Dispersive treatment of $\eta \to 3\pi$ and the determination of light quark masses

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Outline

Introduction

Lattice determination of m_u, m_d and m_s
- FLAG
- FLAG phase 2
- Isospin limit
- Isospin breaking

Quark mass ratios from CHPT

A new dispersive analysis of $\eta \to 3\pi$
- Isospin breaking

Summary and Outlook

Quark masses

QCD Lagrangian:

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4g^2}\mathrm{Tr}G_{\mu\nu}G^{\mu\nu} + \sum_i \bar{q}_i(i\not{D} - m_{q_i})q_i + \sum_j \bar{Q}_j(i\not{D} - m_{Q_j})Q_j$$

- In the limit $m_{q_i} \to 0$ and $m_{Q_j} \to \infty$: $M_{\mathrm{hadrons}} \propto \Lambda$
- Observe that $m_{q_i} \ll \Lambda$ while $m_{Q_j} \gg \Lambda$ $[\Lambda \sim M_N]$
- Quarks do not propagate:
  quark masses are coupling constants! (not observables)

  they depend on the renormalization scale μ (like α_s)
  for light quarks by convention: $\mu = 2\ \mathrm{GeV}$

  In the following

$$m_q \equiv m_q(2\ \mathrm{GeV})$$

How to determine quark masses

- From their influence on the spectrum
  - $m_Q \gg \Lambda$
  $$M_{\bar{Q}q_i} = m_Q + \mathcal{O}(\Lambda)$$
  - $m_q \ll \Lambda$

$$M_{\bar{q}_i q_j} = M_{0\,ij} + \mathcal{O}(m_{q_i}, m_{q_j}) \qquad M_{0\,ij} = \mathcal{O}(\Lambda)$$

In both cases need to understand the $\mathcal{O}(\Lambda)$ term

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$$M_{\bar{q}_i q_j} = M_{0\,ij} + \mathcal{O}(m_{q_i}, m_{q_j}) \qquad M_{0\,ij} = \mathcal{O}(\Lambda)$$

In both cases need to understand the $\mathcal{O}(\Lambda)$ term

- From their influence on any other observable

Quark masses are coupling constants
$\Rightarrow$ exploit the sensitivity to them of any observable
[e.g. η and τ decays]

How to determine quark masses

Approaches

1. Chiral perturbation theory — spectrum + low energy observables
2. Sum rules — spectral functions
3. Lattice QCD — spectrum

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Summary and Outlook

Lattice determinations of quark masses

First principle (and brute force) method:

- QCD Lagrangian as input
- calculate the spectrum of the low-lying states for different quark masses
- tune the values of the quark masses such that the QCD spectrum is reproduced
- the comparison is made for mass ratios — one mass or a dimensionful quantity (*e.g.* F_π) can be used to set the scale

Lattice determinations of quark masses

Systematic effects:

- finite volume
  exponentially suppressed effects, easy to correct for
- finite lattice spacing
  powerlike effects, extrapolate numerically
- unphysical quark masses
  extrapolate numerically with guidance from CHPT
  BMW and PACS-CS: simulations at the physical point!
- renormalization
  relation between bare (input parameter) and renormalized (physically relevant quantity) quark masses not easy

What/Who is FLAG (phase 1)?

FLAG = FLAVIAnet Lattice Averaging Group

Members:

Gilberto Colangelo (Bern)
Stephan Dürr (Jülich, BMW)
Andreas Jüttner (Southampton, RBC/UKQCD)
Laurent Lellouch (Marseille, BMW)
Heiri Leutwyler (Bern)
Vittorio Lubicz (Rome 3, ETM)
Silvia Necco (CERN, Alpha)
Chris Sachrajda (Southampton, RBC/UKQCD)
Silvano Simula (Rome 3, ETM)
Tassos Vladikas (Rome 2, Alpha and ETM)
Urs Wenger (Bern, ETM)
Hartmut Wittig (Mainz, Alpha)

What/Who is FLAG (phase 1)?

FLAG = FLAVIAnet Lattice Averaging Group

History and status:

- Beginning: FLAVIAnet meeting, Orsay, November 2007
- Start of the actual work: Bern, March 2008
- ...
- first paper appeared in November 2010 arXiv.1011.4408
  updated and published in May 2011 on EPJC
- webpage made public in 2011:
  `http://itpwiki.unibe.ch/flag`

What exactly did FLAG-1 offer?

An answer to the questions

- what is the current lattice value for quantity X?
- what is a reliable estimate of the uncertainty?

in a way easily accessible to non-experts

Quantities considered in the first edition:

- light quark masses
- LEC
- decay constants (of pions and kaons)
- form factors (of pions and kaons)
- B_K

Color coding – FLAG-1 definition

[subject to change before each new edition]

- chiral extrapolation
  - ★ $M_{\pi,\text{min}} < 250$ MeV
  - ● $250\text{ MeV} \leq M_{\pi,\text{min}} \leq 400$ MeV
  - ■ $M_{\pi,\text{min}} > 400$ MeV

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- continuum extrapolation
  - ★ 3 or more lattice spacings, at least 2 points below 0.1 fm
  - ● 2 or more lattice spacings, at least 1 point below 0.1 fm
  - ■ otherwise

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- finite volume effects
  - ★ $(M_\pi L)_{\mathrm{min}} > 4$ or at least 3 volumes
  - ● $(M_\pi L)_{\mathrm{min}} > 3$ and at least 2 volumes
  - ■ otherwise

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  - ■ otherwise
- renormalization (where applicable)
  - ★ non-perturbative
  - ● 2-loop perturbation theory (well behaved series)
  - ■ otherwise

Averages

Different lattice results were averaged *if*

- published
  [lattice proceedings not enough]
- no red tags
- same N_f
  [no average of $N_f = 2$ and $N_f = 3$ calculations]

Final FLAG number:

- average or single *no-red-tag* $N_f = 3$ number (if available)
- average or single *no-red-tag* $N_f = 2$ number (if available)

If *both* $N_f = 3$ *and* $N_f = 2$ numbers available:

agreement $\Rightarrow$ more confidence in the final number

Similar initiative: Laiho, Lunghi and Van de Water

- began in 2009 to provide lattice-QCD inputs for the CKM unitarity-triangle analysis and other flavor-physics phenomenology LLVdW, Phys.Rev. D81 (2010) 034503
- main differences wrt FLAG-1:
  - only include $N_f = 2 + 1$ flavor results
  - no strict publication-only rule provided complete and reasonable systematic error budgets
  - heavy-quark quantities included from the start
  - unitarity triangle fits with lattice input
  - whenever a source of error is at all correlated between two lattice calculations (e.g. use the same gauge configurations, same theoretical tools, or experimental inputs), conservatively assume that the degree-of-correlation is 100%
- web page (`www.latticeaverages.org`) also popular

FLAG-2

FLAG = Flavour Lattice Averaging Group
has now entered its phase 2 and has been extended in various directions

- quantities to be reviewed
  main extension: light quarks $\to$ + heavy quarks
- represented lattice collaborations:
  Alpha, BMW, ETMC, RBC/UKQCD $\to$ + CLS, Fermilab, HPQCD, JLQCD, MILC, PACS-CS, SWME
- represented world regions: Europe $\to$ + Japan and US
- number of people: 12 $\to$ 28

