Residual symmetries in the lepton mass matrices

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Tri-bimaximal mixing:

$$U_{\mathrm{TBM}} = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

ruled out!

$$\sin^2\theta_{13} = 0.0227\ {}^{+0.0023}_{-0.0024}$$

Gonzalez-Garcia et al. (2012)

$$\sin^2\theta_{12} = 0.302\ {}^{+0.013}_{-0.012}$$

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

$U = (U_{\alpha j}) = (u_1, u_2, u_3)$ with columns u_j

Albright, Rodejohann (2008): TM_1, TM_2 still valid!

$$\mathrm{TM}_1: \quad u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad \mathrm{TM}_2: \quad u_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathrm{TM}_1: \quad s_{12}^2 = 1 - \frac{2}{3c_{13}^2} < \frac{1}{3}, \quad \cos\delta \tan 2\theta_{23} \simeq -\frac{1}{2\sqrt{2}s_{13}} \left(1 - \frac{7}{2} s_{13}^2\right)$$

$$\mathrm{TM}_2: \quad s_{12}^2 = \frac{1}{3c_{13}^2} > \frac{1}{3}, \quad \cos\delta \tan 2\theta_{23} \simeq \frac{1}{\sqrt{2}s_{13}} \left(1 - \frac{5}{4} s_{13}^2\right)$$

Residual symmetries

Fixing the notation:

Mass terms: Majorana neutrinos

$$\mathcal{L}_{\mathrm{mass}} = -\bar{\ell}_L M_\ell \ell_R + \frac{1}{2}\nu_L^T C^{-1} \mathcal{M}_\nu \nu_L + \mathrm{H.c.}$$

Diagonalization:

$$U_\ell^\dagger M_\ell M_\ell^\dagger U_\ell = \mathrm{diag}\left(m_e^2, m_\mu^2, m_\tau^2\right), \quad U_\nu^T \mathcal{M}_\nu U_\nu = \mathrm{diag}\left(m_1, m_2, m_3\right)$$

Mixing matrix: $U = U_\ell^\dagger U_\nu$

$$V_\ell(\alpha) \equiv U_\ell \,\mathrm{diag}\left(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}\right) U_\ell^\dagger$$

$$V_\nu(\epsilon) \equiv U_\nu \,\mathrm{diag}\left(\epsilon_1, \epsilon_2, \epsilon_3\right) U_\nu^\dagger \quad \text{with} \quad \epsilon_j^2 = 1$$

Residual symmetries

Invariance of the mass matrices:

$$V_\ell(\alpha)^\dagger M_\ell M_\ell^\dagger V_\ell(\alpha) = M_\ell M_\ell^\dagger, \quad V_\nu(\epsilon)^T \mathcal{M}_\nu V_\nu(\epsilon) = \mathcal{M}_\nu$$

Remarks:

- $V_\ell(\alpha) \in U(1) \times U(1) \times U(1)$, $V_\nu(\epsilon) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
- $V_\ell(\alpha)$, $V_\nu(\epsilon)$ depend on VEVs and Yukawa coupling constants
- Invariance of mass matrices V_ℓ, V_ν contains no information beyond diagonalizability

Residual symmetries

Idea of residual symmetries: C.W. Lam

- Weak basis $\Rightarrow \ell_L$, ν_L in same multiplet of G
- G broken to different subgroups in charged-lepton and neutrino sectors:

$$G_\ell \subseteq U(1) \times U(1) \times U(1), \quad G_\nu \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

- For simplicity:
  One generator T of G_ℓ, one generator S of G_ν:

$$T^\dagger M_\ell M_\ell^\dagger T = M_\ell M_\ell^\dagger, \quad S^T \mathcal{M}_\nu S = \mathcal{M}_\nu$$

- For simplicity:
  T has three different eigenvalues
- Then T and S determine one column of U!

Residual symmetries

Why is this so?

1. $S^2 = \mathbb{1} \Rightarrow S = \pm(2uu^\dagger - \mathbb{1})$ with $Su = \pm u$
2. $U_\ell^\dagger T U_\ell = \widetilde{T}$ diagonal
3. $U_\ell^\dagger u$ column in mixing matrix
4. Two matrices S_1, S_2 with $S_j^T \mathcal{M}_\nu S_j = \mathcal{M}_\nu \Rightarrow$ mixing matrix U completely determined

Theorem

If $S^T \mathcal{M}_\nu S = \mathcal{M}_\nu$ with $S = \pm(2uu^\dagger - \mathbb{1})$, then $\mathcal{M}_\nu u \propto u^*$.

