4)$

- 1-loop: *[Tarrach (1981)]*
- 2-loop: *[Gray, Broadhurst, Grafe, Schilcher (1990)]*
- 3-loop: *[Chetyrkin, Steinhauser (1999), Chetyrkin, Steinhauser (2000), Melnikov, Ritbergen (2000), Marquard, Mihaila, Piclum, Steinhauser (2007)]*
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Asymptotic form known for all orders from renormalon behavior

$$a_n^{\mathrm{asy}} = 4\pi\, N_{1/2}(2\beta_0)^{n-1} \sum_{\ell=0}^{\infty} g_\ell\, (1+\hat{b}_1)_{n-1-\ell}$$

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4-loop and higher can be estimated within a few percent

[Lepenik, Hoang, Preisser (2017)], [VM, P.G. Ortega (2017), this talk]

The MSR mass

[Hoang, Jain, Scimemi, Stewart (2008)]

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We exploit the fact that the ambiguity is mass-independent

Since the renormalon only sees light flavors, express the series in terms of $\alpha_s^{(n_\ell)}$

Either by setting $n_h = 0$ (*Natural MSR mass* or MSRn)

Or expressing $\alpha_s^{(n_\ell+1)}(\overline{m}_Q)$ in terms of $\alpha_s^{(n_\ell)}(\overline{m}_Q)$ (*Practical MSR mass* or MSRp)

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$$m_Q^{\rm pole} - m_Q^{\rm MSR}(R) = R \sum_{n=1}^{\infty} a_n^{\rm MSR}(n_\ell) \left(\frac{\alpha_s^{(n_\ell)}(R)}{4\pi} \right)^n$$

same ambiguity as $\overline{\rm MS}$ to pole relation

The MSRn mass can be easily matched to the $\overline{\rm MS}$ mass at $R = \overline{m}_Q$

By construction $m_Q^{\rm MSRp}(\overline{m}_Q) = \overline{m}_Q(\overline{m}_Q)$ to all orders

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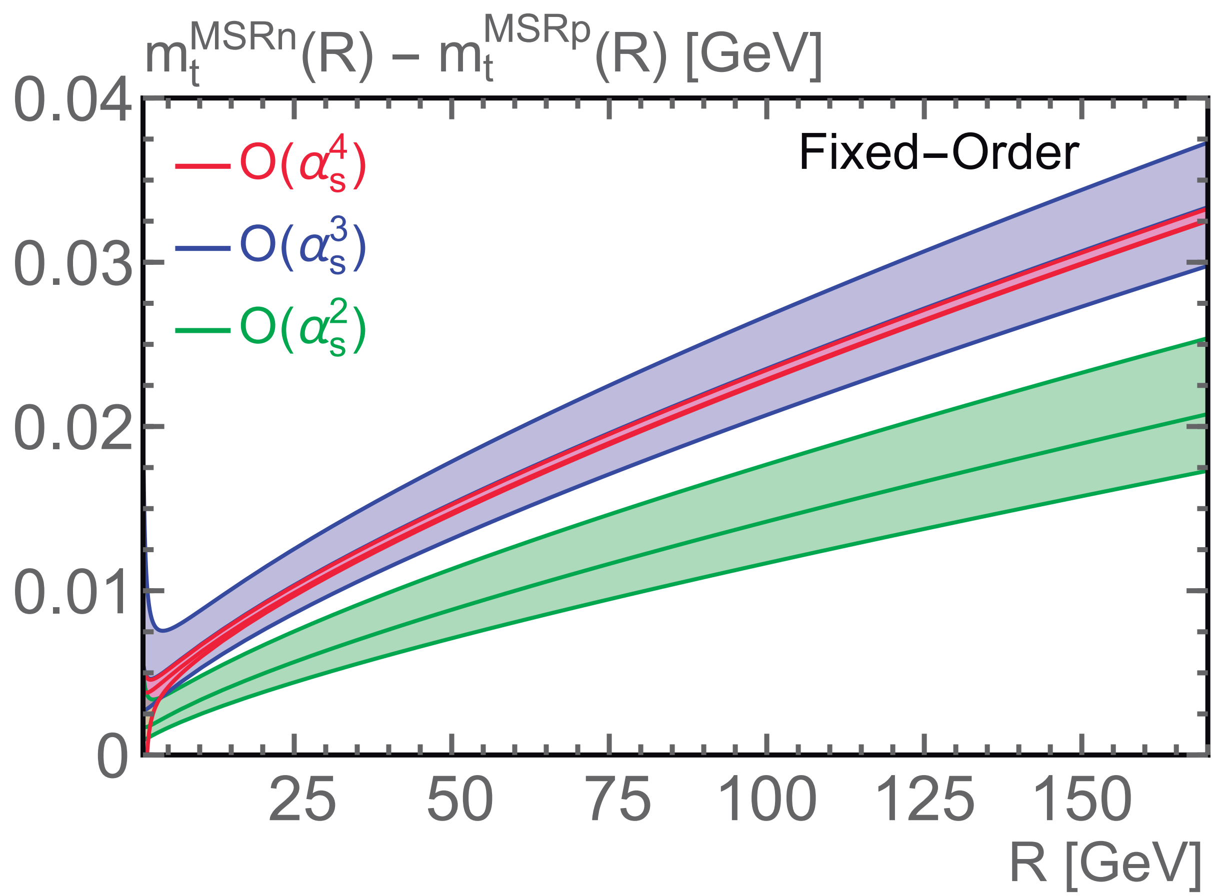

Both realizations coincide at $\mathcal{O}(\alpha_s)$

Difference of masses is renormalon-free as long as series expressed in terms of α_s at the same scale

Last statement true for any series

R-evolution

[Hoang, Jain, Scimemi, Stewart (2008)]

In a given physical situation one has a perturbative expansion in terms of $\alpha_s(\mu)$

Therefore we have to choose $\mu \sim R$

The value of μ is in general much smaller than $\overline{m}_Q$ for the cases we care

Therefore there are large logs of $\overline{m}_Q/R$ that need to be summed up:

$$R\frac{\mathrm{d}}{\mathrm{d}R}m_Q^{\mathrm{MSR}}(R) = -\,R\,\gamma^R[\alpha_s(R)] = -\,R\sum_{n=0}^{\infty}\gamma_n^R\left(\frac{\alpha_s(R)}{4\pi}\right)^{n+1}$$

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pole mass is R-independent ⟶ R-anomalous dimension from MSR definition

ambiguity R-independent ⟶ R-anomalous dimension renormalon-free

general formula $\gamma_n^R = a_{n+1}^{\mathrm{MSR}} - 2\sum_{j=0}^{n-1}(n-j)\,\beta_j\,a_{n-j}^{\mathrm{MSR}}$

R-evolution

[Hoang, Jain, Scimemi, Stewart (2008)]

In a given physical situation one has a perturbative expansion in terms of $\alpha_s(\mu)$

Therefore we have to choose $\mu \sim R$

The value of μ is in general much smaller than $\overline{m}_Q$ for the cases we care

Therefore there are large logs of $\overline{m}_Q/R$ that need to be summed up:

$$R\frac{\mathrm{d}}{\mathrm{d}R}m_Q^{\mathrm{MSR}}(R) = -\,R\,\gamma^R[\alpha_s(R)] = -\,R\sum_{n=0}^{\infty}\gamma_n^R\left(\frac{\alpha_s(R)}{4\pi}\right)^{n+1}$$

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Solution to RGE equation:

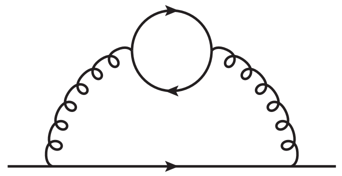
$$m_Q^{\mathrm{MSR}}(R_2) - m_Q^{\mathrm{MSR}}(R_1) = \int_{R_1}^{R_2}\mathrm{d}R\,\gamma_n^R[\alpha_s^{(n_\ell)}(R)]$$

Sums up large logs of R_2/R_1 associated to the renormalon to all orders in pert. theory

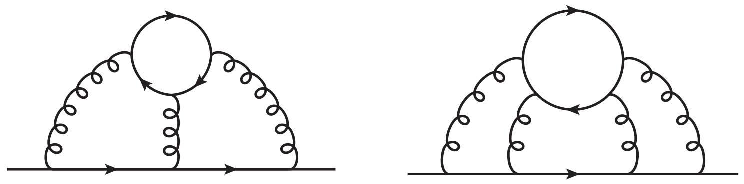
These logs are also summed up e.g. in *[Pineda (2001)]* for the Renormalon Subtracted mass

Massive lighter quarks

$$m_Q^{\text{pole}} - \overline{m}_Q = \delta\overline{m}_Q(\overline{m}_Q) + \overline{m}_Q\,\Delta_{\overline{m}_q}^{\overline{\text{MS}}}(\overline{m}_Q, \xi)$$



$\mathcal{O}(\alpha_s^2)$



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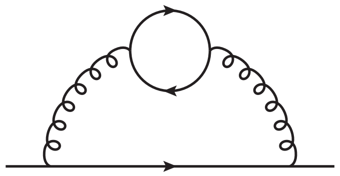
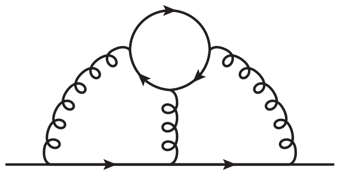
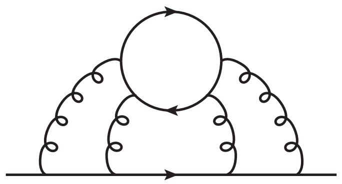


$\mathcal{O}(\alpha_s^2)$

$$\Delta^{\overline{\rm MS}}_{\overline{m}_c}(\overline{m}_Q,\xi) = \sum_{n=2}\left(\frac{\alpha_s^{(n_\ell+n_h)}(\overline{m}_Q)}{4\pi}\right)^n \Delta^{\overline{\rm MS}}_n(n_\ell,n_h,\xi)$$

$\xi = \overline{m}_q/\overline{m}_Q$





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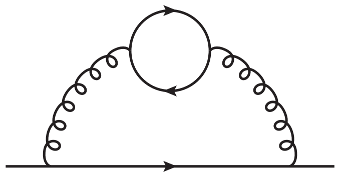
massless | mass corrections



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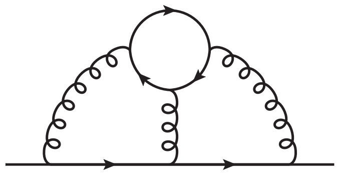

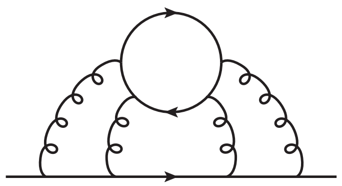

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Two obvious constraints

$$\Delta^{\overline{\text{MS}}}_n(n_\ell,n_h,0) = 0$$

$$\Delta^{\overline{\text{MS}}}_n(n_\ell,n_h,1) = a^{\overline{\text{MS}}}_n(n_\ell-1,n_h+1) - a^{\overline{\text{MS}}}_n(n_\ell,n_h)$$

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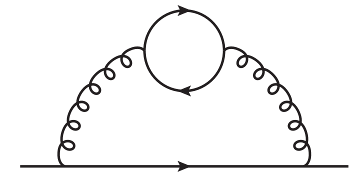

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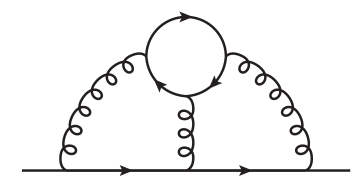

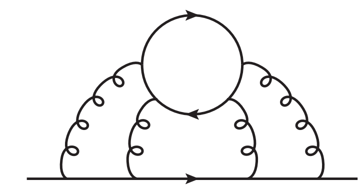

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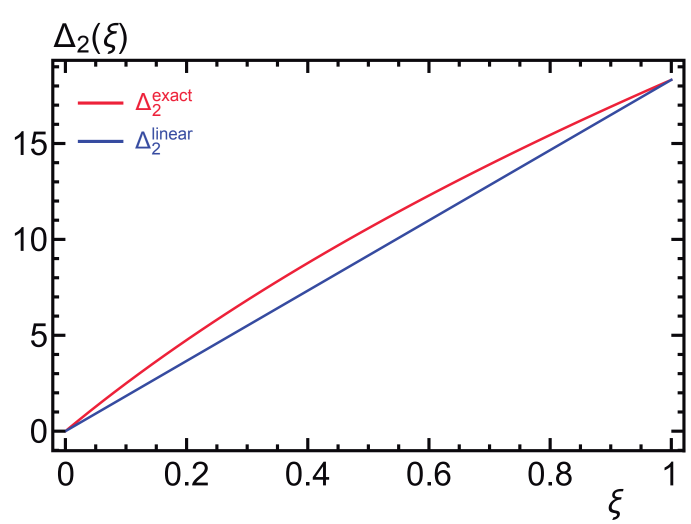