FLAG-2 organization

- Advisory Board:
  S. Aoki, C. Bernard, C. Sachrajda
- Editorial Board:
  GC, H. Leutwyler, T. Vladikas, U. Wenger
- Working Groups
  - Quark masses — L. Lellouch, T. Blum, V. Lubicz
  - V_{us}, V_{ud} — A. Jüttner, T. Kaneko, S. Simula
  - LEC — S. Dürr, H. Fukaya, S. Necco
  - B_K — H. Wittig, J. Laiho, S. Sharpe
  - α_s — R. Sommer, R. Horsley, T. Onogi
  - f_B, B_B — A. El Khadra, Y. Aoki, M. Della Morte
  - $B \to H\ell\nu$ — R.Van de Water, E.Lunghi, C.Pena, J.Shigemitsu

FLAG-2 plans and rules

- next review: work in progress (first half 2013)
- regularly update the webpage
- new published review: every 2nd year

- some internal FLAG rules
  - members of the advisory board have a 4-year mandate
  - AB = EU+J+US
  - regular members can stay longer
  - replacements must keep/improve the balance of FLAG
  - WG members belong to 3 different lattice coll.
  - a paper is not reviewed (color-coded) by an author

FLAG-2 status

- kick-off meeting: Les Houches, May 7-11 2012
- work on the update of the review is in progress
- WG for the new sections are working on defining their own quality criteria
- first draft (internal) of the new review: September 2012
- publication of the new review: early 2013
- it will cover all results published until December 31 2012

FLAG-2 status

Two important policy changes wrt FLAG-1

- use of one-loop renormalization will not necessarily mean a red tag
- error calculation for the averages:
  LLVdW procedure will be adopted as default
  (final error may be stretched if not convincingly conservative)

Quark masses

Collaboration	publ.	$m_{ud} \to 0$	$a \to 0$	FV	renorm.	m_{ud}	m_s
PACS-CS 12	P	★	■	■	★	3.12(24)(8)	83.60(0.58)(2.23)
RBC/UKQCD 12	C	★	●	★	★	3.39(9)(4)(2)(7)	94.2(1.9)(1.0)(0.4)(2.1)
LVdW 11	C	●	★	★	●	3.31(7)(20)(17)	94.2(1.4)(3.2)(4.7)
PACS-CS 10	A	★	■	■	★	2.78(27)	86.7(2.3)
MILC 10A	C	●	★	★	●	3.19(4)(5)(16)	–
HPQCD 10	A	●	★	★	★	3.39(6)	92.2(1.3)
BMW 10A, 10B$^+$	A	★	★	★	★	3.469(47)(48)	95.5(1.1)(1.5)
RBC/UKQCD 10A	A	●	●	★	★	3.59(13)(14)(8)	96.2(1.6)(0.2)(2.1)
Blum 10	A	●	■	●	★	3.44(12)(22)	97.6(2.9)(5.5)
PACS-CS 09	A	★	■	■	★	2.97(28)(3)	92.75(58)(95)
HPQCD 09	A	●	★	★	★	3.40(7)	92.4(1.5)
MILC 09A	C	●	★	★	●	3.25 (1)(7)(16)(0)	89.0(0.2)(1.6)(4.5)(0.1)
MILC 09	A	●	★	★	●	3.2(0)(1)(2)(0)	88(0)(3)(4)(0)
PACS-CS 08	A	★	■	■	■	2.527(47)	72.72(78)
RBC/UKQCD 08	A	●	■	★	★	3.72(16)(33)(18)	107.3(4.4)(9.7)(4.9)
CP-PACS/ JLQCD 07	A	■	★	★	■	$3.55(19)(^{+56}_{-20})$	$90.1(4.3)(^{+16.7}_{-4.3})$
HPQCD 05	A	●	●	●	●	3.2(0)(2)(2)(0)	87(0)(4)(4)(0)
MILC 04, HPQCD/ MILC/UKQCD 04	A	●	●	●	■	2.8(0)(1)(3)(0)	76(0)(3)(7)(0)

$N_f = 2 + 1$

Quark masses

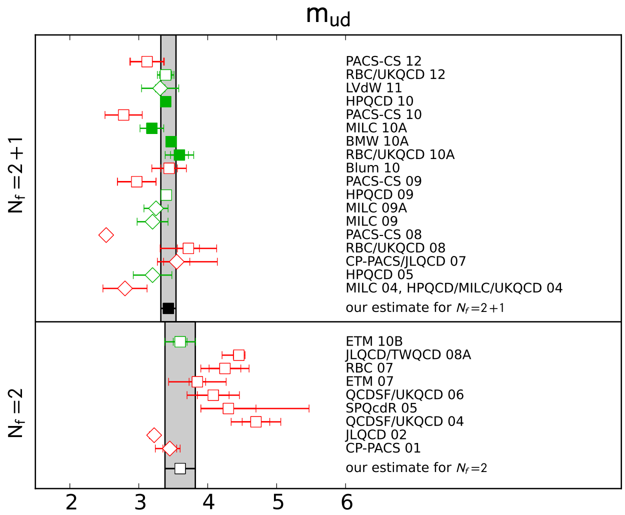
Collaboration	publ.	$m_{ud} \to 0$	$a \to 0$	FV	renorm.	m_{ud}	m_s
ALPHA 12	P	●	★	★	★	–	102(3)(1)
Dürr 11	A	●	★	●	–	3.52(10)(9)	97.0(2.6)(2.5)
ETM 10B	A	●	★	●	★	3.6(1)(2)	95(2)(6)
JLQCD/TWQCD 08A	A	●	■	■	★	$4.452(81)(38)\left(^{+0}_{-227}\right)$	–
RBC 07†	A	■	■	★	★	4.25(23)(26)	119.5(5.6)(7.4)
ETM 07	A	●	■	●	★	3.85(12)(40)	105(3)(9)
QCDSF/ UKQCD 06	A	■	★	■	★	4.08(23)(19)(23)	111(6)(4)(6)
SPQcdR 05	A	■	●	●	★	$4.3(4)(^{+1.1}_{-0.0})$	$101(8)(^{+25}_{-0})$
ALPHA 05	A	■	●	★	★	–	97(4)(18)
QCDSF/ UKQCD 04	A	■	★	■	★	4.7(2)(3)	119(5)(8)
JLQCD 02	A	■	■	●	■	$3.223(^{+46}_{-69})$	$84.5(^{+12.0}_{-1.7})$
CP-PACS 01	A	■	■	★	■	$3.45(10)(^{+11}_{-18})$	$89(2)(^{+2}_{-6})^\star$

