Perturbation Theory

for the Tau Hadronic Width

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Adler Function

The vector correlator is central to the τ hadronic width.

It's general perturbative expansion reads:

$$\Pi_V^{(1+0)}(s) = -\frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_\mu^n \sum_{k=0}^{n+1} c_{n,k} L^k \,, \quad L \equiv \ln \frac{-s}{\mu^2}$$

where $a_\mu \equiv \alpha_s(\mu)/\pi$.

Defining the Adler function as

$$D_V^{(1+0)}(s) \equiv -s \frac{\mathrm{d}}{\mathrm{d}s} \Pi_V^{(1+0)}(s) \,,$$

one arrives at

$$D_V^{(1+0)}(s) = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k \, c_{n,k} L^{k-1} \,.$$

Resumming the Log's with the scale choice $\mu^2 = -s \equiv Q^2$:

$$D_V^{(1+0)}(Q^2) = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} c_{n,1}\, a_Q^n \,.$$

This shows that only the coefficients $c_{n,1}$ are independent.

$$c_{0,1} = c_{1,1} = 1\,, \quad c_{2,1} = 1.640\,, \quad c_{3,1} = 6.371\,,$$

$$c_{4,1} = 49.076$$ (Baikov, Chetyrkin, Kühn 2008)

All other $c_{n,k}$ with $k > 1$ are related to lower $c_{m,1}$ ($m < n$) and β-function coefficients through the RG equation.

Numerically at $Q = M_\tau$:

$$4\pi^2 D_V^{(1+0)}(Q^2) = 1 + 0.1014 + 0.0169 + 0.0066 + 0.0052 + \ldots$$

Tau Moments

Define general τ moments (without factor $|V_{ud}|^2 S_{\mathrm{EW}}$):

$$R^w_{V/A}(s_0) \equiv 6\pi i \oint\limits_{|s|=s_0} \frac{ds}{s_0}\, w(s) \left[\Pi^{(1+0)}_{V/A}(s) + \frac{2s}{(s_0+2s)}\, \Pi^{(0)}_{V/A}(s) \right].$$

For $R_{\tau,V/A}$, the kinematic weight reads:

$$w_\tau(s) = \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + 2\frac{s}{M_\tau^2}\right).$$

And the general decomposition of $R^w_{\tau,V/A}(s_0)$:

$$R^w_{V/A}(s_0) = \frac{N_c}{2} \left[\delta^{\mathrm{tree}}_w + \delta^{(0)}_w(s_0) + \sum_{D\geq 2} \delta^{(D)}_{w,V/A}(s_0) + \delta^{\mathrm{DV}}_{w,V/A}(s_0) \right].$$

Introducing the dimensionless variable $x \equiv s/s_0$:

$$\delta^{(0)} = -2\pi i \oint\limits_{|x|=1} \frac{dx}{x}\,(1-x)^3(1+x)\,D_V^{(1+0)}(M_\tau^2 x)\,.$$

Inserting the general expansion of $D_V^{(1+0)}(M_\tau^2 x)$:

$$\delta^{(0)} = \sum_{n=1}^{\infty} a_\mu^n \sum_{k=1}^{n} k\,c_{n,k}\,\frac{1}{2\pi i}\oint\limits_{|x|=1} \frac{dx}{x}\,(1-x)^3\,(1+x)\ln^{k-1}\left(\frac{-M_\tau^2 x}{\mu^2}\right).$$

Setting the renormalisation scale $\mu = M_\tau$, FOPT follows:

$$\delta_{\rm FO}^{(0)} = \sum_{n=1}^{\infty} a(M_\tau^2)^n \sum_{k=1}^{n} k\,c_{n,k}\,J_{k-1}\,,$$

with

$$J_0 = 1\,, \quad J_1 = -\frac{19}{12}\,, \quad J_2 = \frac{265}{72} - \frac{1}{3}\,\pi^2\,, \quad J_3 = -\frac{3355}{288} + \frac{19}{12}\,\pi^2\,.$$