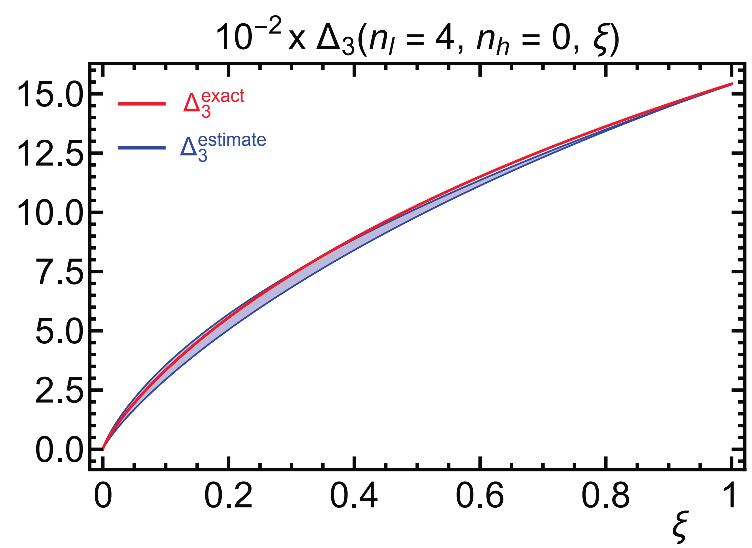

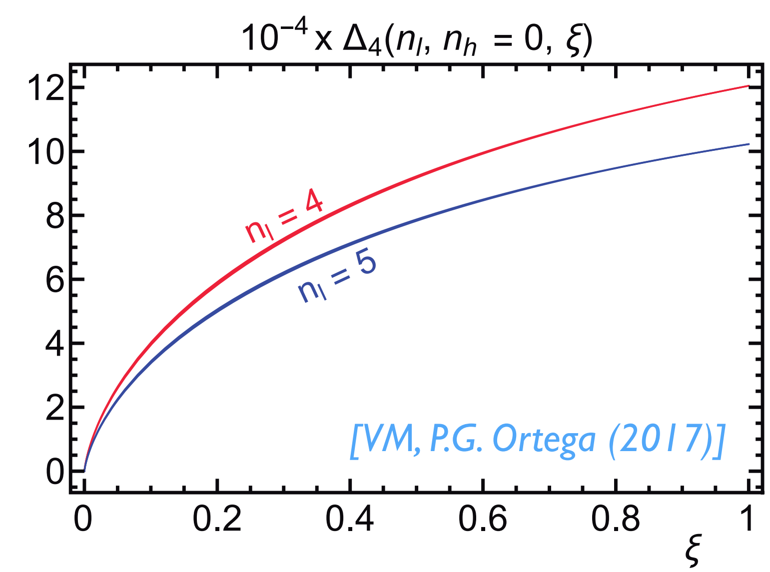

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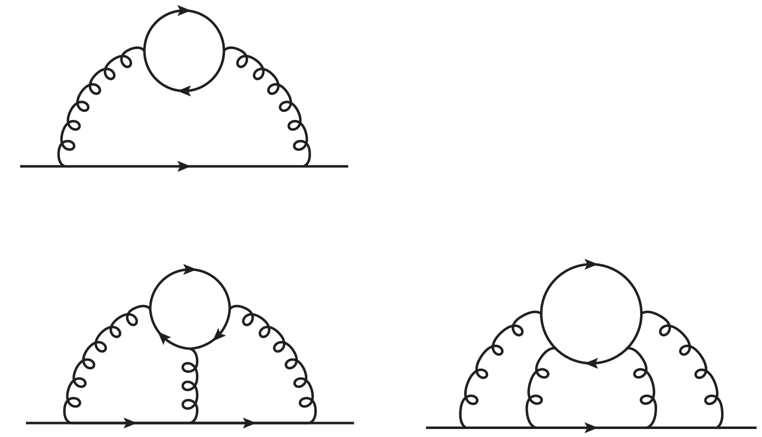

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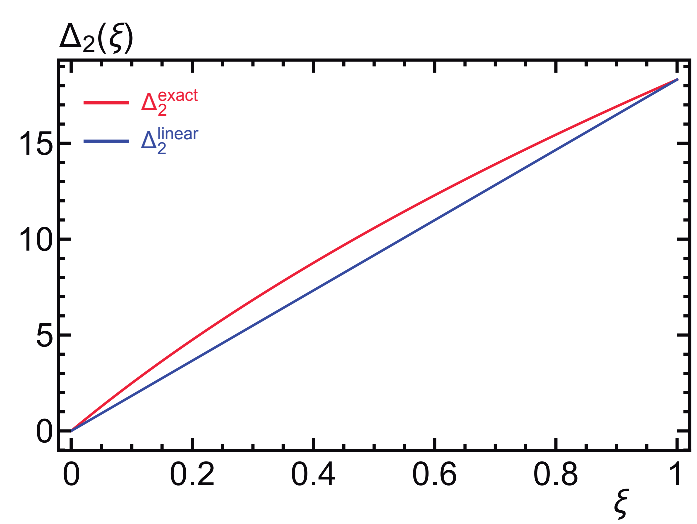

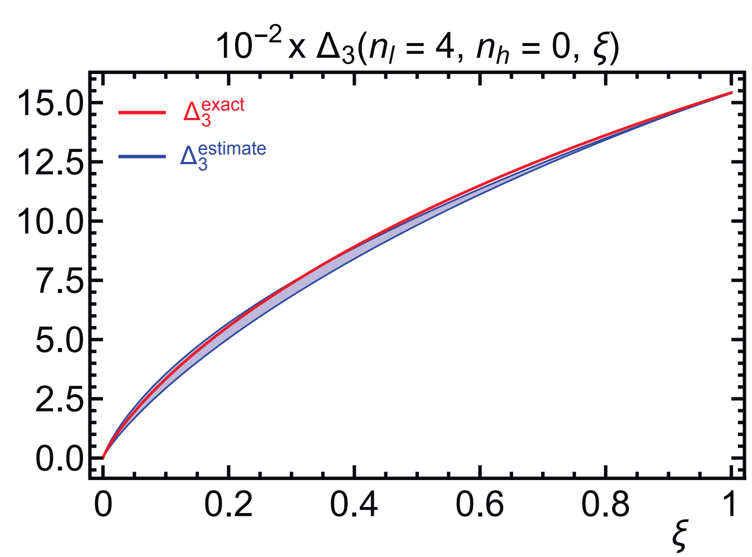

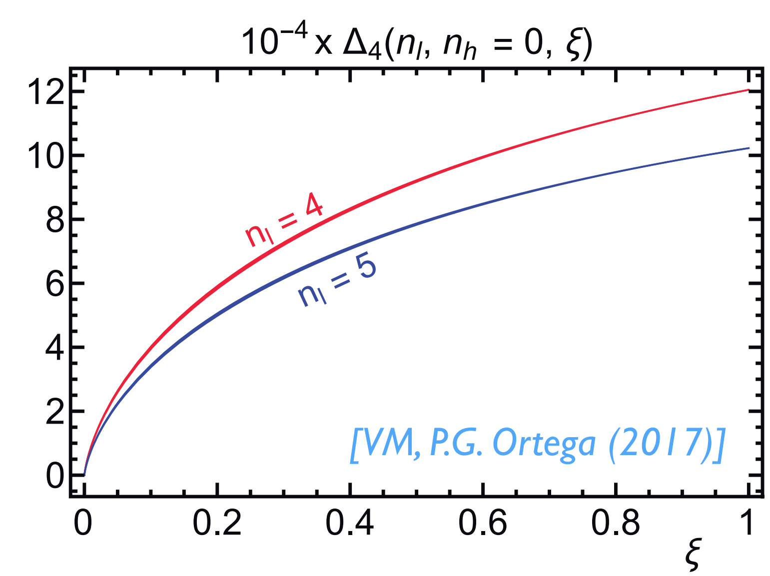

Massive lighter quarks effects on the MSR mass

$$\delta m_Q^{\rm MSR}(R,\overline{m}_q) = \delta m_Q^{\rm MSR}(R) + R\,\Delta_{\overline{m}_q}(R,\xi_R)\,,$$

$$\Delta_{\overline{m}_q}(R,\xi_R) = \sum_{k=2}\Delta^{(k)}_{\overline{m}_q}(\xi_R)\left(\frac{\alpha_s^{(n_\ell)}(R)}{4\pi}\right)^k\,, \qquad \xi_R = \overline{m}_q/R$$

different implementation in [Lepenik, Hoang, Preisser (2017)]

MSR with Massive lighter quarks

[VM, P.G. Ortega (2017)]

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exact Heavy Quark symmetry for MSRn

$$m_Q^{\rm pole} - m_Q^{\rm MSRn}(\overline{m}_q) = m_q^{\rm pole} - \overline{m}_q$$

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mass-dependent R-anomalous dimension

massless R-anomalous dimension

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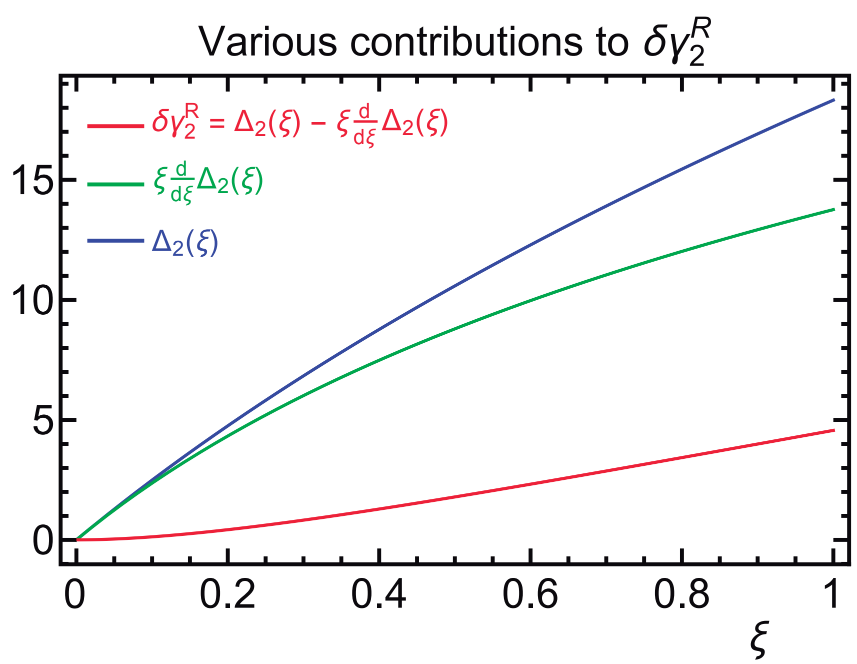

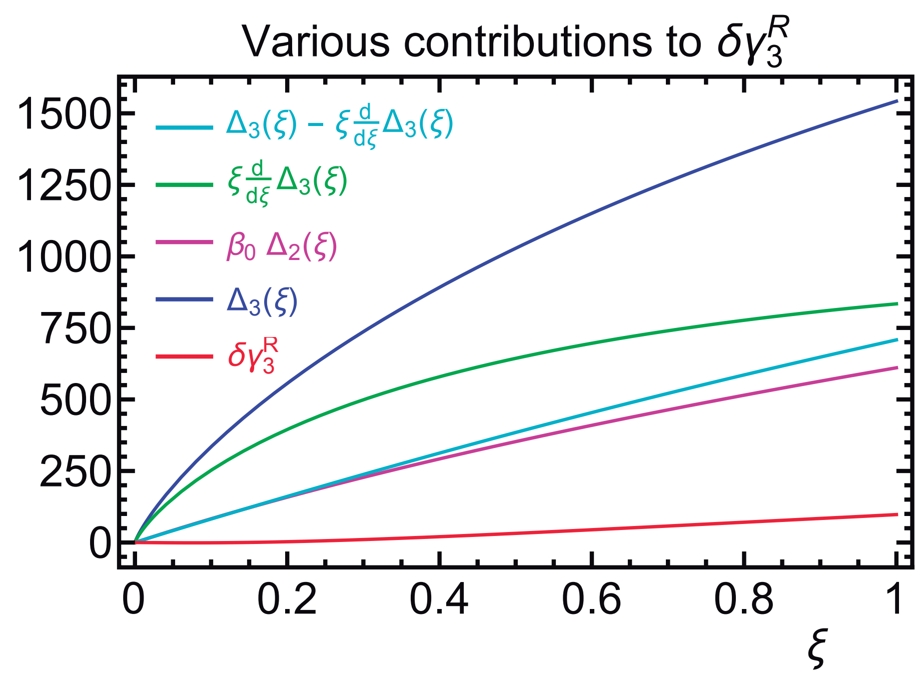

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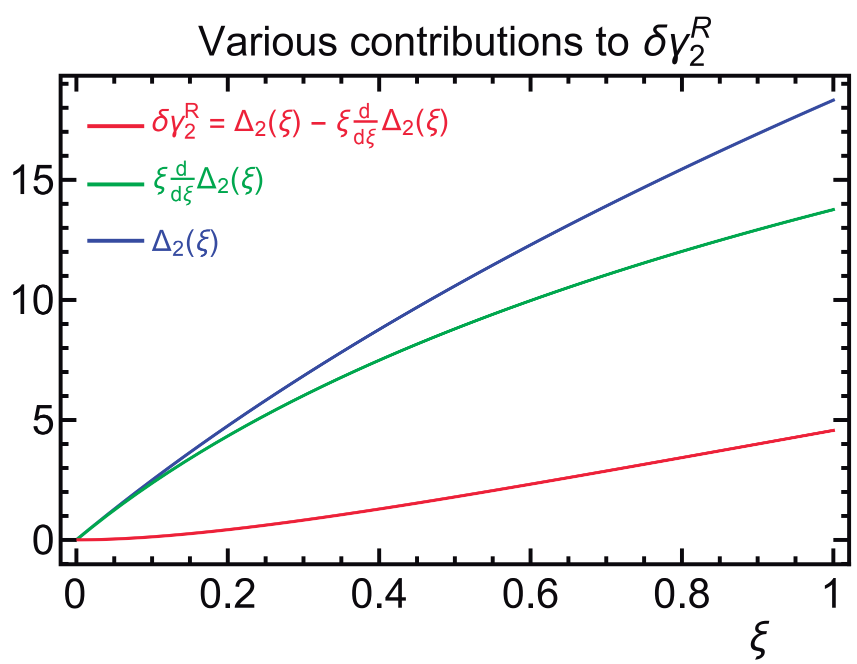

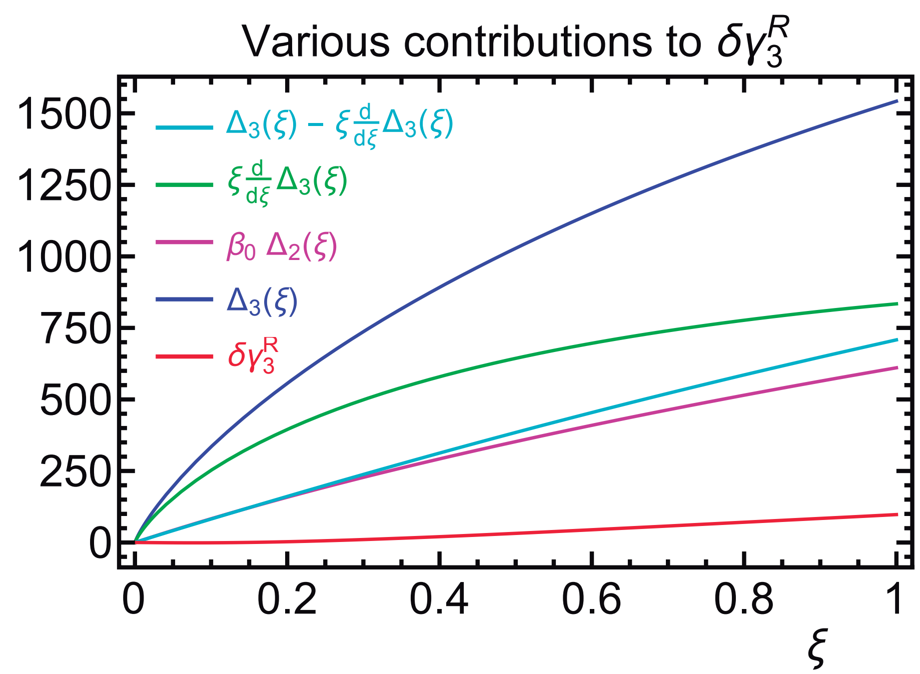

huge cancelations among the various contributions!