$N_f = 2$

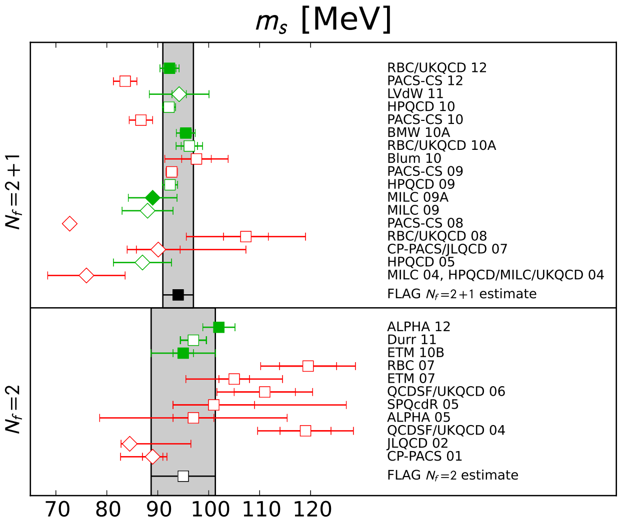
Quark masses

Collaboration	N_f	publ.	$m_{ud} \to 0$	$a \to 0$	FV	m_s/m_{ud}
PACS-CS 12	2+1	P	★	■	■	26.8(2.0)
LVdW 11	2+1	C	●	★	★	28.4(0.5)(1.3)
BMW 10A, 10B	2+1	A	★	★	★	27.53(20)(8)
RBC/UKQCD 10A	2+1	A	●	●	★	26.8(0.8)(1.1)
Blum 10	2+1	A	●	■	●	28.31(0.29)(1.77)
PACS-CS 09	2+1	A	★	■	■	31.2(2.7)
MILC 09A	2+1	C	●	★	★	27.41(5)(22)(0)(4)
MILC 09	2+1	A	●	★	★	27.2(1)(3)(0)(0)
PACS-CS 08	2+1	A	★	■	■	28.8(4)
RBC/UKQCD 08	2+1	A	●	■	★	28.8(0.4)(1.6)
MILC 04, HPQCD/ MILC/UKQCD 04	2+1	A	●	●	●	27.4(1)(4)(0)(1)
ETM 10B	2	A	●	★	●	27.3(5)(7)
RBC 07	2	A	■	■	★	28.10(38)
ETM 07	2	A	●	■	●	27.3(0.3)(1.2)
QCDSF/UKQCD 06	2	A	■	★	■	27.2(3.2)

Quark masses



Quark masses



FLAG estimates of the light quark masses

$N_f = 2+1$:

$$\begin{aligned} m_s &= 94 \pm 3 \text{ MeV} \\ m_{ud} &= 3.43 \pm 0.11 \text{ MeV} \\ \frac{m_s}{m_{ud}} &= 27.4 \pm 0.4 \end{aligned}$$

$N_f = 2$:

$$\begin{aligned} m_s &= 95 \pm 2 \pm 6 \text{ MeV} \\ m_{ud} &= 3.6 \pm 0.1 \pm 0.2 \text{ MeV} \\ \frac{m_s}{m_{ud}} &= 27.3 \pm 0.5 \pm 0.7 \end{aligned}$$

PDG 2010: $m_s = 101^{+29}_{-21}$ $m_{ud} = 3.77^{+1.03}_{-0.77}$ $\frac{m_s}{m_{ud}} = 26 \pm 4$

Isospin breaking on the lattice

- all lattice results described so far: isospin limit

$$m_u = m_d \qquad \text{and} \qquad \alpha_{em} = 0$$

- need as input M_π and M_K in this limit
  at the current level of precision

  the difference $\qquad M_P - M_P(\alpha_{em} = 0) \qquad$ matters!

  Remark: $M_{\pi^+}(\alpha_{em} = 0) \simeq M_{\pi^0}(\alpha_{em} = 0) \simeq M_{\pi^0}$
  $\Rightarrow$ need to know $(M_{K^+} - M_{K^0})_{em}$

- alternatively:
  simulate QCD+QED on the lattice and compare to the real-world spectrum

Electromagnetic effects in lattice calculations

Lattice calculations with QCD+QED

- Duncan, Eichten, Thacker (96)
  QCD quenched approximation, estimate for meson mass em splittings; systematic error not estimated

$$(M_{K^+} - M_{K^0})_{em} = 1.9 \text{ MeV}$$

- Blum et al. (RBC 07)
  QCD with 2 dynamical quarks, but QED quenched

$$(M_{K^+} - M_{K^0})_{em} = 1.443(55) \text{ MeV}$$

- Blum et al. (10)
  QCD with 3 dynamical quarks (RBC/UKQCD configurations), but QED quenched

$$(M_{K^+} - M_{K^0})_{em} = 1.87(10) \text{ MeV}$$

Mixed approach

1. take as an input the phenomenological estimates of $(M_{K^+} - M_{K^0})_{em}$
2. use the kaon masses to determine m_u/m_d

MILC adopts the value

Bijnens-Prades 97, Donoghue-Perez 97

$$(M_{K^+} - M_{K^0})_{em} = 2.8(6)\ \mathrm{MeV}$$

and obtains

$$\frac{m_u}{m_d} = 0.432(1)(9)(0)(39)$$

Quark masses from lattice $+Q$ from $\eta \to 3\pi$

Alternative approach to determine m_u, m_d and m_s:

- determine m_{ud} and m_s on the lattice in the isospin limit
- determine Q from $\eta \to 3\pi$
- combine the two pieces of information

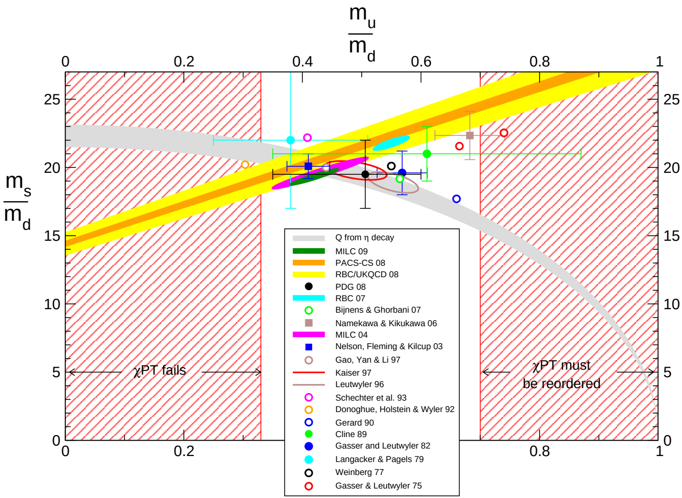
Example: BMW 10

$$m_s = 95.5(1.1)(1.5)\ \mathrm{MeV} \qquad m_{ud} = 3.469(47)(48)\ \mathrm{MeV}$$

Add Q from $\eta \to 3\pi$:

$$m_u = 2.15(03)(10)\ \mathrm{MeV} \qquad m_d = 4.79(07)(12)\ \mathrm{MeV}$$

Quark masses from lattice $+Q$ from $\eta \to 3\pi$



Leutwyler PoS, CD09

Lattice determinations of m_u and m_d

Collaboration	publ.	$m_{u,d} \to 0$	$a \to 0$	FV	renorm.	m_u	m_d
PACS-CS 12	A	★	■	■	★	2.57(26)(7)	3.68(29)(10)
LVdW 11	C	●	★	★	●	1.90(8)(21)(10)	4.73(9)(27)(24)
HPQCD 10	A	●	★	★	★	2.01(14)	4.77(15)
BMW 10A, 10B	A	★	★	★	★	2.15(03)(10)	4.79(07)(12)
Blum et al. 10	P	●	■	●	★	2.24(10)(34)	4.65(15)(32)
MILC 09A	C	●	★	★	●	1.96(0)(6)(10)(12)	4.53(1)(8)(23)(12)
MILC 09	P	●	★	★	●	1.9(0)(1)(1)(1)	4.6(0)(2)(2)(1)
MILC 04, HPQCD/ MILC/UKQCD 04	A	●	●	●	■	1.7(0)(1)(2)(2)	3.9(0)(1)(4)(2)
RM123 11	A	●	★	●	★	2.43(20)(12)	4.78(20)(12)
Dürr 11	A	●	★	●	–	2.18(6)(11)	4.87(14)(16)
RBC 07	A	■	■	★	★	3.02(27)(19)	5.49(20)(34)