Setting the renormalisation scale $\mu^2 = -M_\tau^2 x$, CIPT follows:

$$\delta_{\rm CI}^{(0)} = \sum_{n=1}^{\infty} c_{n,1}\, J_n^a(M_\tau^2)\,,$$

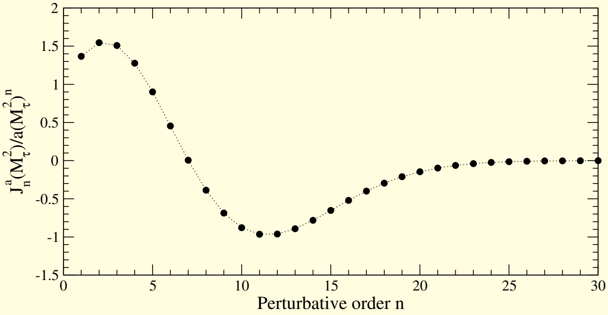
with

$$J_n^a(M_\tau^2) \equiv \frac{1}{2\pi i} \oint\limits_{|x|=1} \frac{dx}{x}\,(1-x)^3\,(1+x)\,a^n(-M_\tau^2 x)\,.$$

Numerically at $\alpha_s(M_\tau) = 0.3186$:

	a^1	a^2	a^3	a^4	a^5	
$\delta_{\rm FO}^{(0)} =$	0.101	$+0.054$	$+0.027$	$+0.013$	$(+0.006)$	$= 0.196\,(0.202)$
$\delta_{\rm CI}^{(0)} =$	0.137	$+0.026$	$+0.010$	$+0.007$	$(+0.003)$	$= 0.181\,(0.185)$

Geometric growth of $\delta_{\rm FO}^{(0)}$: $\Rightarrow c_{5,1} \approx 283$. (Also $\Rightarrow c_{4,1} \approx 52$!)



The contour integrals in CIPT display a problematic suppression around the 7th order.

A model for higher orders is required to gain further insight into the nature of the CIPT versus FOPT controversy.

Large-β_0 limit

Introduce new function to discuss Borel transform:

$$4\pi^2\, D_V^{(1+0)}(s) \equiv 1 + \widehat{D}(s) = 1 + \sum_{n=1}^{\infty} c_{n,1}\, a(Q^2)^n\,.$$

Then the Borel transform is defined by:

$$B[\widehat{D}](u) \equiv \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{c_{n+1,1}}{n!} \left(\frac{2u}{\beta_1}\right)^n .$$

$\widehat{D}(a)$ is given by the integral representation:

$$\widehat{D}(a) = \frac{2\pi}{\beta_1} \int_0^{\infty} \mathrm{d}u\, \mathrm{e}^{-\frac{2u}{\beta_1 a}} B[\widehat{D}](u)\,.$$

In the large-β_0 limit closed solution for $B[\widehat{D}](u)$:

$$B[\widehat{D}](u) = \frac{32}{3\pi}\,\frac{\mathrm{e}^{-Cu}}{(2-u)}\sum_{k=2}^{\infty}\frac{(-1)^k k}{[k^2-(1-u)^2]^2}\,.$$

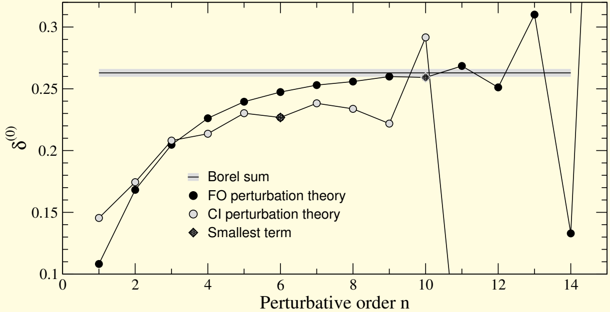
Scheme-dependent constant: $C_{\overline{\mathrm{MS}}} = -5/3$.

Poles for positive integer $u \geq 2$ (IR renormalons) and negative integer u (UV renormalons).

IR renormalons: fixed-sign contribution to $c_{n,1}$.

UV renormalons: alternating-sign contribution to $c_{n,1}$.