requiring exactly $\delta\gamma_n^R(\xi) = 0$

prediction for higher orders

$$\Delta_n(\xi) \approx \xi\,\Delta_n(1) + 2\,\xi\sum_{j=0}^{n-2}(n-j)\,\beta_j\int_\xi^1{\rm d}x\,\frac{\Delta_{n-j}(x)}{x^2}$$

satisfies $\xi = 0$ and $\xi = 1$ constraints

MSR with $n_\ell - 1$ active flavors

[Hoang, Lepenik, Preisser, '17]
[VM, P.G. Ortega (2017)]

Physical situations in which one runs to scales $R < \overline{m}_q$ and $\overline{m}_q$ is integrated out

Therefore we must integrate the quark q in the MSR mass as well

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It is smoothly matched with the $\mathrm{MSR}^{(n_\ell)}$ mass at $R = \overline{m}_q$

Essential to study the ambiguity of the pole mass [Hoang, Lepenik, Preisser, '17]

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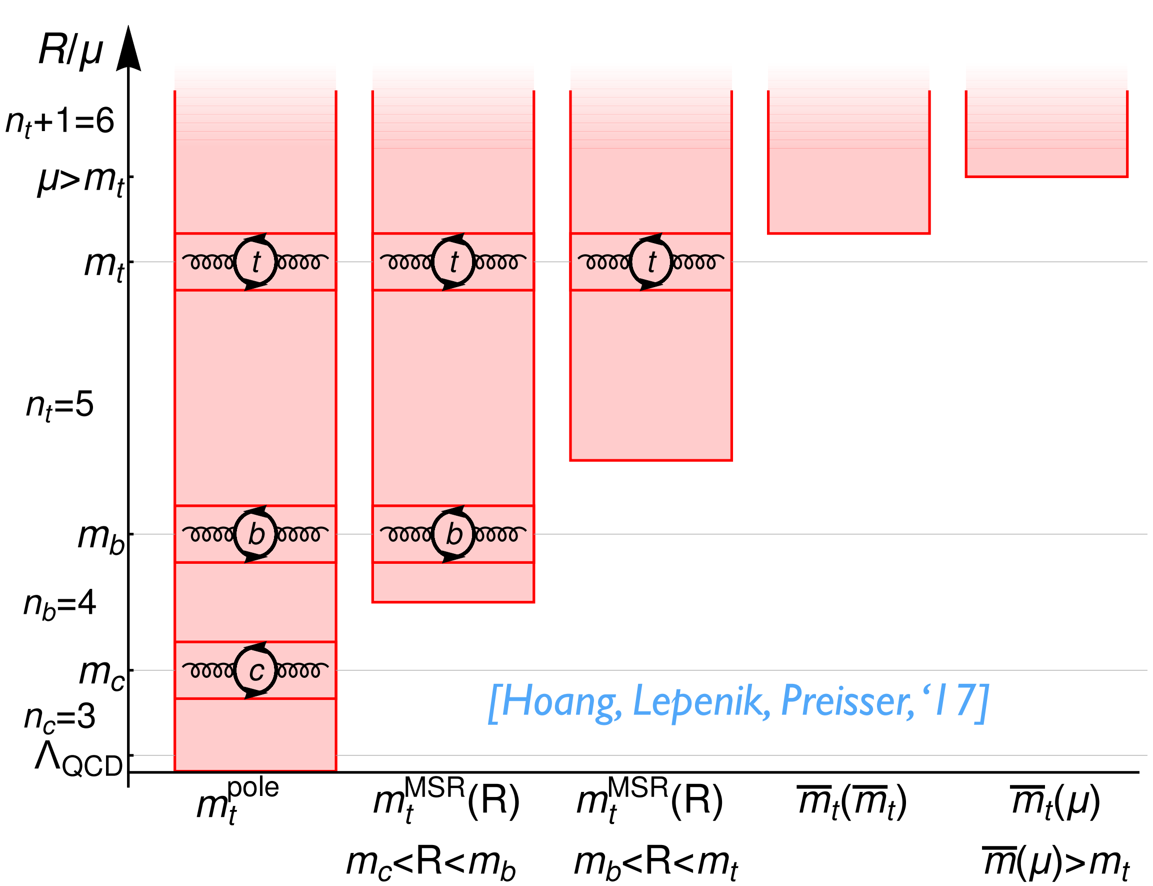

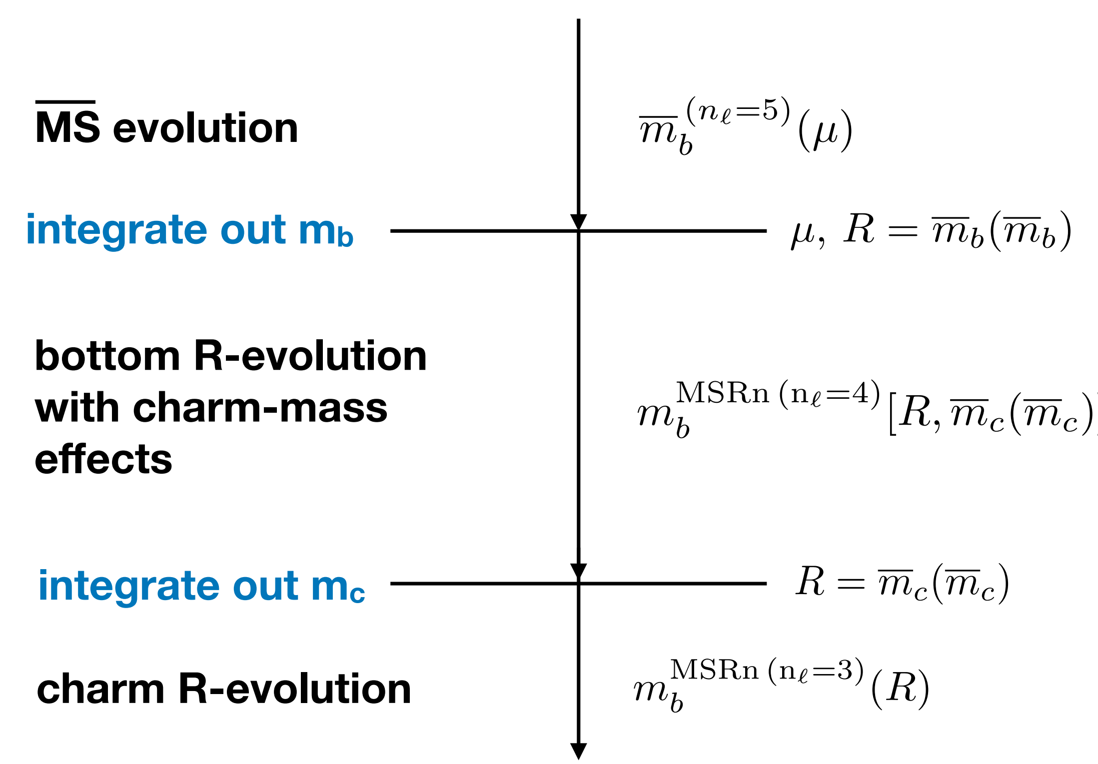

VFNS-like sequence of running and matching

Comment on “Vienna implementation” *[Hoang, Lepenik, Preisser, ‘17]*

While we worked on our MSR mass with massive lighter quarks the article by Hoang, Lepenik and Preisser appeared

$$m_Q^{\text{pole}} - m_Q^{\text{MSR}}(R, \overline{m}_q) = \delta m_Q^{\text{MSR}}(R) + \overline{m}_Q \, \Delta_{\overline{m}_q}(\overline{m}_Q, \overline{m}_q/\overline{m}_Q)$$

massless term

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Therefore R-evolutions is the same as for massless quarks

Identical matching to $\overline{\mathrm{MS}}$ mass at $R = \overline{m}_Q$

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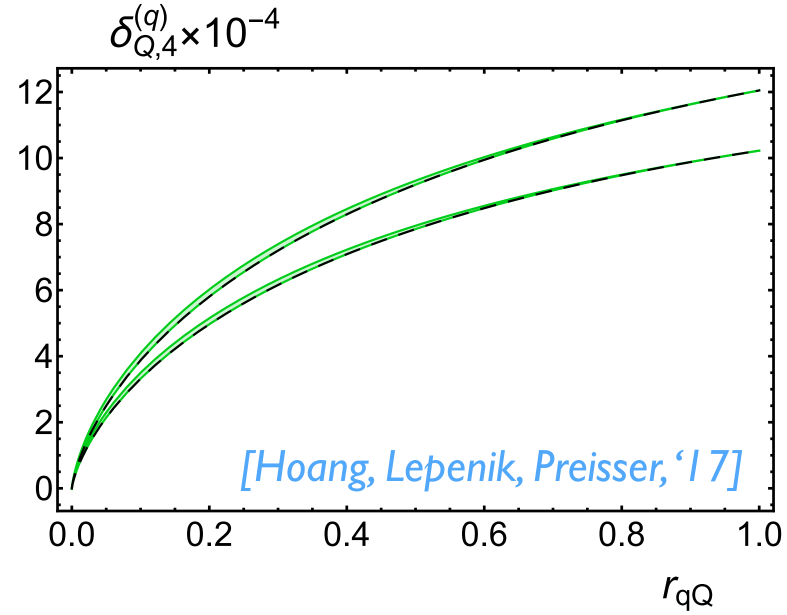
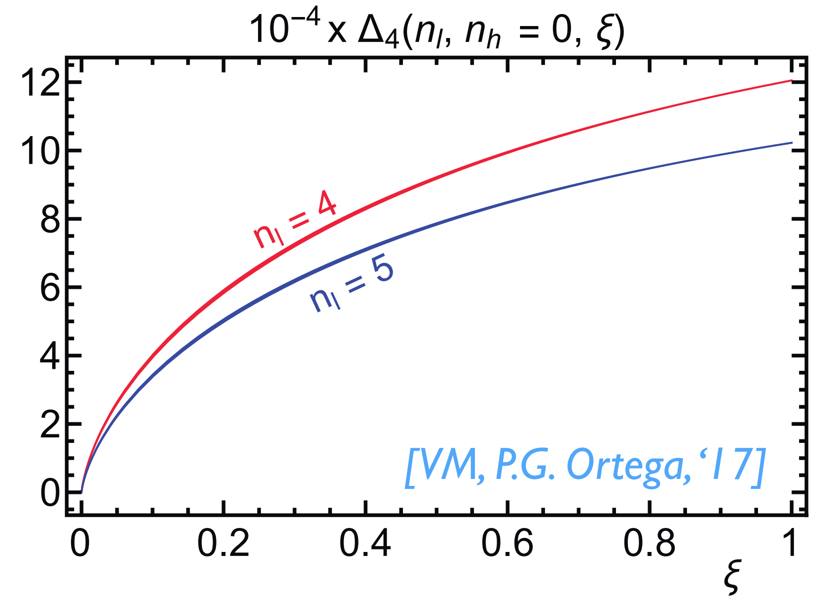
Prediction for higher order corrections from imposing exact Heavy Quark Symmetry





almost identical to our prediction

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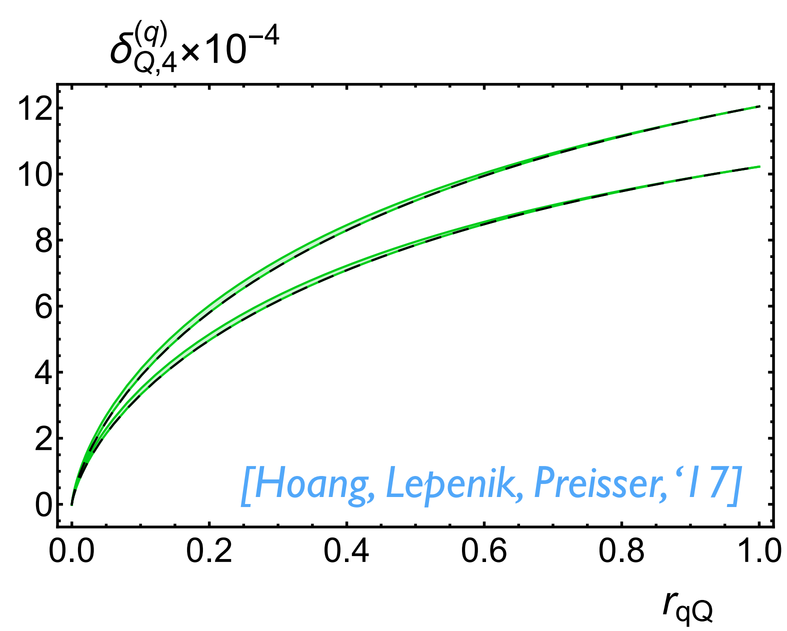
massless term | R-independent

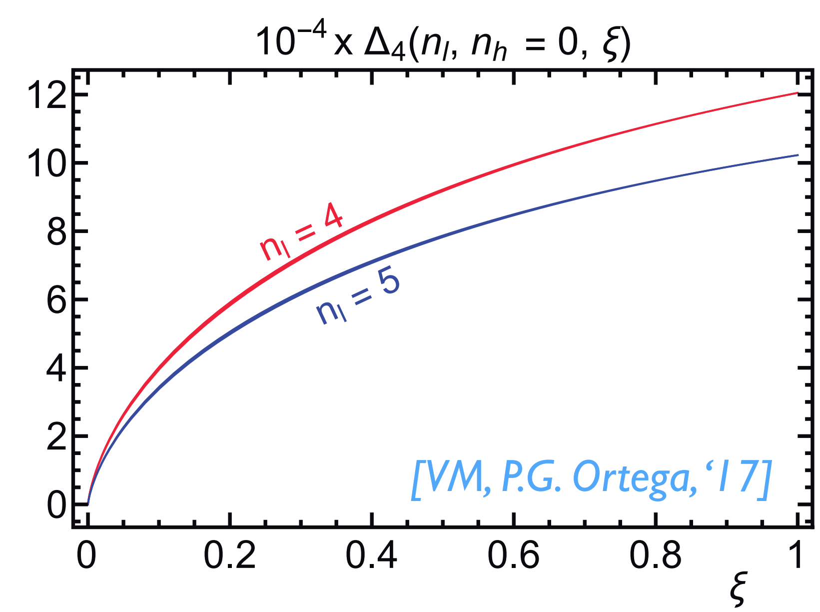
Therefore R-evolutions is the same as for massless quarks

Identical matching to $\overline{\text{MS}}$ mass at $R = \overline{m}_Q$

Different matching for $\text{MSR}^{(n_\ell)}$ and $\text{MSR}^{(n_\ell - 1)}$ masses at $R = \overline{m}_q$