FLAG-1 summary of the quark masses

all masses in MeV

N_f	m_u	m_d	m_s	m_{ud}
2+1	2.19(15)	4.67(20)	94(3)	3.43(11)
2	–	–	95(6)	3.6(2)

N_f	m_u/m_d	m_s/m_{ud}	R	Q
2+1	0.47(4)	27.4(4)	36.6(3.8)	22.8(1.2)
2	–	27.3(9)	–	–

Outline

Expansion around the chiral limit

$$\mathcal{H}_{\mathrm{QCD}} = \mathcal{H}^0_{\mathrm{QCD}} + \mathcal{H}_m \qquad \mathcal{H}_m := \bar{q}\mathcal{M}q \quad \mathcal{M} = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

$$\bar{q} = (\bar{u}, \bar{d}, \bar{s})$$

the mass term $\mathcal{H}_m$ will be treated as a small perturbation

Expansion around the chiral limit

$$\mathcal{H}_{\mathrm{QCD}} = \mathcal{H}^0_{\mathrm{QCD}} + \mathcal{H}_m \qquad \mathcal{H}_m := \bar{q}\mathcal{M}q \quad \mathcal{M} = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

Expansion around $\mathcal{H}^0_{\mathrm{QCD}} \equiv$ expansion in powers of m_q

General quark mass expansion for a particle P:

$$M^2 = M_0^2 + (m_u + m_d)\langle P|\bar{q}q|P\rangle + O(m_q^2)$$

Expansion around the chiral limit

$$\mathcal{H}_{\mathrm{QCD}} = \mathcal{H}^0_{\mathrm{QCD}} + \mathcal{H}_m \qquad \mathcal{H}_m := \bar{q}\mathcal{M}q \quad \mathcal{M} = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

Expansion around $\mathcal{H}^0_{\mathrm{QCD}} \equiv$ expansion in powers of m_q

For a Goldstone bosons $M_0^2 = 0$:

$$M_\pi^2 = -(m_u + m_d)\frac{1}{F_\pi^2}\langle 0|\bar{q}q|0\rangle + O(m_q^2)$$

where we have used a Ward identity: Gell-Mann, Oakes and Renner (68)

$$\langle \pi|\bar{q}q|\pi\rangle = -\frac{1}{F_\pi^2}\langle 0|\bar{q}q|0\rangle =: B_0$$

Quark masses

Consider the whole pseudoscalar octet:

$$M_\pi^2 = B_0(m_u + m_d) \quad M_{K^+}^2 = B_0(m_u + m_s) \quad M_{K^0}^2 = B_0(m_d + m_s)$$

Quark masses

Consider the whole pseudoscalar octet:

$$M_\pi^2 = B_0(m_u + m_d) \quad M_{K^+}^2 = B_0(m_u + m_s) \quad M_{K^0}^2 = B_0(m_d + m_s)$$

Quark mass ratios:

$$\frac{m_u}{m_d} \simeq \frac{M_{\pi^+}^2 - M_{K^0}^2 + M_{K^+}^2}{M_{\pi^+}^2 + M_{K^0}^2 - M_{K^+}^2} \simeq 0.67$$

$$\frac{m_s}{m_d} \simeq \frac{M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \simeq 20$$

Quark masses

Consider the whole pseudoscalar octet:

$$M_\pi^2 = B_0(m_u + m_d) \quad M_{K^+}^2 = B_0(m_u + m_s) \quad M_{K^0}^2 = B_0(m_d + m_s)$$

Quark mass ratios:

$$\frac{m_u}{m_d} \simeq \frac{M_{\pi^+}^2 - M_{K^0}^2 + M_{K^+}^2}{M_{\pi^+}^2 + M_{K^0}^2 - M_{K^+}^2} \simeq 0.67$$

$$\frac{m_s}{m_d} \simeq \frac{M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \simeq 20$$

$\hat{m} \equiv (m_u + m_d)/2 \simeq 5.4$ MeV

SU(6) relation, Leutwyler (75)

$$m_u \simeq 4 \text{ MeV} \qquad m_d \simeq 6 \text{ MeV} \qquad m_s \simeq 135 \text{ MeV}$$

Gasser and Leutwyler (75)

Electromagnetic corrections to the masses

According to Dashen's theorem

$$\begin{aligned}
M^2_{\pi^0} &= B_0(m_u + m_d) \\
M^2_{\pi^+} &= B_0(m_u + m_d) + \Delta_{\mathrm{em}} \\
M^2_{K^0} &= B_0(m_d + m_s) \\
M^2_{K^+} &= B_0(m_u + m_s) + \Delta_{\mathrm{em}}
\end{aligned}$$

Extracting the quark mass ratios gives

Weinberg (77)

$$\begin{aligned}
\frac{m_u}{m_d} &= \frac{M^2_{K^+} - M^2_{K^0} + 2M^2_{\pi^0} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}} &= 0.56 \\
\frac{m_s}{m_d} &= \frac{M^2_{K^0} + M^2_{K^+} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}} &= 20.1
\end{aligned}$$

Weinberg (77) estimated m_s from the splitting in baryon octet

$$m_u = 4.2 \text{ MeV} \qquad m_d = 7.5 \text{ MeV} \qquad m_s = 150 \text{ MeV}$$

Higher order chiral corrections

Mass formulae to second order

Gasser-Leutwyler (85)

$$\frac{M_K^2}{M_\pi^2} = \frac{m_s + \hat{m}}{2\hat{m}} \left[1 + \Delta_M + \mathcal{O}(m^2)\right]$$

$$\frac{M_{K^0}^2 - M_{K^+}^2}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}} \left[1 + \Delta_M + \mathcal{O}(m^2)\right]$$

$$\Delta_M = \frac{8(M_K^2 - M_\pi^2)}{F_\pi^2}(2L_8 - L_5) + \chi\text{-logs}$$

The same $\mathcal{O}(m)$ correction appears in both ratios
$\Rightarrow$ this double ratio is free from $\mathcal{O}(m)$ corrections

$$Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} = \frac{M_K^2}{M_\pi^2} \frac{M_K^2 - M_\pi^2}{M_{K^0}^2 - M_{K^+}^2} \left[1 + \mathcal{O}(m^2)\right]$$

Higher order chiral corrections

Mass formulae to second order

Gasser-Leutwyler (85)

$$\begin{aligned}
\frac{M_K^2}{M_\pi^2} &= \frac{m_s+\hat{m}}{2\hat{m}}\left[1+\Delta_M+\mathcal{O}(m^2)\right] \\
\frac{M_{K^0}^2-M_{K^+}^2}{M_K^2-M_\pi^2} &= \frac{m_d-m_u}{m_s-\hat{m}}\left[1+\Delta_M+\mathcal{O}(m^2)\right] \\
\Delta_M &= \frac{8(M_K^2-M_\pi^2)}{F_\pi^2}(2L_8-L_5)+\chi\text{-logs}
\end{aligned}$$