High orders dominated by poles close to $u = 0$.
($u = 2$ and $u = -1$.)



$$\delta^{(0)}_{\rm FO} = \sum_{n=1}^{\infty} \left[c_{n,1} + g_n \right] a(M_\tau^2)^n .$$

$$c_{n+1,1} = \left(\frac{\beta_1}{2}\right)^n n! \left[\frac{4}{9}\, \mathrm{e}^{-5/3} (-1)^n \left(n + \frac{7}{2}\right) + \frac{\mathrm{e}^{10/3}}{2^n} + \ldots \right],$$

$$g_{n+1} = \left(\frac{\beta_1}{2}\right)^n n! \left[-\frac{4}{9}\, \mathrm{e}^{-5/3} (-1)^n \left(n + \frac{16}{5}\right) - \frac{\mathrm{e}^{10/3}}{2^n} + \ldots \right].$$

IR Renormalon-Poles

General term in the Operator Product Expansion:

$$\widehat{C}_{O_d}(a_Q)\,\frac{\langle \widehat{O}_d \rangle}{Q^d} = [a_Q]^{\frac{\gamma^{(1)}_{O_d}}{\beta_1}} \left[\widehat{C}^{(0)}_{O_d} + \widehat{C}^{(1)}_{O_d}\, a_Q + \widehat{C}^{(2)}_{O_d}\, a_Q^2 + \ldots \right] \frac{\langle \widehat{O}_d \rangle}{Q^d} \,.$$

Express Q-dependence in terms of a_Q:

$$\frac{\widehat{C}_{O_d}(a_Q)}{Q^d} \sim \widehat{C}_{O_d}(a_Q)\,\mathrm{e}^{-\frac{d}{\beta_1 a_Q}}\,[a_Q]^{-d\frac{\beta_2}{\beta_1^2}} \exp\left\{ d \int\limits_0^{a_Q} \left[\frac{1}{\beta(a)} - \frac{1}{\beta_1 a^2} + \frac{\beta_2}{\beta_1^2 a} \right] \mathrm{d}a \right\}$$

$$\sim \widehat{C}_{O_d}(a_Q)\,\mathrm{e}^{-\frac{d}{\beta_1 a_Q}}\,[a_Q]^{-d\frac{\beta_2}{\beta_1^2}} \left[1 + b_1\, a_Q + b_2\, a_Q^2 + \ldots \right],$$

with

$$b_1 = \frac{d}{\beta_1^3}\left(\beta_2^2 - \beta_1\beta_3\right), \quad b_2 = \frac{b_1^2}{2} - \frac{d}{2\beta_1^4}\left(\beta_2^3 - 2\beta_1\beta_2\beta_3 + \beta_1^2\beta_4\right).$$

Take Ansatz for Borel transform of IR renormalon pole:

$$B[\widehat{D}_p^{\rm IR}](u) \equiv \frac{d_p^{\rm IR}}{(p-u)^{1+\tilde\gamma}} \Big[1 + \tilde b_1 (p-u) + \tilde b_2 (p-u)^2 + \ldots \Big] .$$

The imaginary ambiguity takes the form:

$$\mathrm{Im}\Big[\widehat{D}_p^{\rm IR}(a_Q)\Big] \sim \mathrm{e}^{-\frac{2p}{\beta_1 a_Q}} \, [a_Q]^{-\tilde\gamma} \left[1 + \tilde b_1 \frac{\beta_1}{2}\, \tilde\gamma\, a_Q + \tilde b_2 \frac{\beta_1^2}{4}\, \tilde\gamma(\tilde\gamma-1)\, a_Q^2 + \ldots \right] .$$

We can identify:

$$p = \frac{d}{2} , \quad \tilde\gamma = 2p\, \frac{\beta_2}{\beta_1^2} - \frac{\gamma_{O_d}^{(1)}}{\beta_1} ,$$

$$\tilde b_1 = \frac{2(b_1 + c_1)}{\beta_1 \tilde\gamma} , \quad \tilde b_2 = \frac{4(b_2 + b_1 c_1 + c_2)}{\beta_1^2\, \tilde\gamma(\tilde\gamma - 1)} .$$

with

$$c_1 \equiv \widehat{C}_{O_d}^{(1)} / \widehat{C}_{O_d}^{(0)} , \quad c_2 \equiv \widehat{C}_{O_d}^{(2)} / \widehat{C}_{O_d}^{(0)} .$$