Prediction for higher order corrections from imposing exact Heavy Quark Symmetry





almost identical to our prediction

Different aims lead to slightly different versions of the MSR mass

For all practical purposes can be considered identical

Analysis

Scale variation and charm mass dependence

Scale dependence investigation

"Popular" scheme choices in the literature $\overline{\mathrm{MS}}$: large logs of $\frac{\overline{m}_b}{\mu}$ in subtractions

[Brambilla, Vairo, Sumino], [Kiyo, Mishima, Sumino]

RS mass (Pineda): no smooth transition to (n_l -1) scheme [Ayala, Czvetic, Pineda (2016)]

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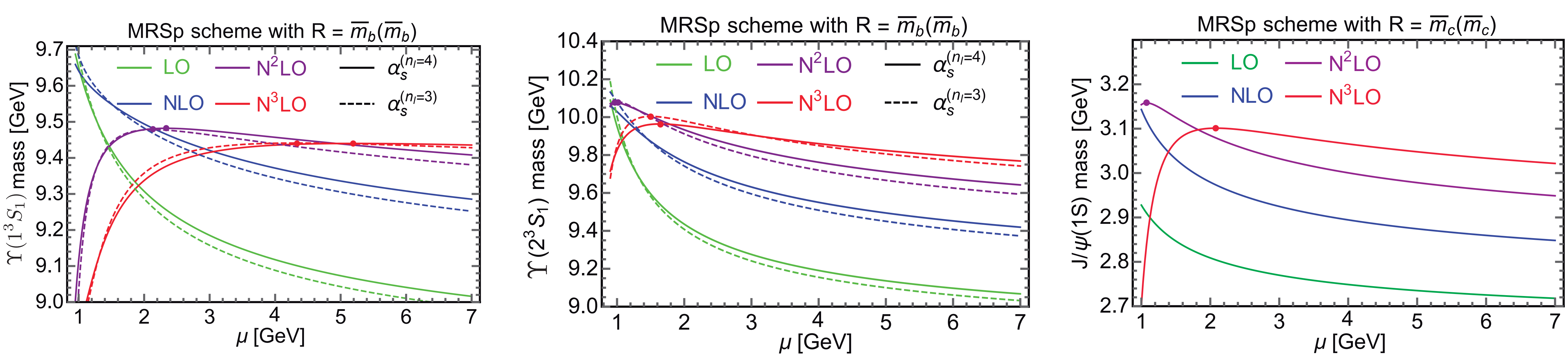
Principle of minimal sensitivity [Brambilla, Vairo, Sumino]

Take the scale at maximum or minimum, double that scale to estimate uncertainties

Not defined at all orders. Large dependence on order and quantum numbers other than n.

Results in ranges that cover relativistic scales. Renders small perturbative uncertainties.



Scale dependence investigation

Scale variation should (only) depend on the principal quantum number n, since the argument of perturbative logs depends on n (but not on other numbers)

Should not depend on the perturbative order

It should also depend on bottomonium vs charmonium

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$$(\mu_{\rm nat} \pm \Delta\mu) \sim \frac{2^{\pm\phi} C_F \alpha_s^{(n_\ell)}(\mu) m_Q}{n}$$

We take $\phi = 0.5$ but extend the upper limit to 4 GeV (similar to scale variation in relativistic sum rules *[Dehnadi, Hoang, Mateu (2013, 2015)]*

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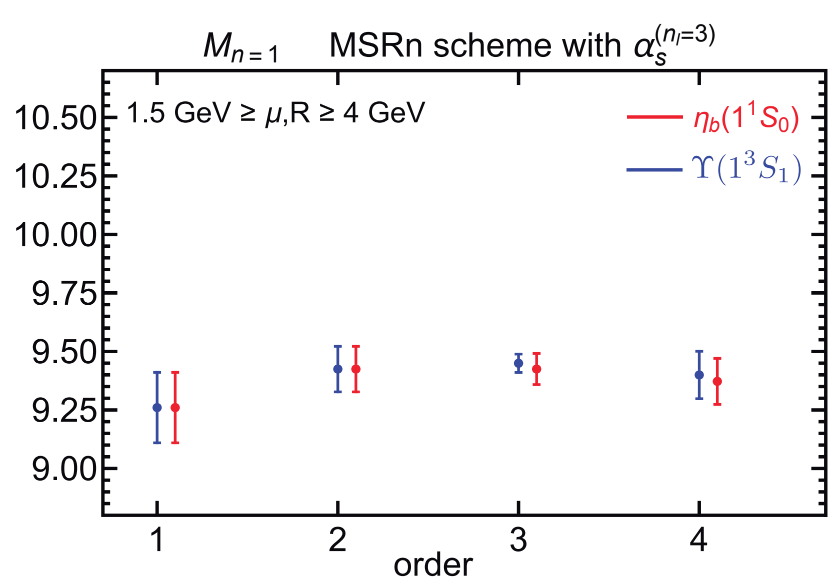
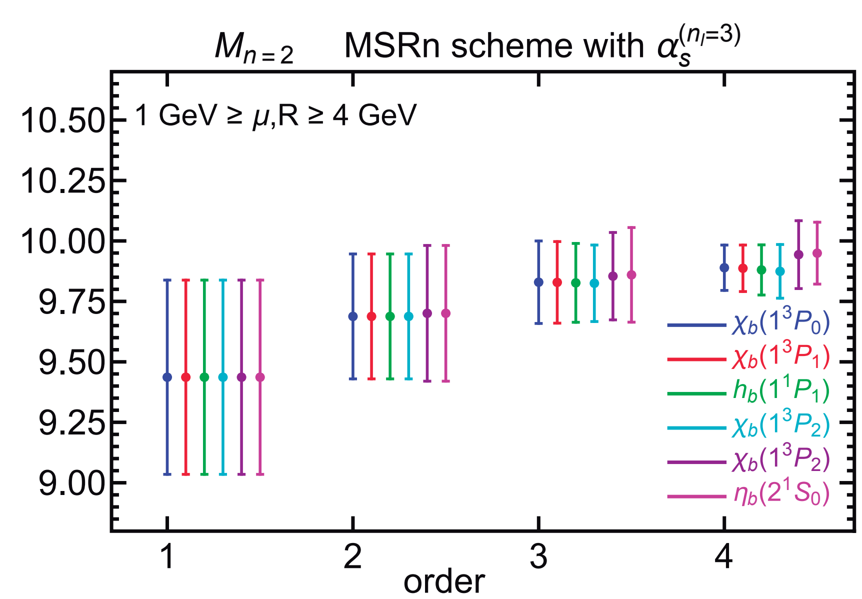
$$\mu_{n=1} \sim 1.9^{+1.6}_{-0.4}\,\text{GeV} \qquad \mu_{n=2} \sim 1.25 \pm 0.25\,\text{GeV}$$





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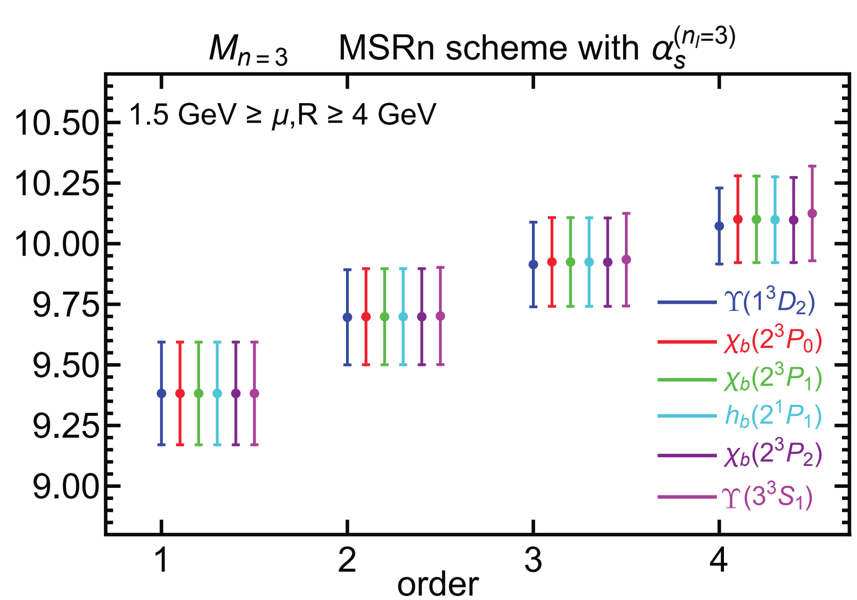
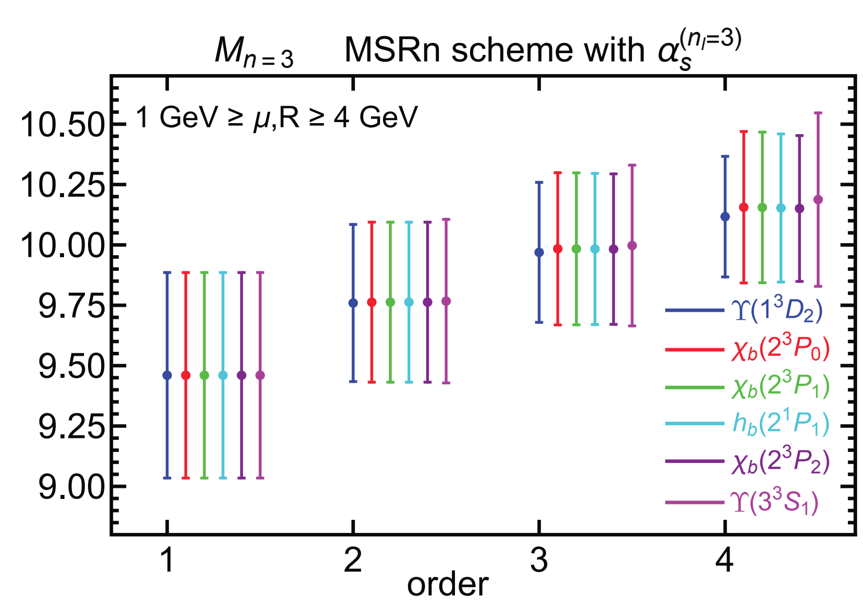
For n = 3 one gets a lower scale below 1GeV. It seems no scale choice can make the perturbative series both convergent and compatible.





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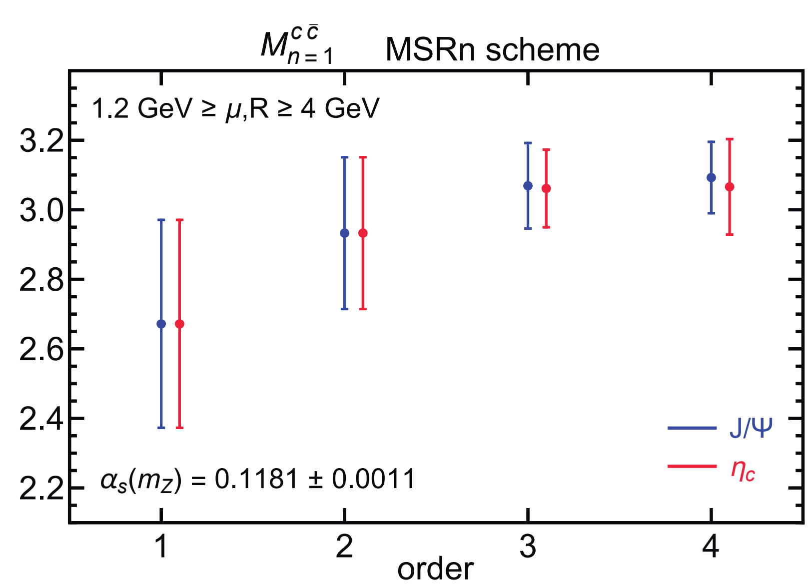
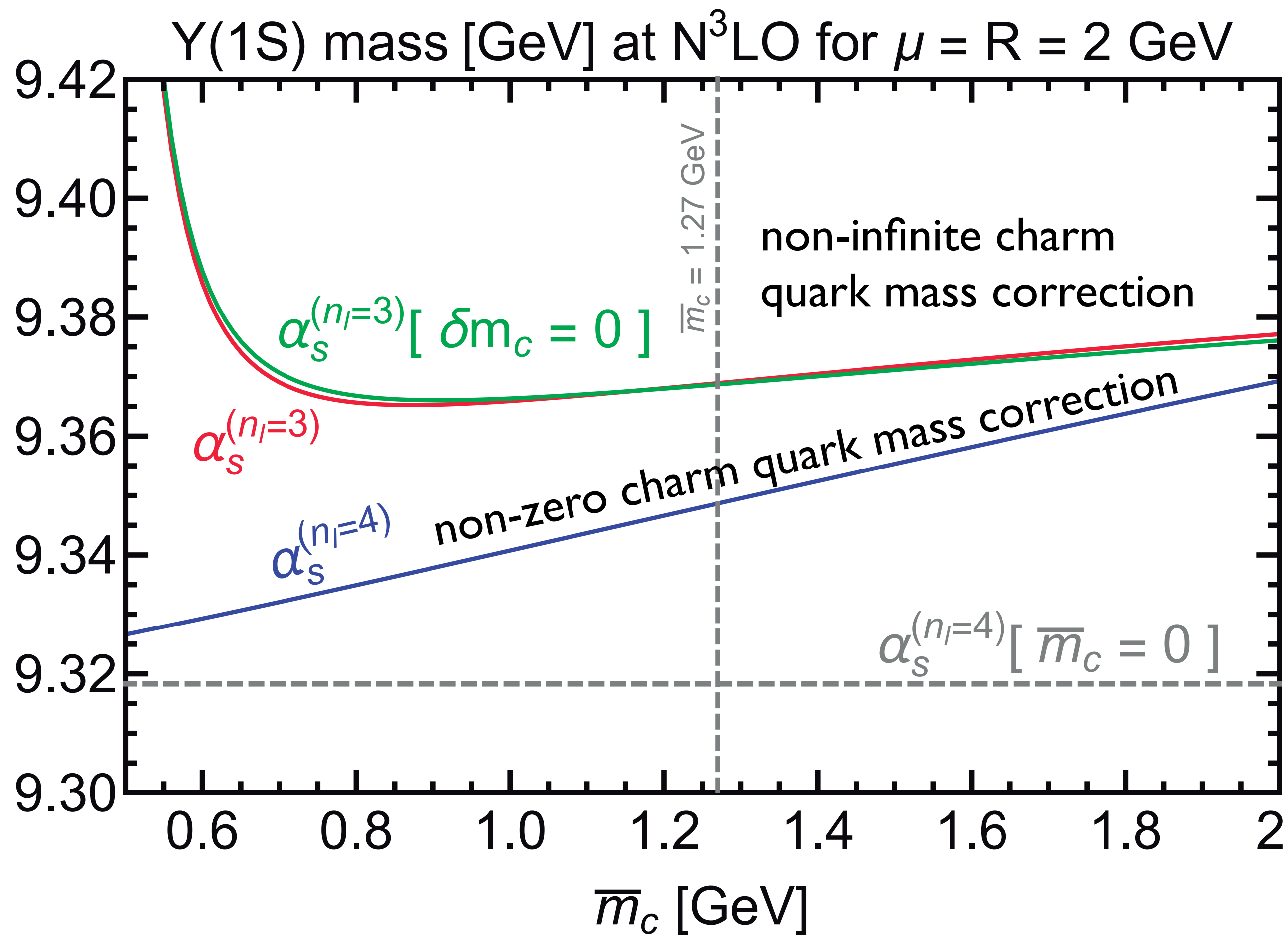
For charm this criterion renders the lower scale below 1GeV. However the following choice $1.2\,{\rm GeV} \geq \mu_{\rm charm} \geq 4\,{\rm GeV}$ makes for a convergent and compatible perturbative series