The same $\mathcal{O}(m)$ correction appears in both ratios
$\Rightarrow$ this double ratio is free from $\mathcal{O}(m)$ and em corrections

$$Q_D^2 \equiv \frac{(M_{K^0}^2+M_{K^+}^2-M_{\pi^+}^2+M_{\pi^0}^2)(M_{K^0}^2+M_{K^+}^2-M_{\pi^+}^2-M_{\pi^0}^2)}{4M_{\pi^0}^2(M_{K^0}^2-M_{K^+}^2+M_{\pi^+}^2-M_{\pi^0}^2)} = 24.2$$

Leutwyler's ellipse

Information on Q amounts to an elliptic constraint in the plane of $\frac{m_s}{m_d}$ and $\frac{m_u}{m_d}$ Leutwyler

$$\left(\frac{m_s}{m_d}\right)^2 \frac{1}{Q^2} + \left(\frac{m_u}{m_d}\right)^2 = 1$$

Leutwyler's ellipse

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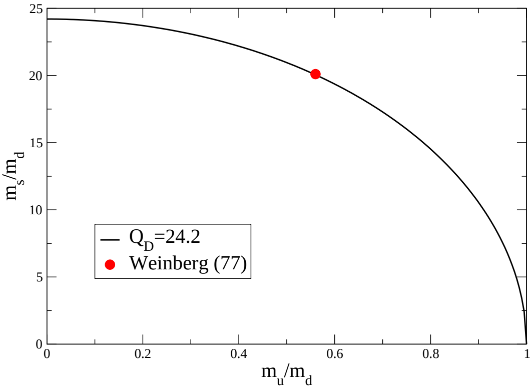

Estimate of Q: violation of Dashen's theorem

$$\left(M_{K^+}^2 - M_{K^0}^2\right)_{\mathrm{em}} = \left(M_{\pi^+}^2 - M_{\pi^0}^2\right)_{\mathrm{em}} \Rightarrow (M_{K^+} - M_{K^0})_{\mathrm{em}} = 1.3\ \mathrm{MeV}$$

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Higher order corrections change the numerical value

$$(M_{K^+} - M_{K^0})_{\text{em}} = \begin{cases} 1.9\ \text{MeV} & \text{Duncan et al. (96)} & Q = 22.8 \\ & \text{(Lattice)} & \\ 2.3\ \text{MeV} & \text{Bijnens-Prades (97)} & Q = 22 \\ & \text{(ENJL model)} & \\ 2.6\ \text{MeV} & \text{Donoghue-Perez (97)} & Q = 21.5 \\ & \text{(VMD)} & \\ 3.2\ \text{MeV} & \text{Anant-Moussallam (04)} & Q = 20.7 \\ & \text{(Sum rules)} & \end{cases}$$

Most recent evaluation: Kastner-Neufeld (08): $Q = 20.7 \pm 1.2$

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Lattice determination of m_u, m_d and m_s
- FLAG
- FLAG phase 2
- Isospin limit
- Isospin breaking

Quark mass ratios from CHPT

A new dispersive analysis of $\eta \to 3\pi$
- Isospin breaking

Summary and Outlook

Q from the decay $\eta \to 3\pi$

Decay amplitude at leading order

$$\mathcal{A}(\eta \to \pi^0\pi^+\pi^-) = -\frac{\sqrt{3}}{4}\frac{m_u - m_d}{m_s - \hat{m}}\frac{s - 4M_\pi^2/3}{F_\pi^2}$$

Q from the decay $\eta \to 3\pi$

Decay amplitude

$$\mathcal{A}(\eta \to \pi^0\pi^+\pi^-) = -\frac{1}{Q^2}\frac{M_K^2(M_K^2 - M_\pi^2)}{3\sqrt{3}M_\pi^2 F_\pi^2} M(s,t,u)$$

The decay width can be written as

$$\Gamma(\eta \to \pi^0\pi^+\pi^-) = \Gamma_0 \left(\frac{Q_D}{Q}\right)^4 = (295 \pm 20)\ \text{eV} \quad \text{PDG (08)}$$

- isospin-breaking sensitive process
- em contributions suppressed (Sutherland's theorem)
  $\Rightarrow$ mainly sensitive to $m_u - m_d$
- strong decay width Γ_0 difficult to estimate

Q from the decay $\eta \to 3\pi$

Decay amplitude

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$$\Gamma_0 = \begin{cases} (167 \pm 50)\ \text{eV} & \text{Gasser-Leutwyler (85)} & Q = 21.1 \pm 1.6 \\ (219 \pm 22)\ \text{eV} & \text{Anisovich-Leutwyler (96)} & Q = 22.6 \pm 0.7 \\ (209 \pm 20)\ \text{eV} & \text{Kambor et al (96)} & Q = 22.3 \pm 0.6 \end{cases}$$

Gasser Leutwyler (85) based on one-loop CHPT
The other two evaluations based on dispersion relations

See also: analysis of KLOE data on $\eta \to 3\pi$ Martemyanov-Sopov (05)

$$Q = 22.8 \pm 0.4$$

Q from the decay $\eta \to 3\pi$

Decay amplitude

$$\mathcal{A}(\eta \to \pi^0\pi^+\pi^-) = -\frac{1}{Q^2}\frac{M_K^2(M_K^2 - M_\pi^2)}{3\sqrt{3}M_\pi^2 F_\pi^2} M(s,t,u)$$

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See also: full two-loop calculation of $\eta \to 3\pi$ Bijnens-Ghorbani (07)

$$Q = 23.2$$

Q from the decay $\eta \to 3\pi$: new analysis

A new analysis is in progress

S. Lanz PhD thesis (11)

GC, Lanz, Leutwyler, Passemar

- recent measurements of the Dalitz plot
  $\Rightarrow$ test the calculation of the strong dynamics of the decay
- dispersive analysis based on $\pi\pi$ scattering phases
  recent improvements must be taken into account

  GC, Gasser, Leutwyler (01)
- recent progress in dealing with isospin breaking (NREFT)
  can be applied also here

  Gasser, Rusetsky et al.

  Schneider, Kubis, Ditsche (11)

Dispersion relation for $\eta \to 3\pi$

Based on the representation Fuchs, Sazdjian, Stern (93), Anisovich, Leutwyler (96)

$$M(s,t,u) = M_0(s) - \frac{2}{3}M_2(s) + [(s-u)M_1(t) + M_2(t) + (t \leftrightarrow u)]$$

valid if the discontinuities of D waves and higher are neglected

Dispersion relation for the M_I's

$$M_I(s) = \Omega_I(s)\left\{P_I(s) + \frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_I(s')\hat{M}_I(s')}{|\Omega_I(s)|s'^n(s'-s)}\right\}$$

where

$$\Omega(s) = \exp\left[\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_I(s')}{s'(s'-s)}\right]$$

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where

$$\Omega(s) = \exp\left[\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_I(s')}{s'(s'-s)}\right]$$

given $\delta_I(s)$, the solution depends on subtraction constants only

Subtraction constants

Extended the number of parameters w.r.t. Anisovich and Leutwyler (96):

$$
\begin{aligned}
P_0(s) &= \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3 \\
P_1(s) &= \alpha_1 + \beta_1 s + \gamma_1 s^2 \\
P_2(s) &= \alpha_2 + \beta_2 s + \gamma_2 s^2 + \delta_2 s^3
\end{aligned}
$$

Solution linear in the subtraction constants: Anisovich, Leutwyler, unpublished

$$M(s,t,u) = \alpha_0 M^{\alpha_0}(s,t,u) + \beta_0 M^{\beta_0}(s,t,u) + \dots$$

makes fitting of data very easy.