Remark: $U_\ell^\dagger u$ determined by the group! It does not contain parameters of the model.

Residual symmetries

Two ways to tackle residual symmetries for the purpose of determination of possible flavour symmetry groups:

1. Scanning finite groups
2. Solving relations involving roots of unity

Holthausen, Lim, Lindner (2013):

$G_\nu = \mathbb{Z}_2 \times \mathbb{Z}_2$, group results within 3σ of fitted s^2_{ij}

a) Assumptions: ord $G < 1536 = 3 \times 2^9$ (with one exception), G_ℓ generated by $\widetilde{T} = \mathrm{diag}(1, \omega, \omega^2)$ with $\omega = e^{2\pi i/3}$

n	G	s^2_{12}	s^2_{13}	s^2_{23}
5	$\Delta(6 \times 10^2)$	0.3432	0.0288	0.3791
		0.3432	0.0288	0.6209
9	$(\mathbb{Z}_{18} \times \mathbb{Z}_6) \rtimes S_3$	0.3402	0.0201	0.3992
		0.3402	0.0201	0.6008
16	$\Delta(6 \times 16^2)$	0.3420	0.0254	0.3867
		0.3420	0.0254	0.6133

b) Assumptions: ord $G < 512$, G_ℓ Abelian $\Rightarrow$ no candidates!

Group scan using GAP

Model-building addendum to group scan (Grimus, Lavoura (2013)):

$$s_{23}^2 = \frac{1}{2}\left(1 \pm \frac{\sqrt{2s_{13}^2 - 3s_{13}^4}}{c_{13}^2}\right), \quad \cos\delta = \mp 1$$

$$U = \begin{pmatrix} \frac{1}{\sqrt{6}}\left(1 + e^{i\alpha}\right) & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\left(1 - e^{-i\alpha}\right) \\ \frac{1}{\sqrt{6}}\left(\omega^2 + \omega e^{i\alpha}\right) & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\left(\omega - \omega^2 e^{-i\alpha}\right) \\ \frac{1}{\sqrt{6}}\left(\omega + \omega^2 e^{i\alpha}\right) & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\left(\omega^2 - \omega e^{-i\alpha}\right) \end{pmatrix}$$

$\alpha/(2\pi)$	s_{13}^2	s_{23}^2
2/5, 3/5	0.028818	0.379101 or 0.620899
1/16, 15/16	0.025373	0.386653 or 0.613347
1/18, 5/18, ..., 13/18, 17/18	0.020102	0.399242 or 0.600758

Residual symmetries and roots of unity

Basic assumption: Flavour group G finite! (finitely generated)
Mixing matrix: $U = (U_{\alpha j})$ $(\alpha = e, \mu, \tau,\ j = 1, 2, 3)$

- G_ℓ generated by T, G_ν generated by S
- $\det S = 1 \Rightarrow S = 2uu^\dagger - \mathbb{1}$
- Finiteness $\Rightarrow \exists m, n \in \mathbb{N}$ such that $T^m = S^2 = (ST)^n = \mathbb{1}$

T has eigenvalues $e^{i\phi_\alpha}$, ST has eigenvalues $\lambda_j \Rightarrow$
$\mathrm{Tr}\,(ST) = \lambda_1 + \lambda_2 + \lambda_3$

Trace and determinant of ST

Hernandez, Smirnov (2012)
u i-th column of $U \Rightarrow$ two equations for 6 roots of unity:

$$\sum_{\alpha=e,\mu,\tau} \left(2\,|U_{\alpha i}|^2 - 1\right) e^{i\phi_\alpha} = \lambda_1 + \lambda_2 + \lambda_3 \quad \text{and} \quad \prod_\alpha e^{i\phi_\alpha} = \lambda_1\lambda_2\lambda_3$$

TM_1 and roots of unity

Which finite group can enforce TM_1? Grimus (2013)

$$\text{TM}_1: \quad u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \Rightarrow \begin{array}{l} 2\,|U_{e1}|^2 - 1 = \frac{1}{3} \\ 2\,|U_{\mu 1}|^2 - 1 = -\frac{2}{3} \\ 2\,|U_{\tau 1}|^2 - 1 = -\frac{2}{3} \end{array}$$

Vanishing sum of roots of unity:

$$-e^{i\phi_e} + 2e^{i\phi_\mu} + 2e^{i\phi_\tau} + 3\lambda_1 + 3\lambda_2 + 3\lambda_2 = 0$$

Solution by theorem of Conway and Jones (1976)

TM_1 and roots of unity

Formal sums of roots of unity: ring over rational numbers
$\omega = e^{2\pi i/3}$, $\beta = e^{2\pi i/5}$, $\gamma = e^{2\pi i/7}$