Charm mass dependence



This confirms that the (n_l - 1) scheme is the most accurate to describe finite charm quark mass effect

Perturbative correlations

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$$\mu_2(\mu) = \mu, \qquad R_2(R) = R \qquad\qquad 1\,\mathrm{GeV} \leq \mu, R \leq 4\,\mathrm{GeV}$$

$$\mu_{1,3}(\mu) = 1.5\,\mathrm{GeV} + 2.5\,(\mu - 1\,\mathrm{GeV})/3\,, \quad R_{1,3}(\mu) = 1.5\,\mathrm{GeV} + 2.5\,(R - 1\,\mathrm{GeV})/3$$

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Perturbative covariance matrix approach: severely affected by D'Agostini bias

We make our χ^2 function depend on (μ, R) and scan on the range shown above

$$\chi^2(\overline{m}_Q, \mu, R) = \sum_i \left(\frac{M_i^{\mathrm{exp}} - M_i^{\mathrm{pert}}(\mu, R, \overline{m}_Q)}{\sigma_i^{\mathrm{exp}}} \right)^2$$

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This approach correctly propagates the theoretical correlations and avoids de bias

We also vary the strong coupling constant and the charm (bottom) mass for bottomonium (charmonium)

Fits to data

Different data sets

Bottomonium

1. $\mathrm{Set}_{n=1} = \{\ \eta_b(1S),\ \Upsilon(1S)\ \}.$
2. $\mathrm{Set}_{n=2} = \{\ \chi_{b0}(1P),\ \chi_{b1}(1P),\ h_b(1P),\ \chi_{b2}(1P),\ \eta_b(2S),\ \Upsilon(2S)\ \}.$
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4. $\mathrm{Set}_{L=P} = \{\ \chi_{b0}(1P),\ \chi_{b1}(1P),\ h_b(1P),\ \chi_{b2}(1P)\ \}.$
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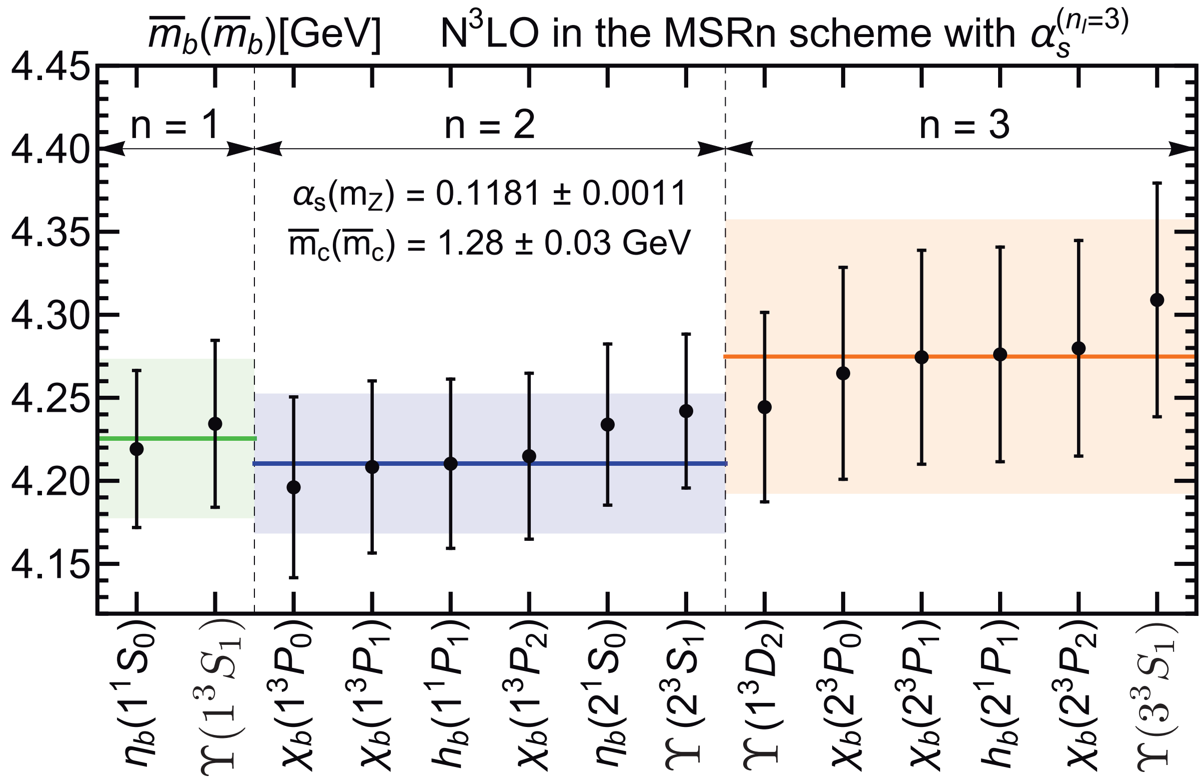
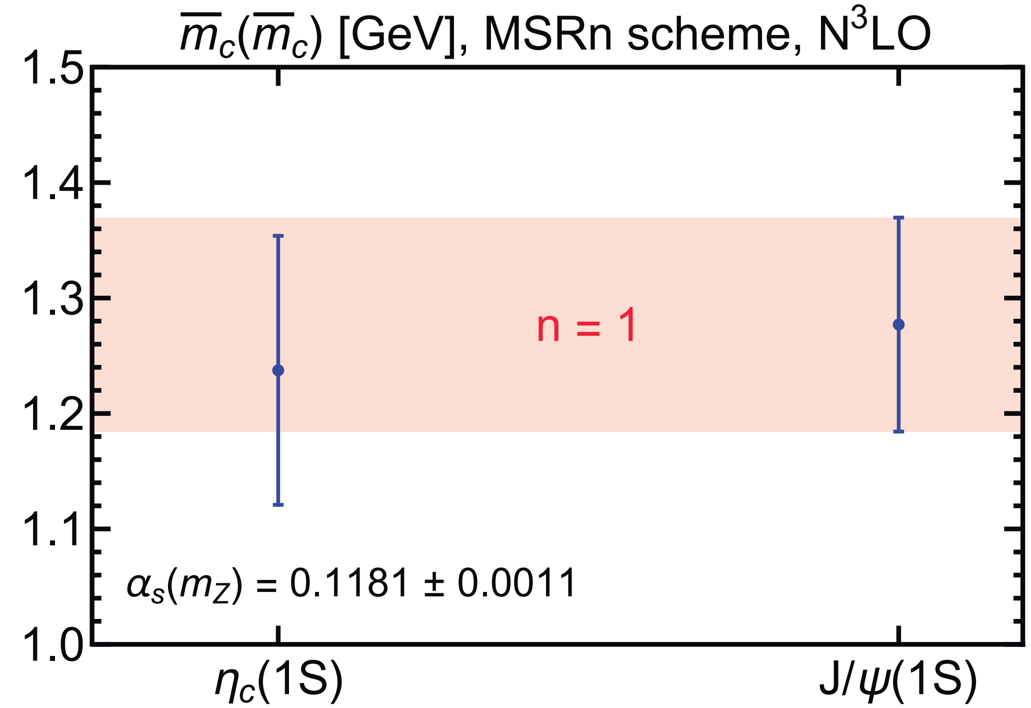
Charmonium

$\eta_c(1S),\ J/\psi(1S)$ + determinations from individual states

Results for bottom and charm





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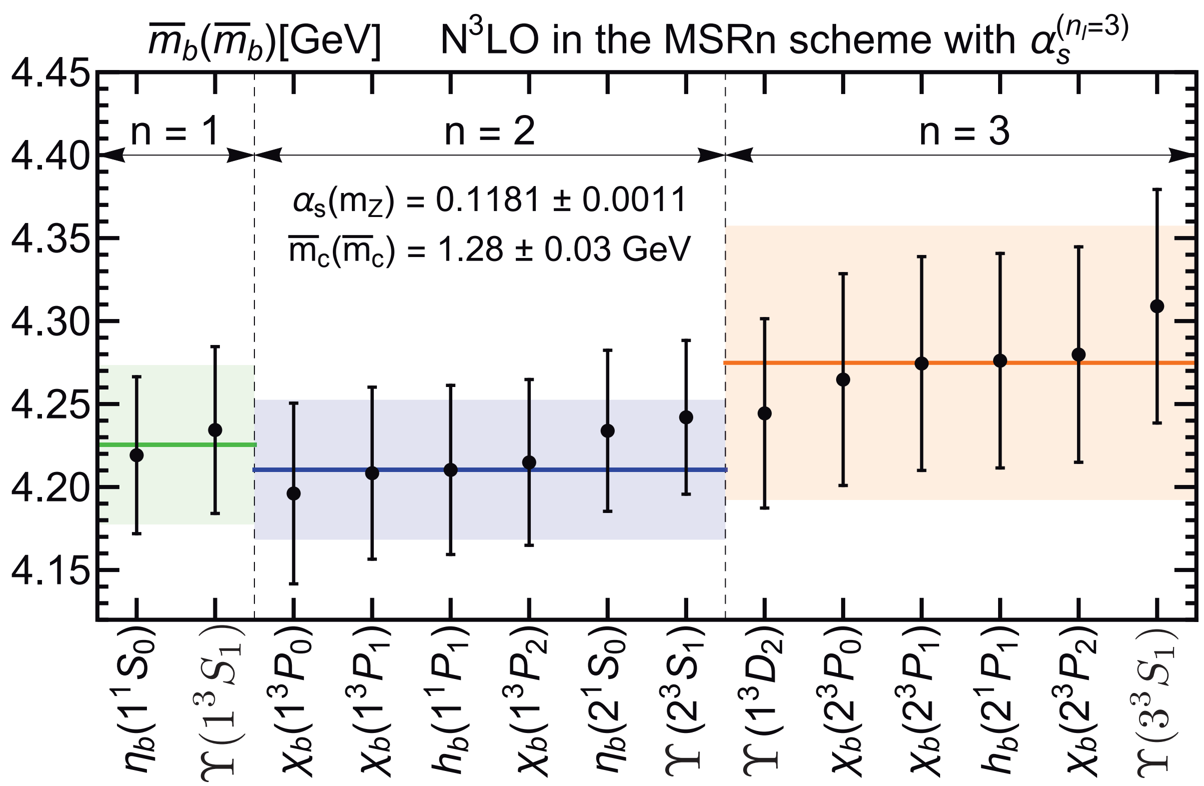
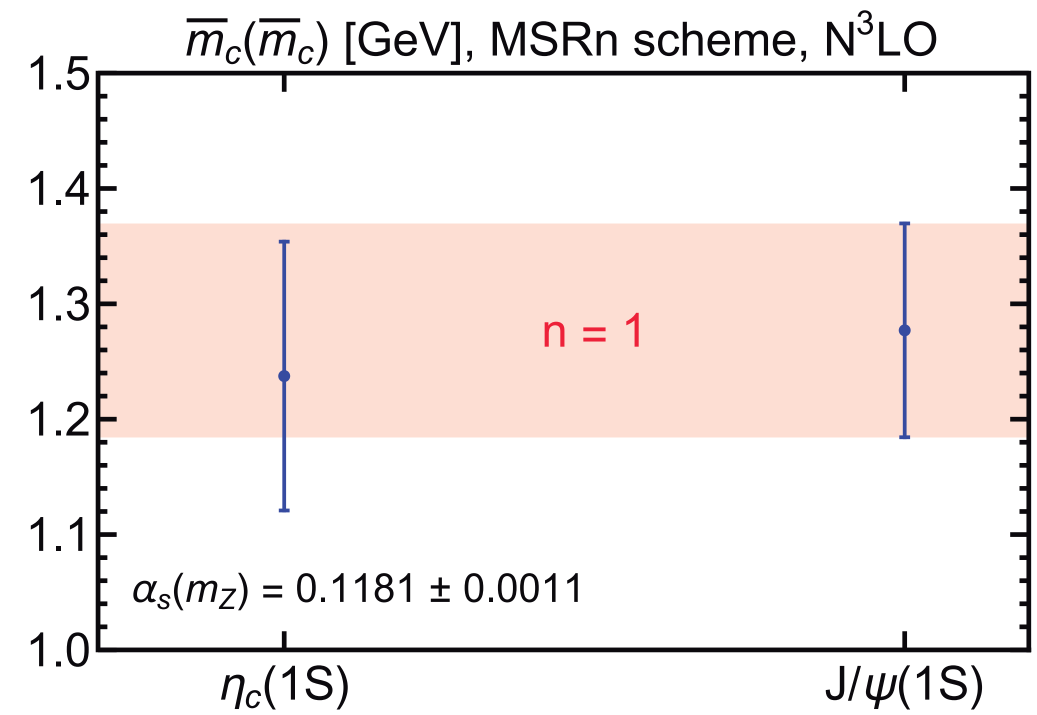
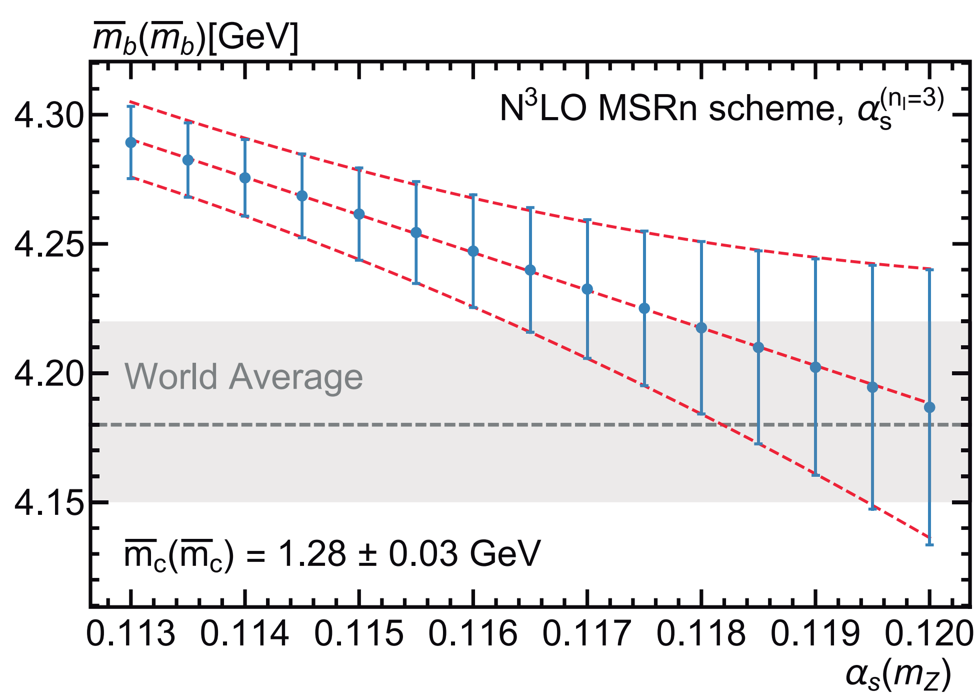
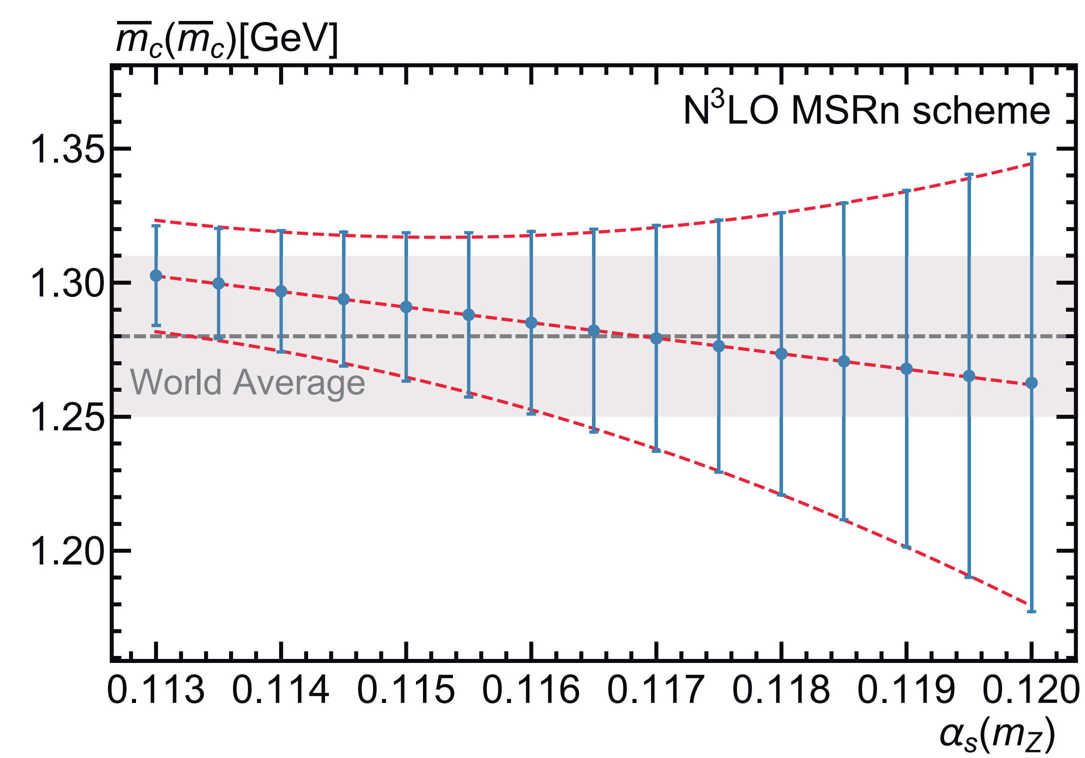