Taylor coefficients

Subtraction constants $\alpha_I,\ \beta_I,\ \gamma_I,\ \ldots$ can be replaced by Taylor coefficients: the relation between the two sets is *linear*

$$
\begin{aligned}
M_0(s) &= a_0 + b_0 s + c_0 s^2 + d_0 s^3 + \ldots \\
M_1(s) &= a_1 + b_1 s + c_1 s^2 + \ldots \\
M_2(s) &= a_2 + b_2 s + c_2 s^2 + d_2 s^3 + \ldots
\end{aligned}
$$

Not all Taylor coefficients are physically relevant:
$\exists$ 5-parameter family of polynomials $\delta M_I(s)$ that added to $M_I(s)$ do not change $M(s,t,u)$ (reparametrization invariance)

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- fix the remaining ones with one-loop ChPT

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  or fix d_0, c_1 *by fitting data* $\Rightarrow$ dispersive, fit to KLOE

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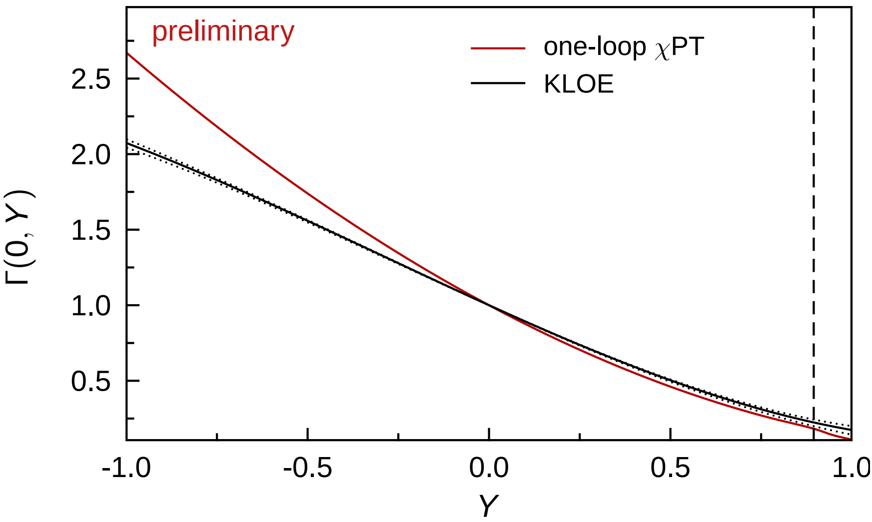
- either set $d_0 = c_1 = 0 \Rightarrow$ *dispersive, one loop*
  or fix d_0, c_1 *by fitting data* $\Rightarrow$ dispersive, fit to KLOE
- Dalitz-plot data are insensitive to the normalization:
  ChPT fixes the normalization and allows the extraction of Q

Dalitz distribution for $\eta \to \pi^+\pi^-\pi^0$



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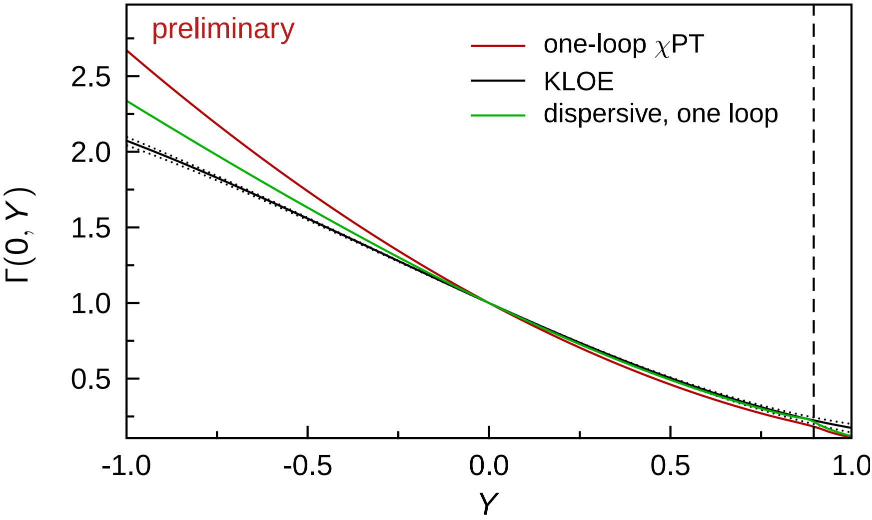

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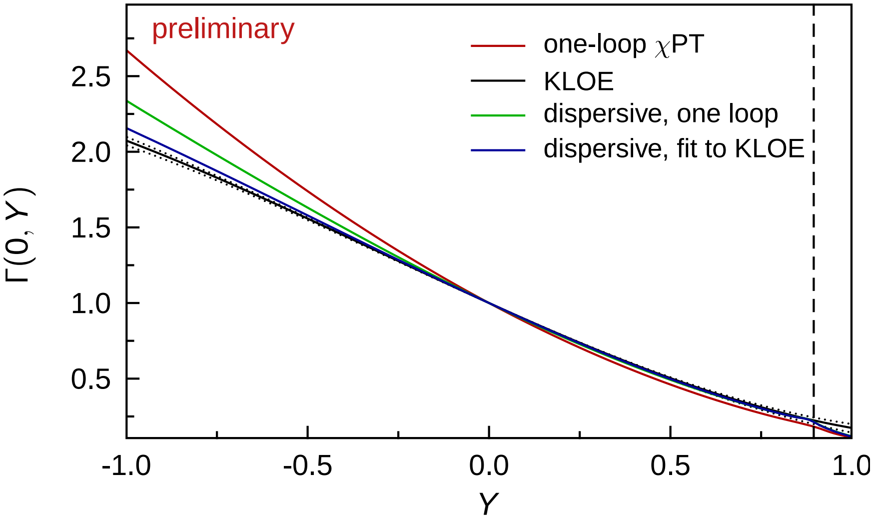
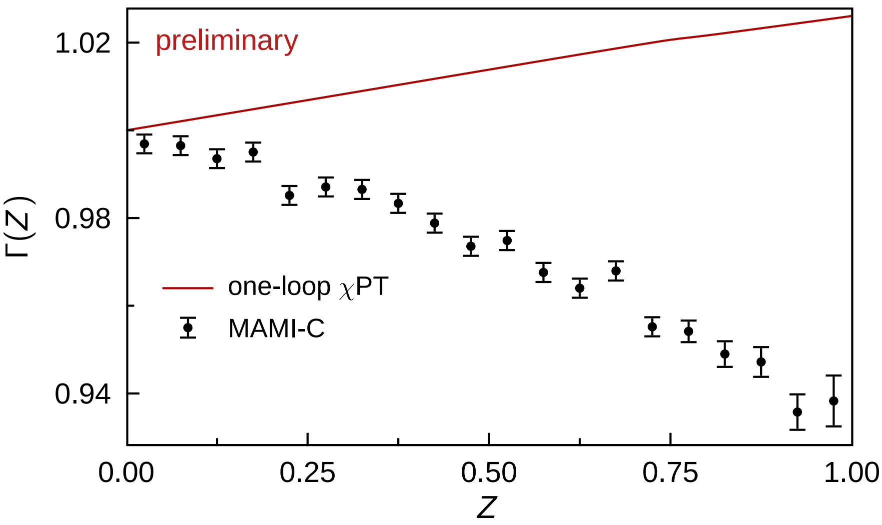
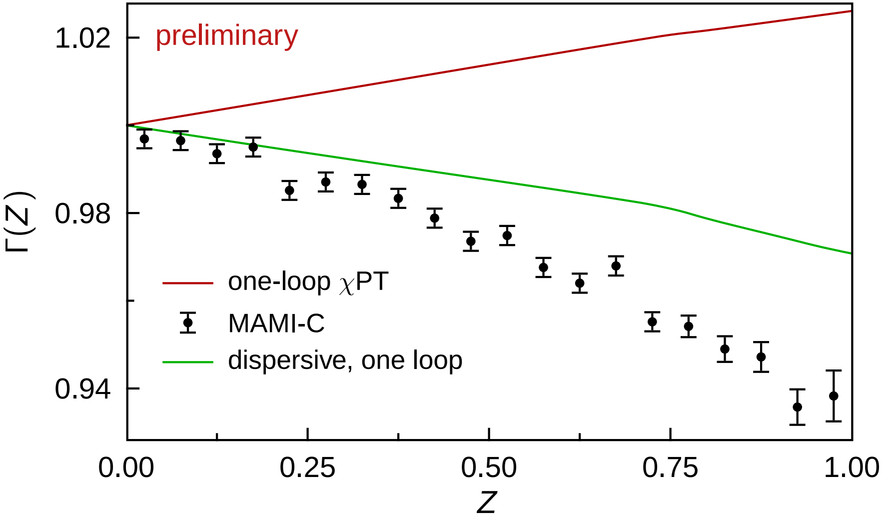