Adler Function Model

(Beneke, MJ 2008)

To incorporate known renormalon structure, use Ansatz:

$$B[\widehat{D}](u) = B[\widehat{D}_1^{\mathrm{UV}}](u) + B[\widehat{D}_2^{\mathrm{IR}}](u) + B[\widehat{D}_3^{\mathrm{IR}}](u) + d_0^{\mathrm{PO}} + d_1^{\mathrm{PO}} u \, .$$

Fitting $c_{1,1}$ to $c_{5,1}$, the parameters are found to be:

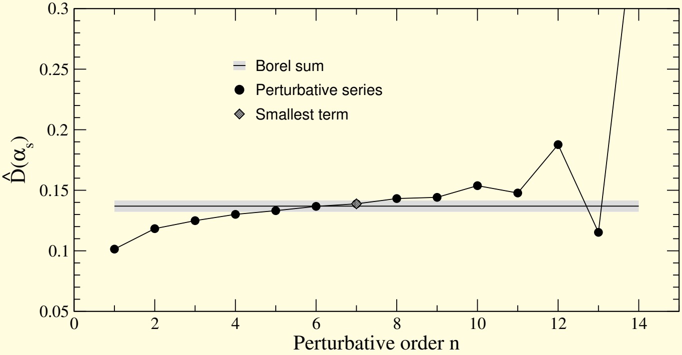
$$d_1^{\mathrm{UV}} = -1.56 \cdot 10^{-2} \, , \qquad d_2^{\mathrm{IR}} = 3.16 \, , \qquad d_3^{\mathrm{IR}} = -13.5 \, ,$$

$$d_0^{\mathrm{PO}} = 0.781 \, , \qquad d_1^{\mathrm{PO}} = 7.66 \cdot 10^{-3} \, .$$

Dropping input for $c_{5,1}$ and d_1^{PO} yields: $c_{5,1} \approx 280$.

Also stable result adding IR pole at $u = 4$ and dropping d_1^{PO}.

Adler Function



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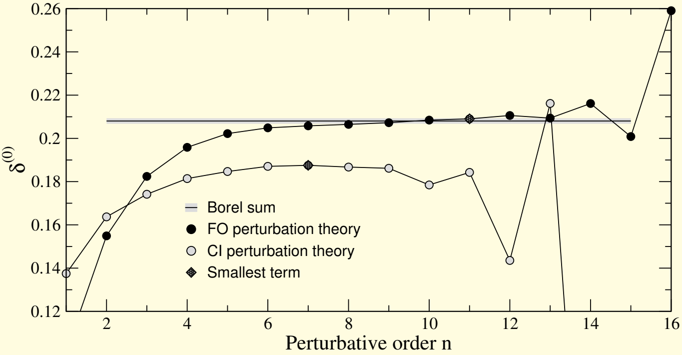
Tau width



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Model Dependence

The behaviour of the Borel model crucially depends on the residue of the gluon-condensate renormalon pole.

Assuming some sensitivity to the $u=2$ pole at intermediate orders (3-5), a fit to the known $c_{n,1}$ yields $d_2^{\mathrm{IR}} \approx 3.2$.

For small d_2^{IR}, models can be constructed for which Contour-improved PT is the preferred resummation.

Hence, to make progress the value of d_2^{IR} should be corroborated. Two possible routes:

i) As the renormalon ambiguity is universal, employ PT series of other correlators to obtain additional information.

ii) Determine d_2^{IR} from the lattice. Not possible directly for the Adler function, but for the plaquette.

Outlook

- Lecture 3: Multi-moment analysis for α_s, duality violation.
- Lecture 4: Description of exclusive τ decay distributions.

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Frohe Weihnacht!