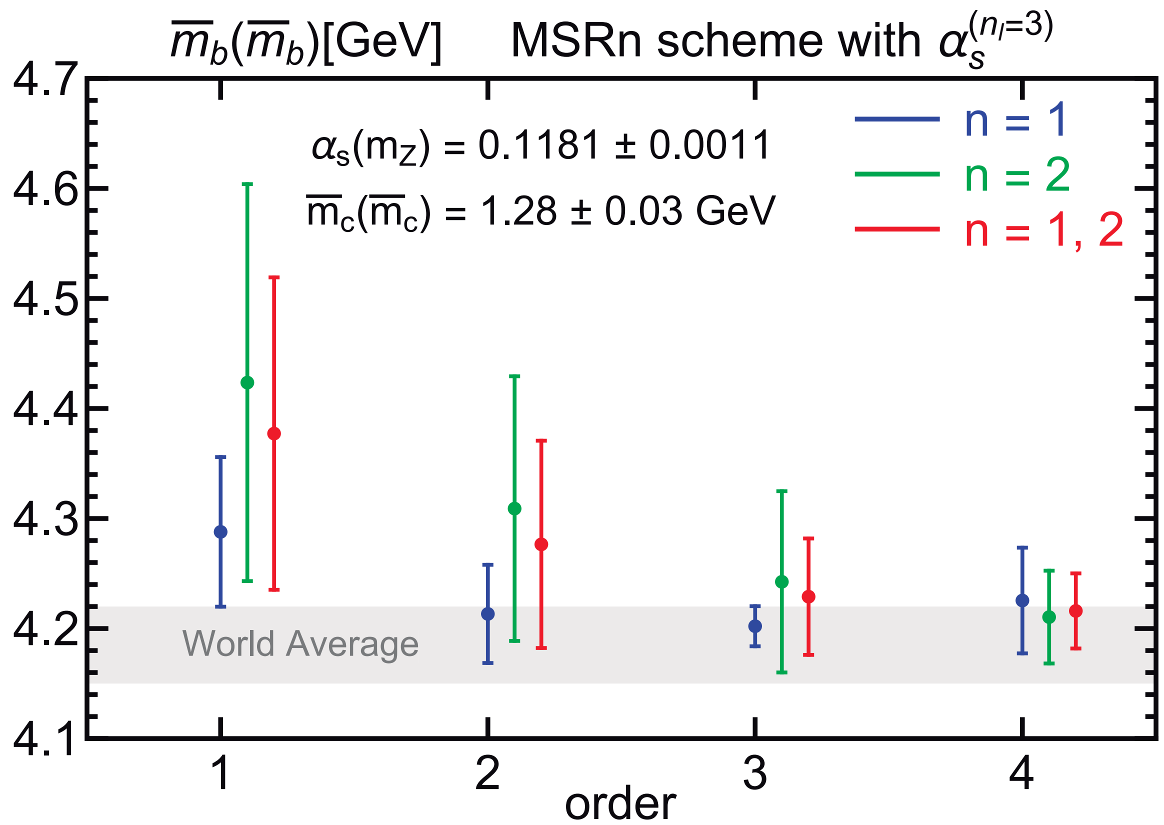


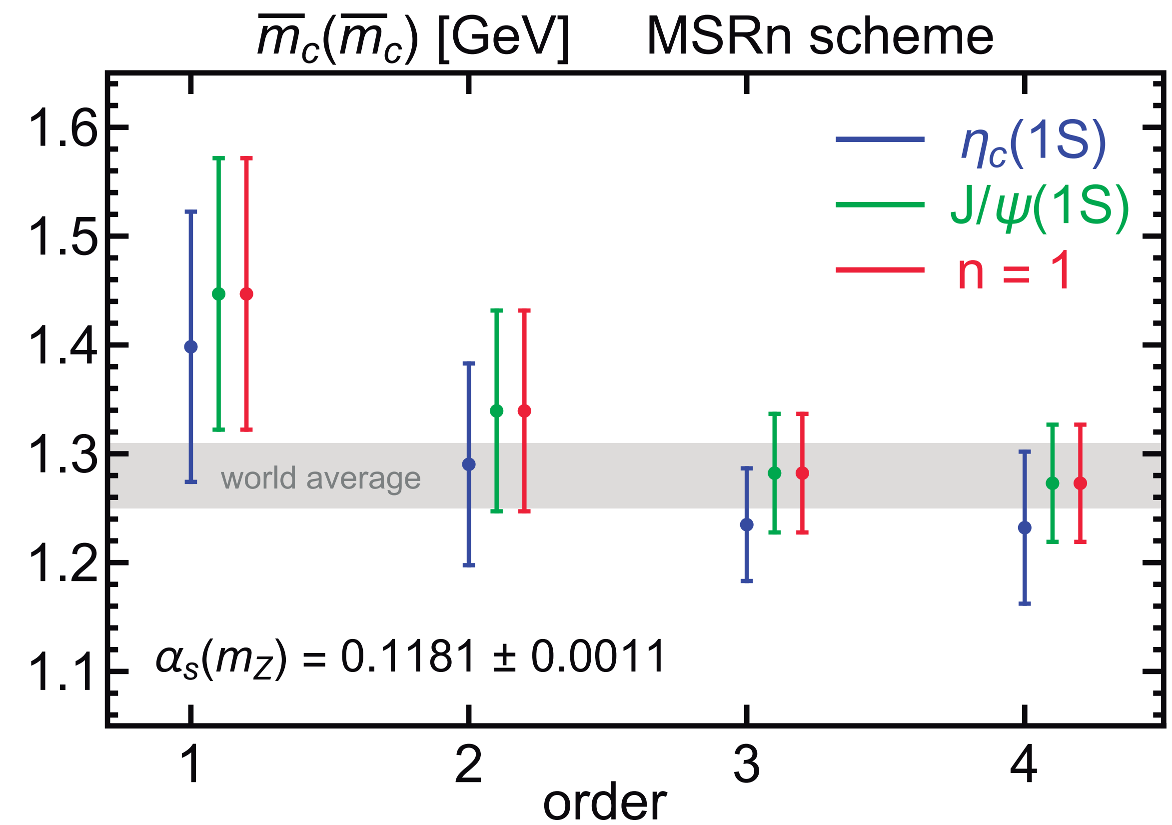
Dependence with α_s





Convergence





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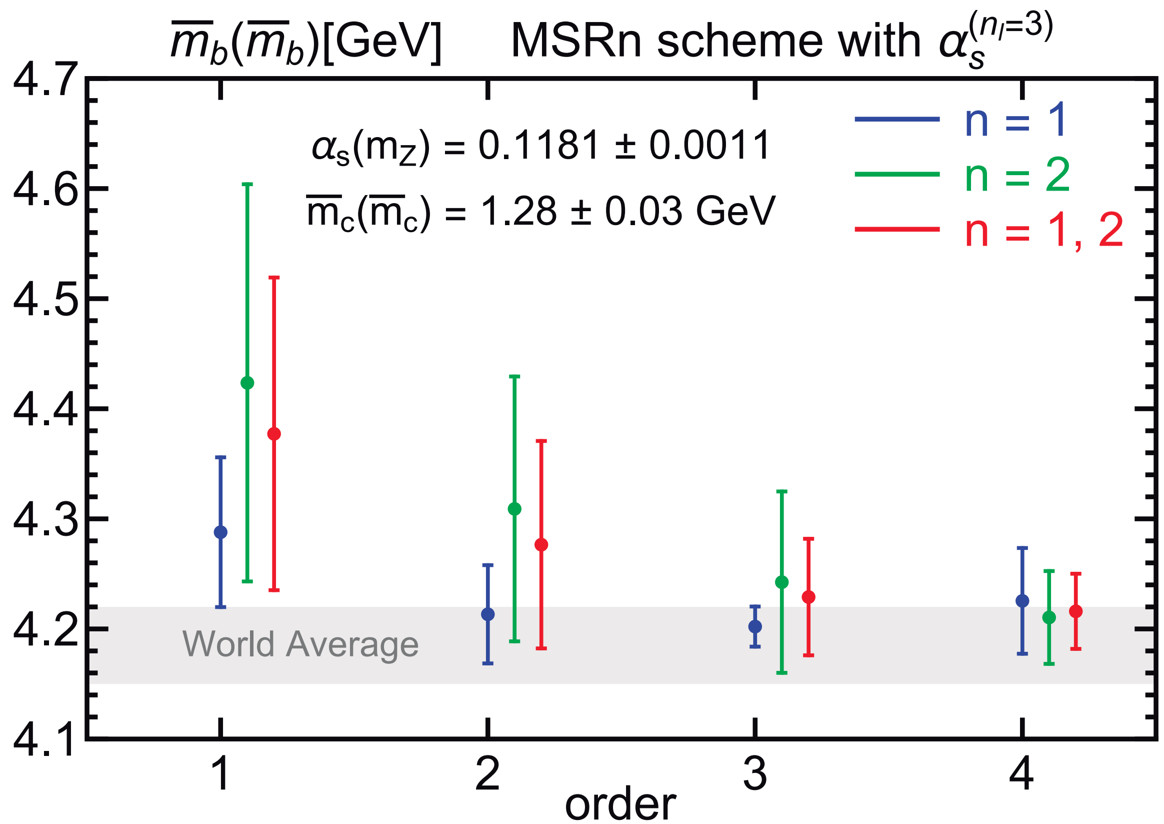
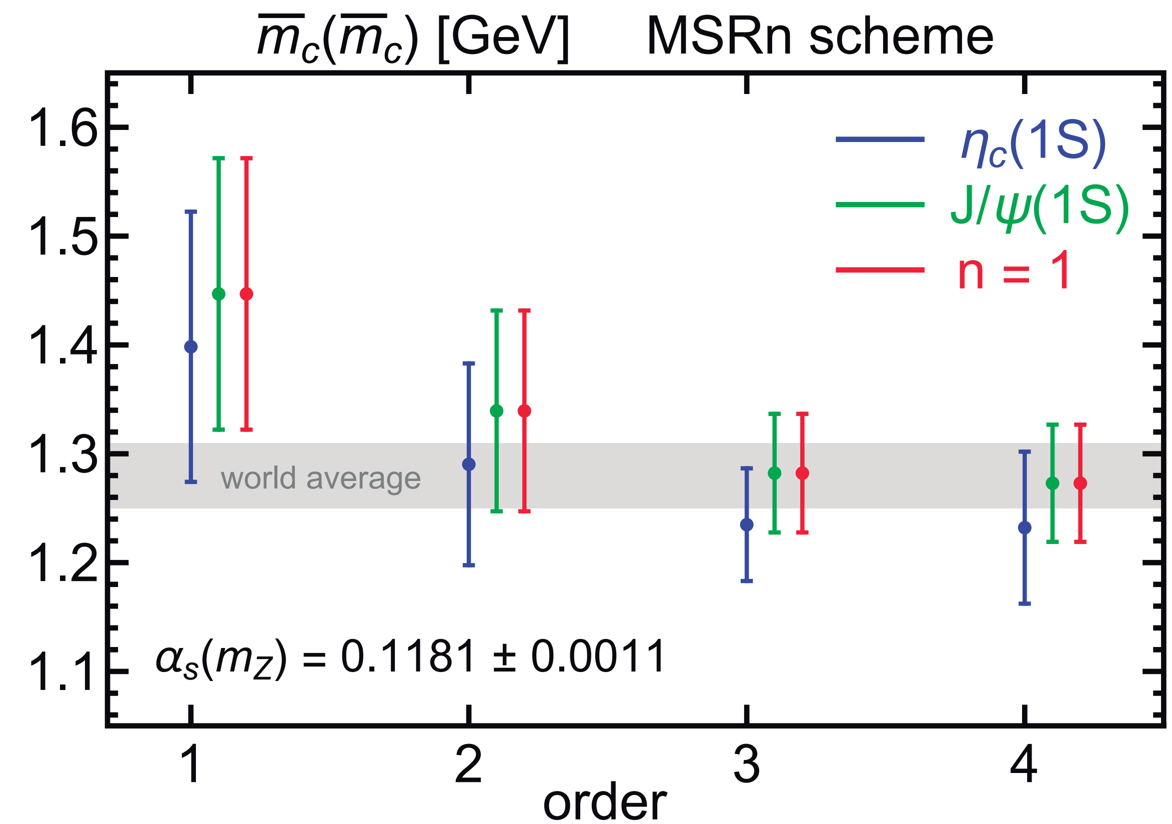
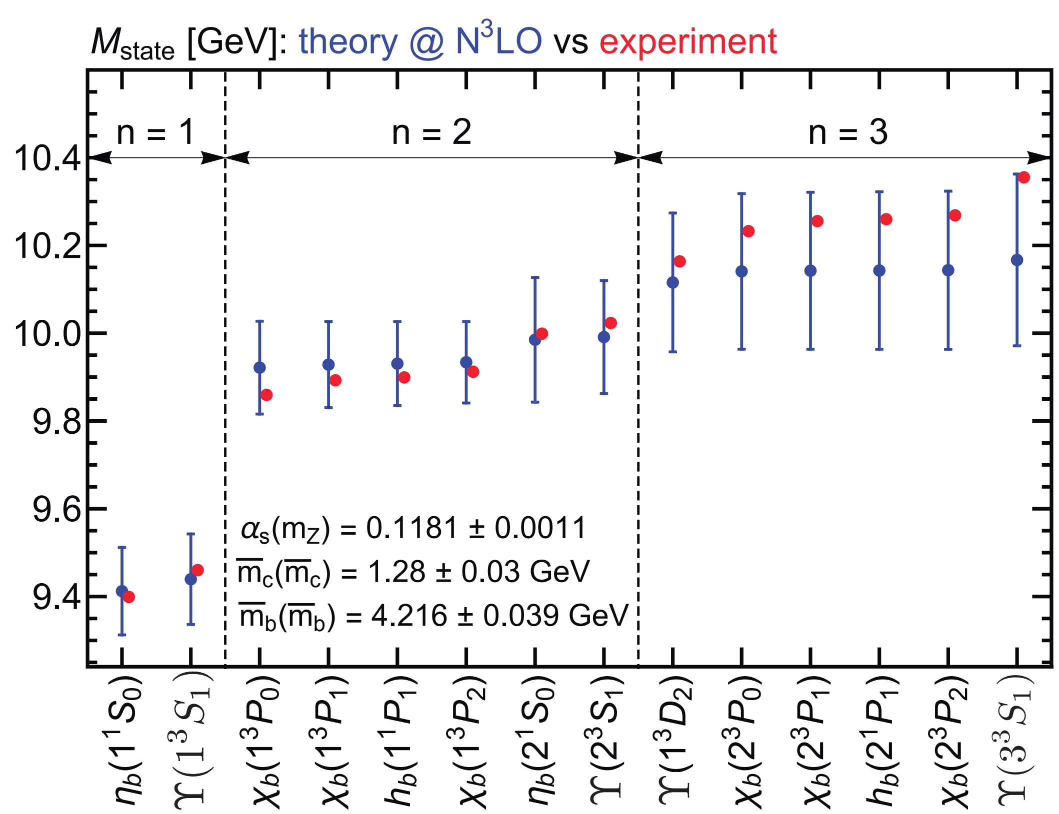
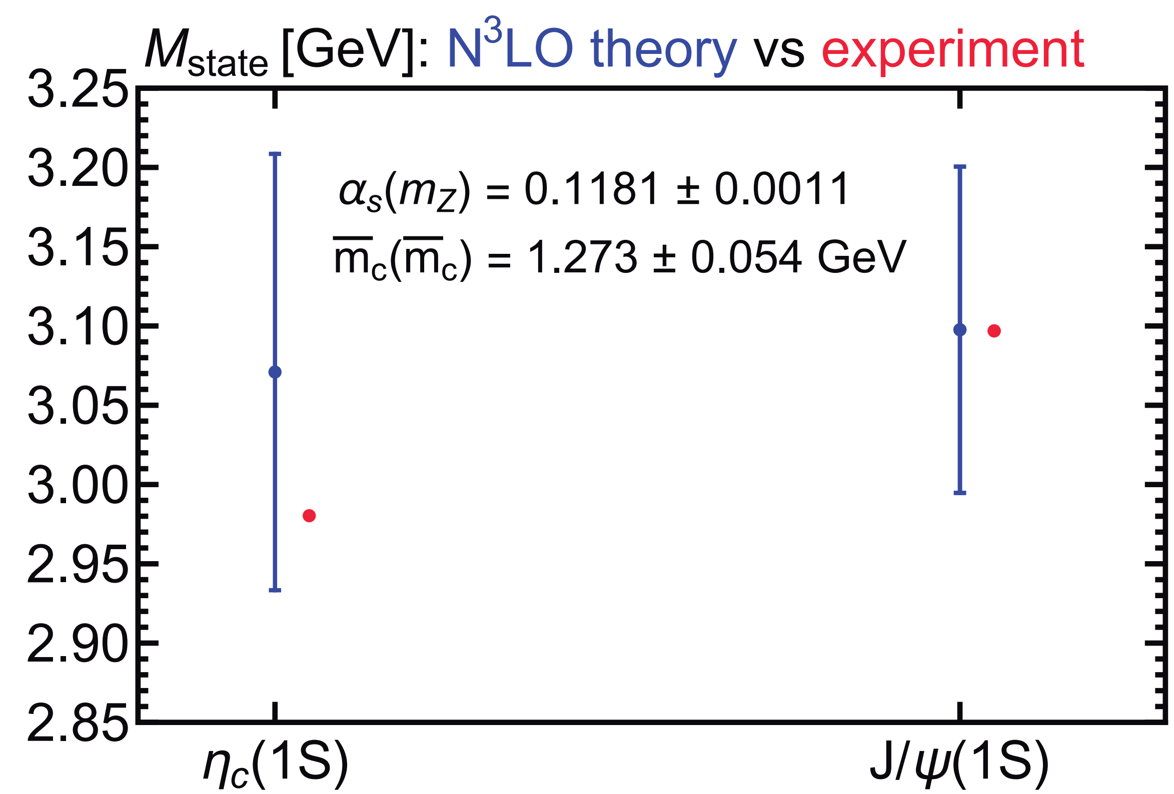




Comparison to data





Final results

$$\overline{m}_b(\overline{m}_b) = 4.216 \pm 0.009_{\rm exp} \pm 0.034_{\rm pert} \pm 0.017_{\alpha_s} \pm 0.0008_{\overline{m}_c}\,{\rm GeV}$$
$$= 4.216 \pm 0.039\,{\rm GeV}$$

$$\overline{m}_c(\overline{m}_c) = 1.273 \pm 0.0005_{\rm exp} \pm 0.054_{\rm pert} \pm 0.006_{\alpha_s} \pm 0.0001_{\overline{m}_b}\,{\rm GeV}$$