Dalitz distribution for $\eta \to 3\pi^0$



Dalitz distribution for $\eta \to 3\pi^0$



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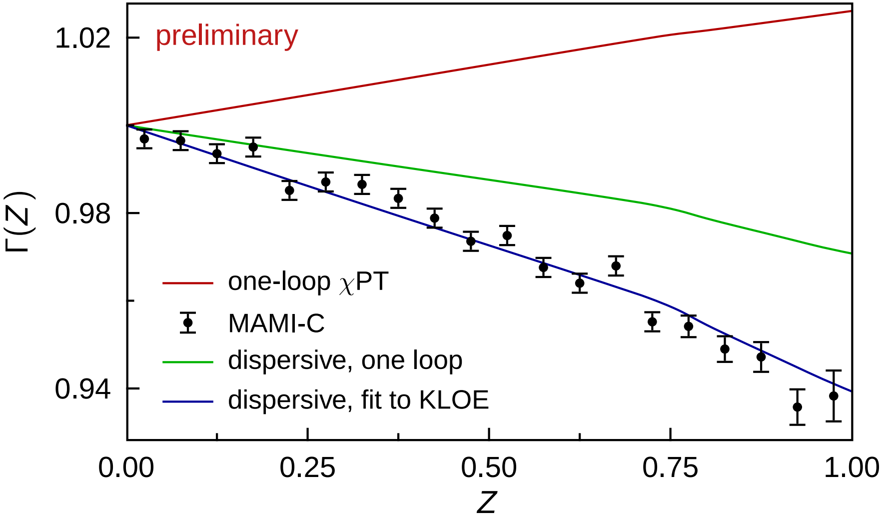
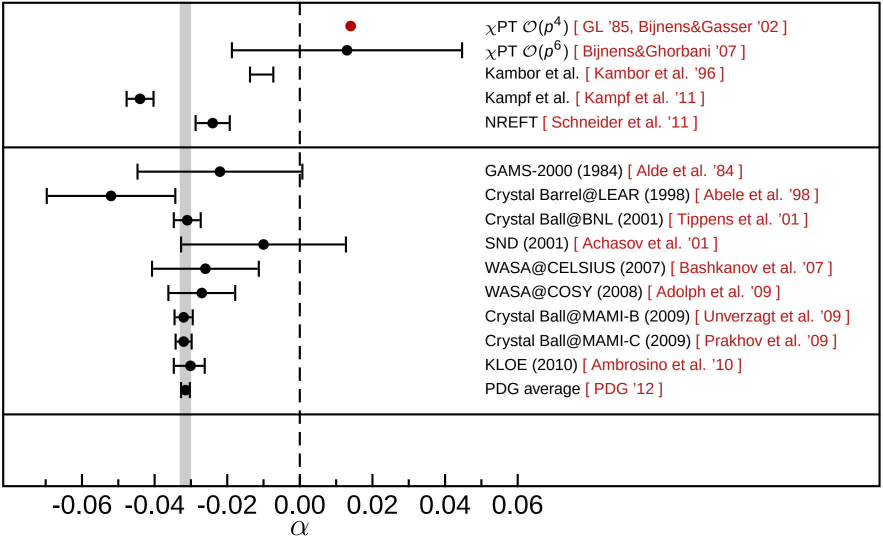


Comparison of α



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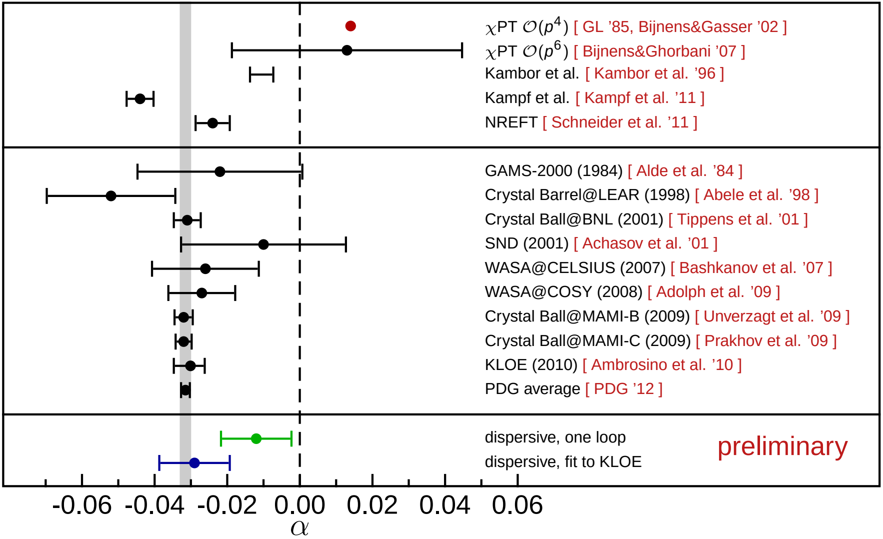
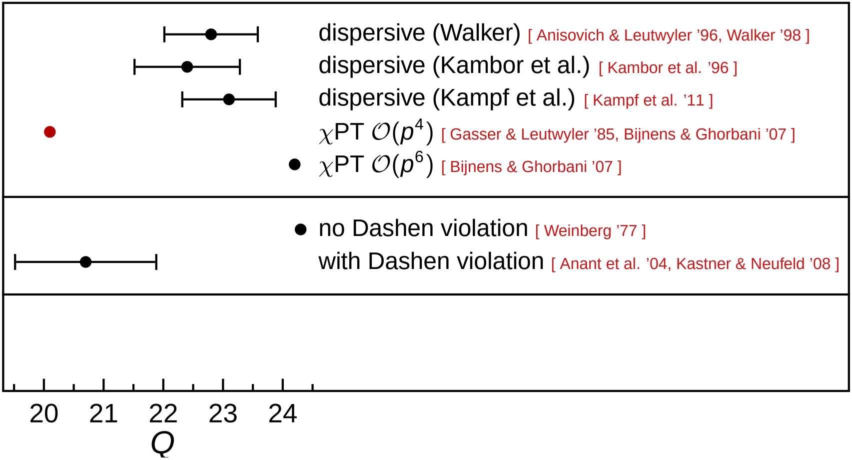