Theorem (Conway and Jones (1976))

Let $\mathcal{S}$ be a non-empty vanishing sum of length at most 9.
Then either $\mathcal{S}$ involves θ, $\theta\omega$, $\theta\omega^2$ for some root θ,
or $\mathcal{S}$ is similar to one of

$$
\begin{aligned}
&1 + \beta + \beta^2 + \beta^3 + \beta^4, \\
&-\omega - \omega^2 + \beta + \beta^2 + \beta^3 + \beta^4, \\
&1 + \beta + \beta^2 - (\omega + \omega^2)(\beta^2 + \beta^3), \\
&1 + \gamma + \gamma^2 + \gamma^3 + \gamma^4 + \gamma^5 + \gamma^6, \\
&-\omega - \omega^2 + \gamma + \gamma^2 + \gamma^3 + \gamma^4 + \gamma^5 + \gamma^6, \\
&\beta + \beta^4 - (\omega + \omega^2)(1 + \beta^2 + \beta^3), \\
&1 + \gamma^2 + \gamma^3 + \gamma^4 + \gamma^5 - (\omega + \omega^2)(\gamma + \gamma^6), \\
&1 - (\omega + \omega^2)(\beta + \beta^2 + \beta^3 + \beta^4).
\end{aligned}
$$

TM_1 and roots of unity

Solution:

$e^{i\phi_e} = \eta$, $e^{i\phi_\mu} = \eta\omega$, $e^{i\phi_\tau} = \eta\omega^2$, $\lambda_1 = \epsilon$, $\lambda_2 = -\epsilon$, $\lambda_3 = \eta$
η is an arbitrary root of unity, $\epsilon = \pm i\eta$

In basis where charged lepton mass matrix is diagonal:

$$\widetilde{T} = \eta \, \mathrm{diag}\left(1, \omega, \omega^2\right)$$

$$u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \Rightarrow \widetilde{S} = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & -2 & 1 \\ -2 & 1 & -2 \end{pmatrix}$$

$\widetilde{T}$ and $\widetilde{S}$ generate group $\mathbb{Z}_q \times S_4$ with η being a primitive root of order q.

TM_1 and roots of unity

Another basis:

$$U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \quad \text{with} \quad \omega = e^{2\pi i/3} = \frac{-1 + i\sqrt{3}}{2}$$

$$S = U_\omega \tilde{S} U_\omega^\dagger = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T = U_\omega \tilde{T} U_\omega^\dagger = \eta \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$E^\dagger \left(M_\ell M_\ell^\dagger \right) E = M_\ell M_\ell^\dagger \;\Rightarrow\; U_\omega^\dagger \left(M_\ell M_\ell^\dagger \right) U_\omega \text{is diagonal}$$

TM_1 and roots of unity

$$Su = u \quad \Rightarrow \quad u = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Mechanism for TM_1:

Lavoura, de Madeiros Varzielas (2012); Grimus (2013)

U_ω diagonalizes $M_\ell M_\ell^\dagger$ and u eigenvector of $\mathcal{M}_\nu \Rightarrow$

$$U_\omega^\dagger u = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \text{ is column in mixing matrix}$$

Example: S_4 and type II seesaw mechanism
Needs 7 scalar gauge doublets in $\mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}'$
and 4 gauge triplets in $\mathbf{1} \oplus \mathbf{3}'$ + VEV alignment

$$M_\ell = \begin{pmatrix} a & b+c & b-c \\ b-c & a & b+c \\ b+c & b-c & a \end{pmatrix}, \quad \mathcal{M}_\nu = \begin{pmatrix} A & B & -B \\ B & A & C \\ -B & C & A \end{pmatrix}$$

Residual symmetries and caveats

Notation:

G = flavour symmetry group of the Lagrangian

$\bar{G}$ = group determined by residual symmetries in $M_\ell M_\ell^\dagger$ and $\mathcal{M}_\nu$

- Restriction:
  - Symmetry group G of Lagrangian is finitely generated
  - Neutrinos have Majorana nature
- Possible relationship between G and $\bar{G}$:
  - $\bar{G} \subset U(3)$ due to 3 families
  - Method is purely group-theoretical and uses only information contained in the mass matrices $\Rightarrow$ $\bar{G}$ can at most yield $D(G)$
  - Accidental symmetries in the mass matrices $\Rightarrow$ $\bar{G}$ not even a subgroup of $D(G)$
- Total breaking of G:
  Method not applicable, though model might be predictive

Thank you for your attention!