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Results in MSRn and MSRp schemes nearly identical

Estimate uncertainty from non-perturbative corrections comparing fits from various datasets

Estimate uncertainty from missing finite charm mass effects by comparing fits in n_l and n_l - 1 schemes

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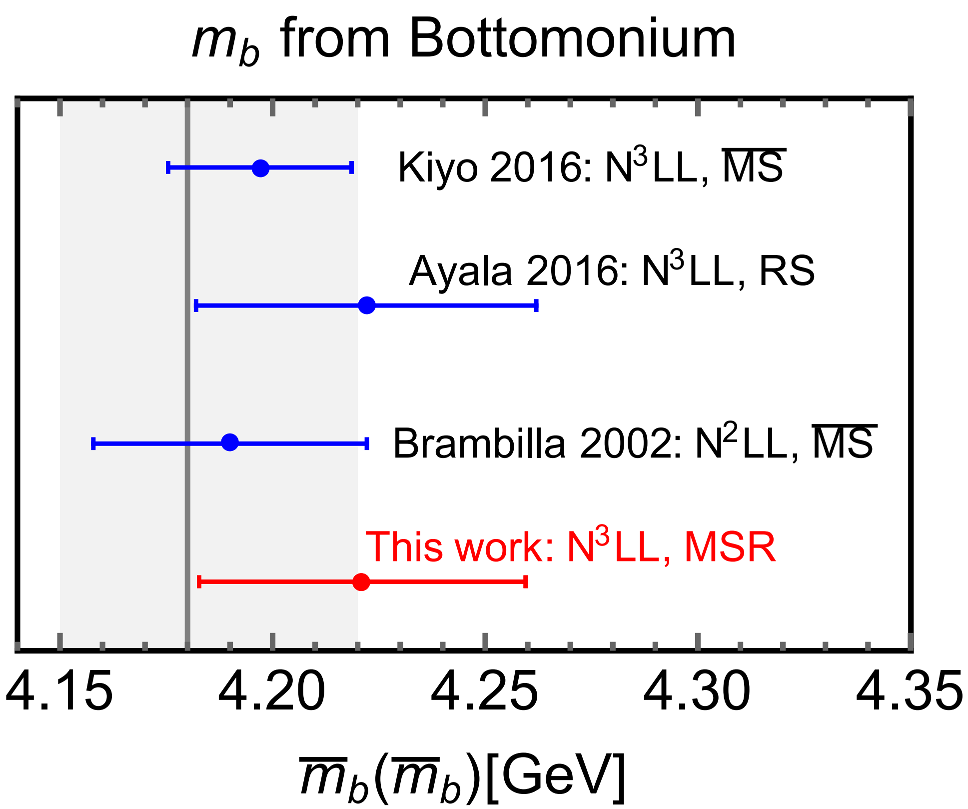
Simultaneous determination

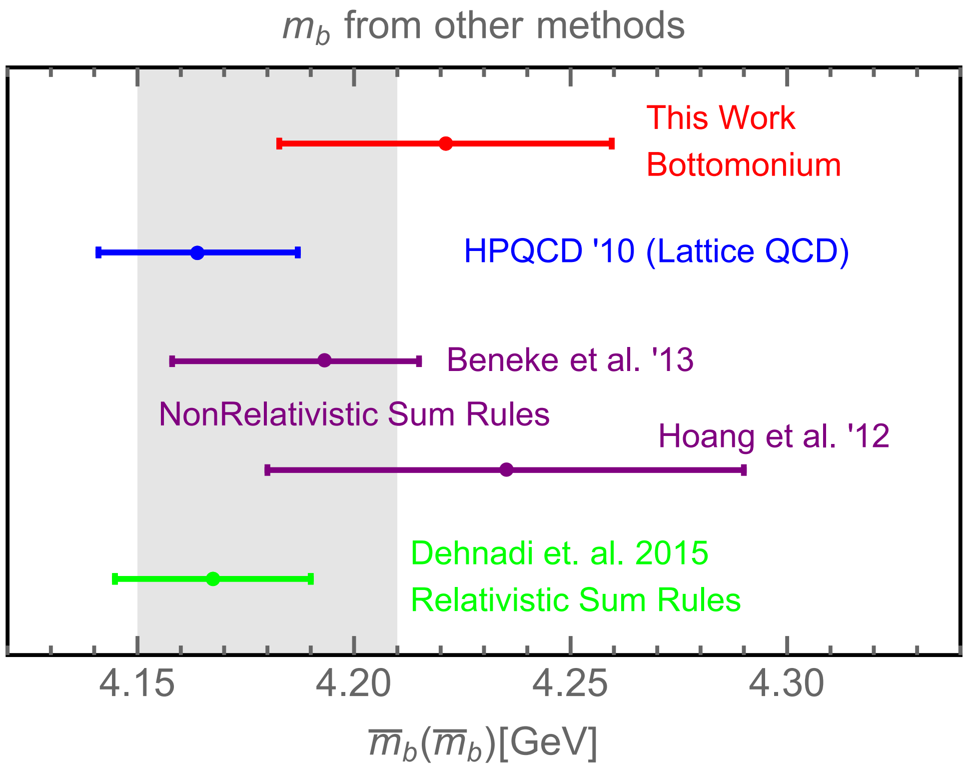
$$\begin{aligned}\overline{m}_b(\overline{m}_b) &= 4.219 \pm 0.0002_{\text{exp}} \pm 0.062_{\text{pert}}\,\text{GeV}\,,\\ \alpha_s^{(n_f=5)}(m_Z) &= 0.1178 \pm 0.00001_{\text{exp}} \pm 0.0050_{\text{pert}}\,.\end{aligned}$$

Very strong correlation between these two parameters
If correlation broken, competitive α_s possible

Comparison to previous determinations

Comparison to other determinations





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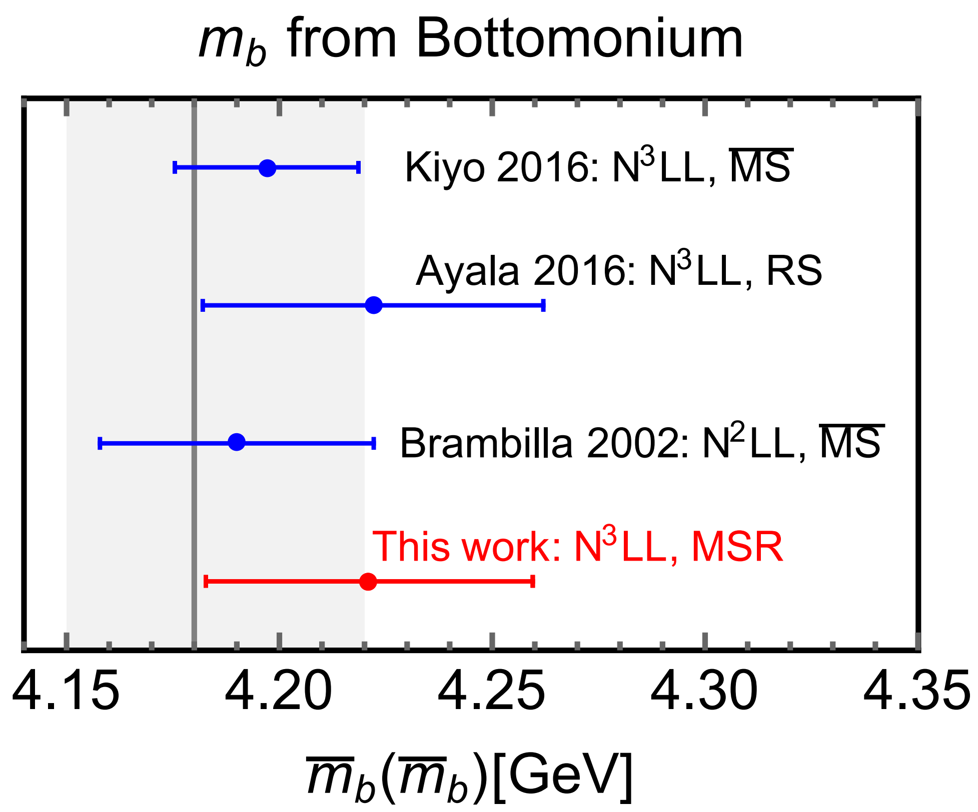

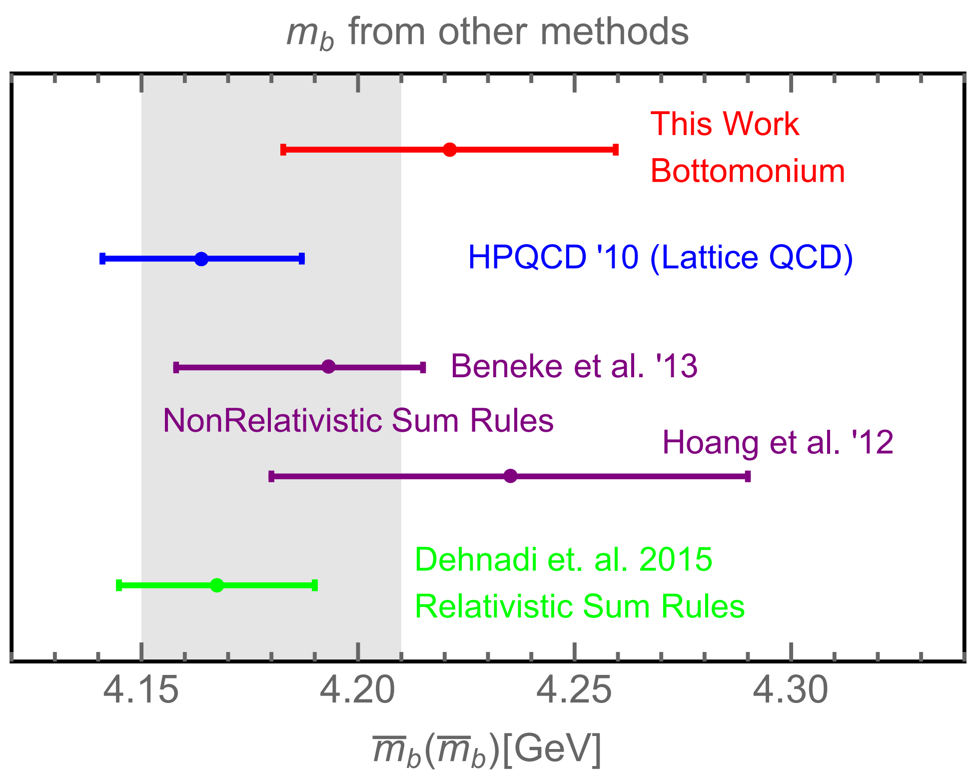

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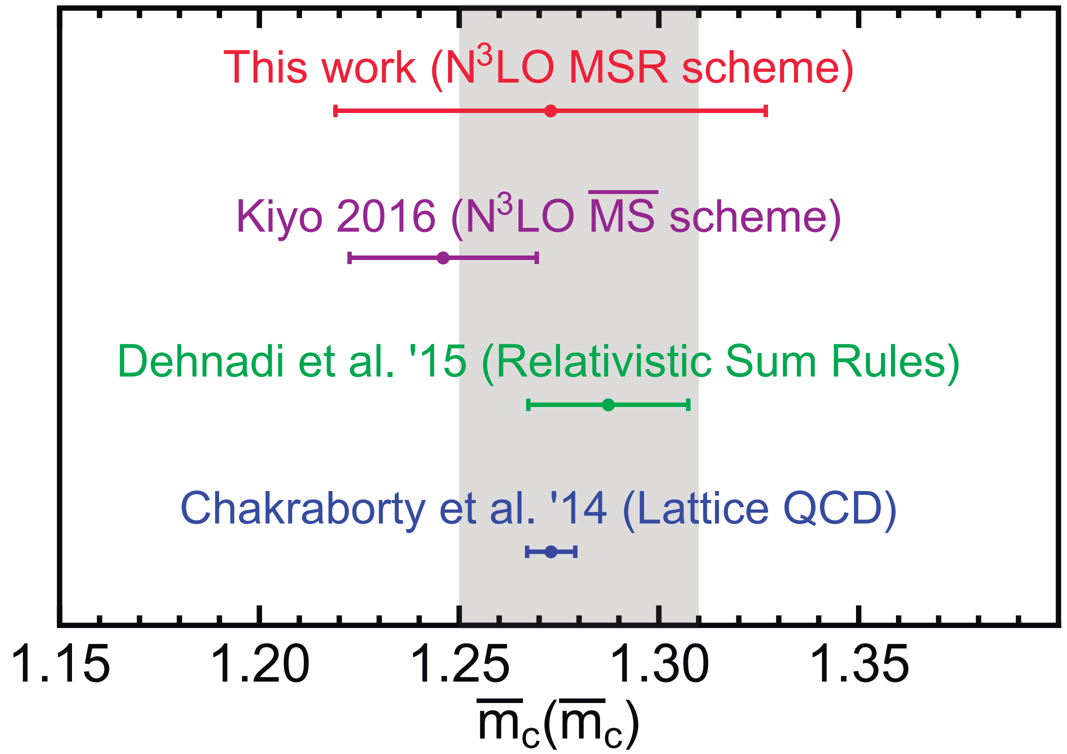

Same statement does not hold true for charm mass

Calibration of the Cornell model

Very preliminary!

Simple idea: can I relate the Cornell model parameters with QCD parameters?

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Strategy: generate QCD predictions for bottomonium masses up to n = 2 and scan over the bottom mass for a wide range. We keep $m_c = 0$ and $\alpha_s(1.3\text{ GeV})$ fixed.

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Cornell model is solved numerically with the Numerov algorithm

The Cornell model parameters are fit to the QCD predictions. Perturbative uncertainties propagated with a scan on scales

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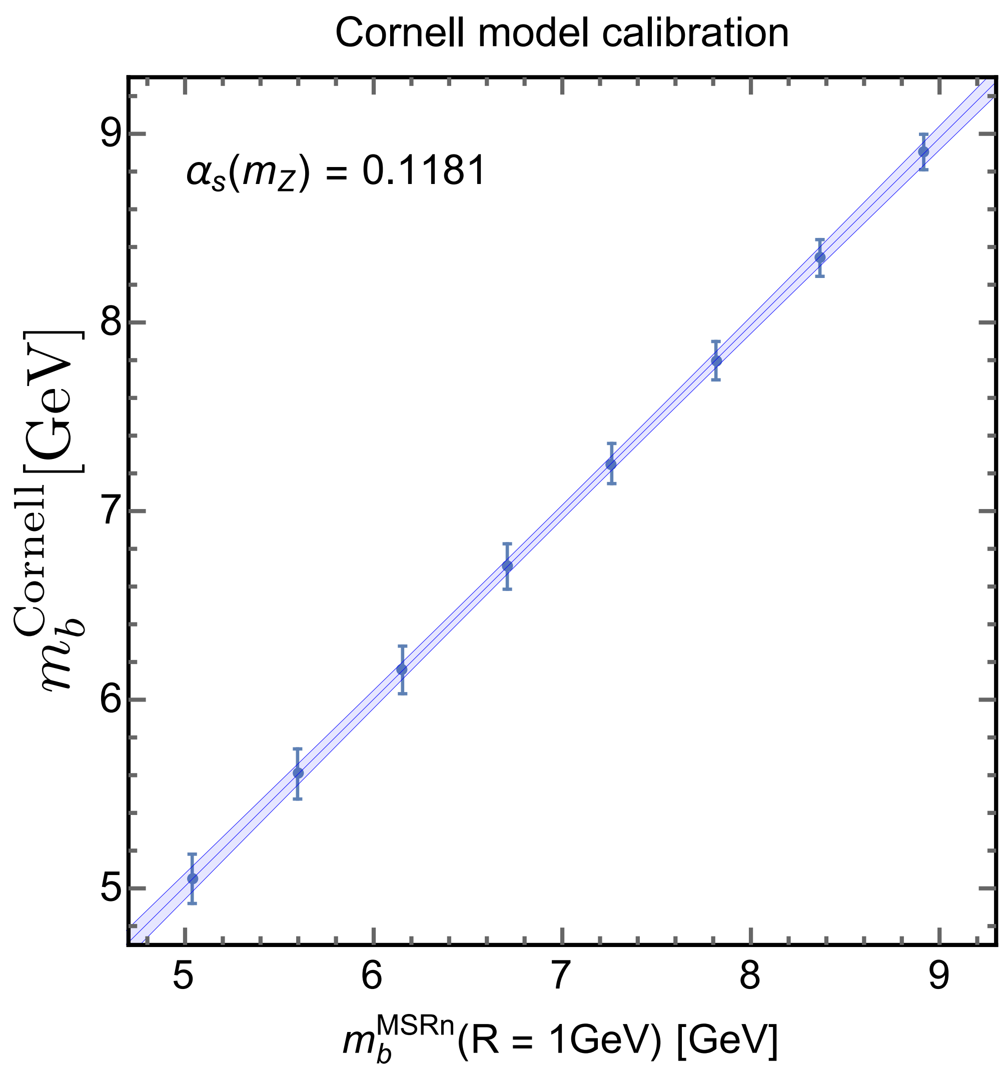

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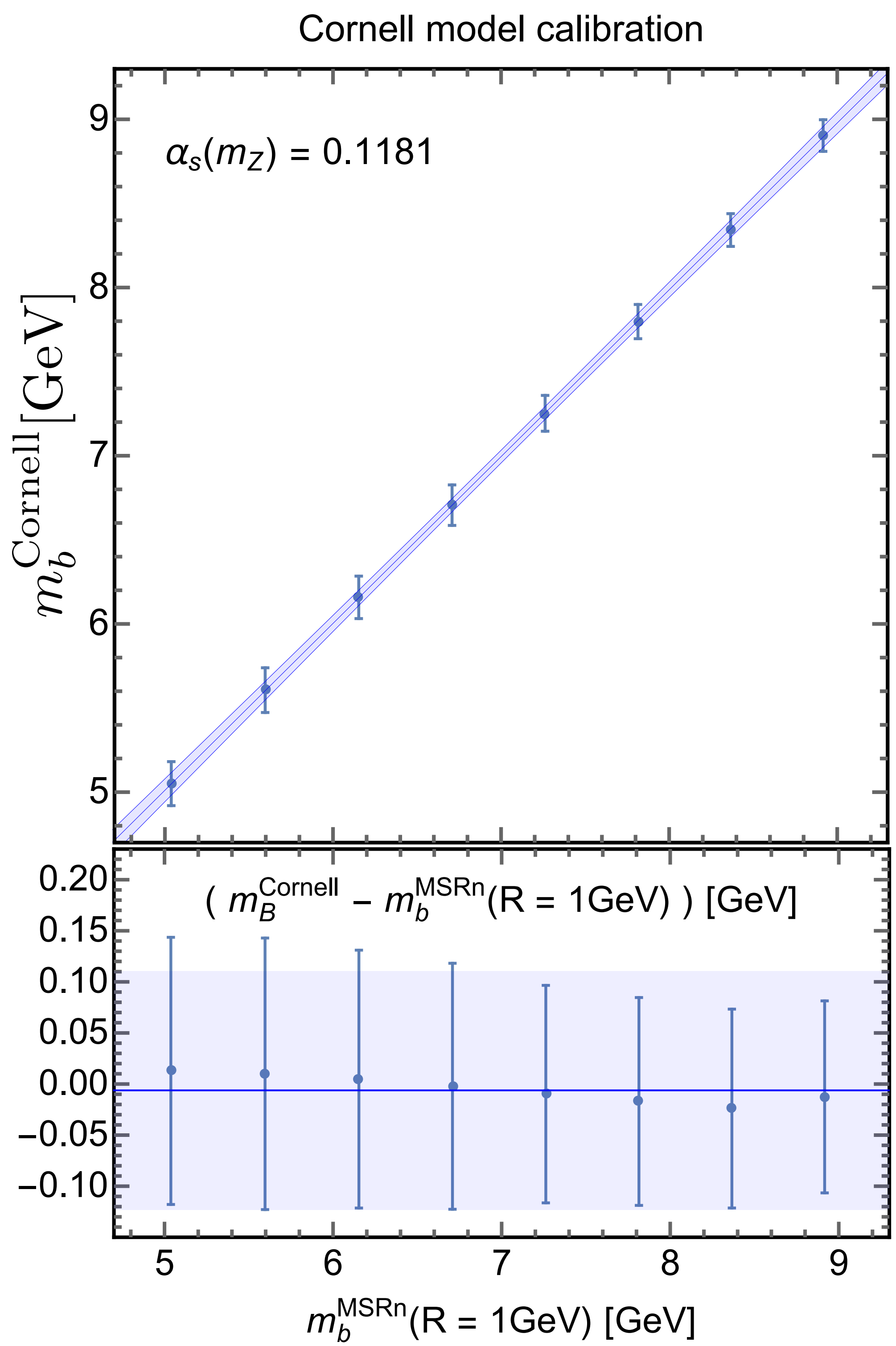

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If we choose R = 1GeV the intersect is very close to zero, and fully compatible with zero within errors.

Conclusions

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- Quarkonia masses are a good place to determine the quark masses with high precision.
- Employing a low-scale short-distance mass as the MSR is crucial to cancel de renormalon and avoid large logs.
- Charm mass effects in bottomonium are close to the decoupling limit: integrate out charm and add power corrections
- Effects from massive lighter quarks must then be incorporated into MSR mass, and the lighter quark can be integrated out.
- Very precise bottom mass determination, charm also good.
- This setup can be used to calibrate quark models such as Cornell