Comparison of Q



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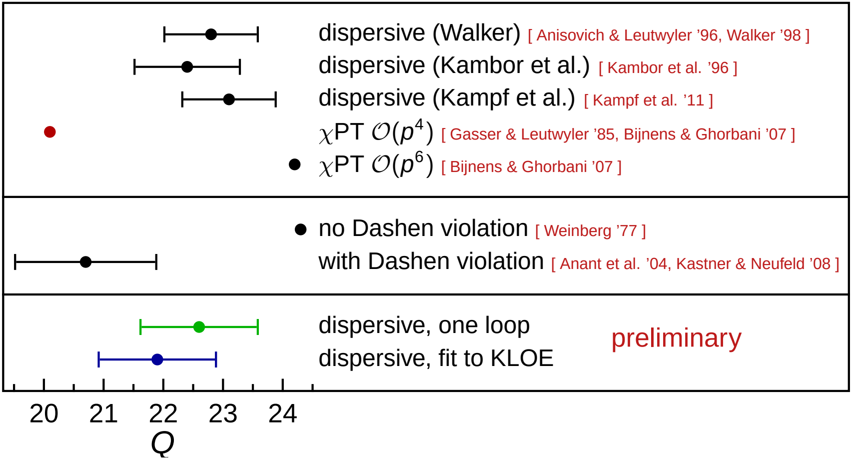
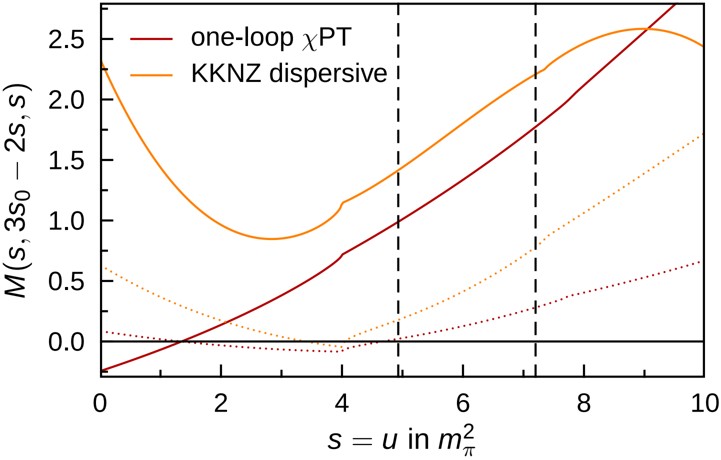


Dispersive analysis by Kampf et al.



The Adler zero has not been imposed as constraint

Intermediate summary

- dispersive representation with purely theoretical input fails to correctly describe the momentum dependence both in the charged and neutral channel
- extend the framework by two more parameters (higher chiral order) $\Rightarrow$ good description of Dalitz plot data
- fit of momentum dependence in the charged channel leads to a correct prediction for α
- the value of Q is also affected by the fit to data

A different determination of the Taylor coefficients

$$
\begin{aligned}
M_0(s) &= a_0 + b_0 s + c_0 s^2 + d_0 s^3 + \dots \\
M_1(s) &= a_1 + b_1 s + c_1 s^2 + \dots \\
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- reparametrization invariance $\Rightarrow$ fix 5 Taylor coefficients

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- reparametrization invariance $\Rightarrow$ fix 5 Taylor coefficients
- fix the others by requiring:
  1. one-loop ChPT value of c_0, b_1, b_2 and c_2 holds to 30%
  2. one-loop prediction for Adler zero and the amplitude derivative at the zero hold to 10%

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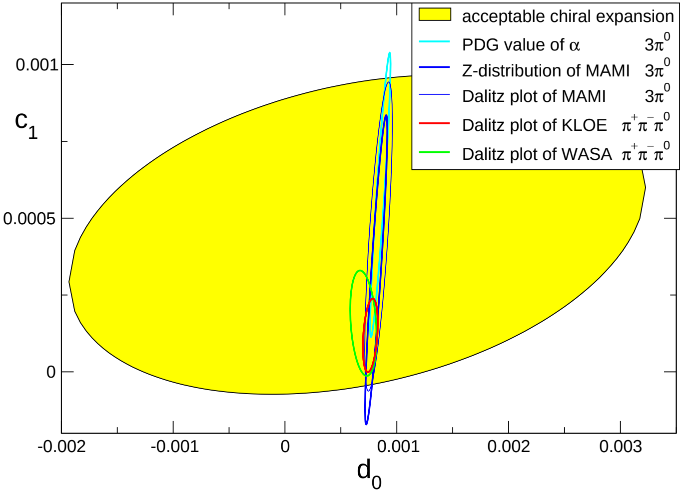
  1. one-loop ChPT value of c_0, b_1, b_2 and c_2 holds to 30%
  2. one-loop prediction for Adler zero and the amplitude derivative at the zero hold to 10%
- fit the data in addition:
  - PDG value for α
  - KLOE in charged channel
  - MAMi in the neutral channel
  - WASA in the charged channel

Importance of data vs theory



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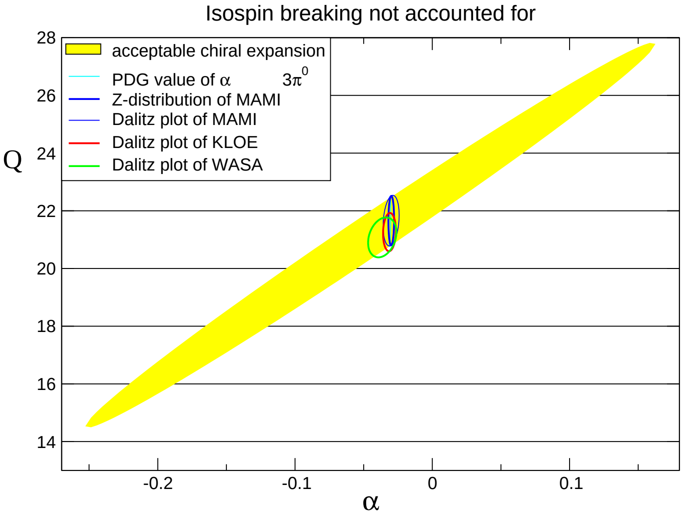

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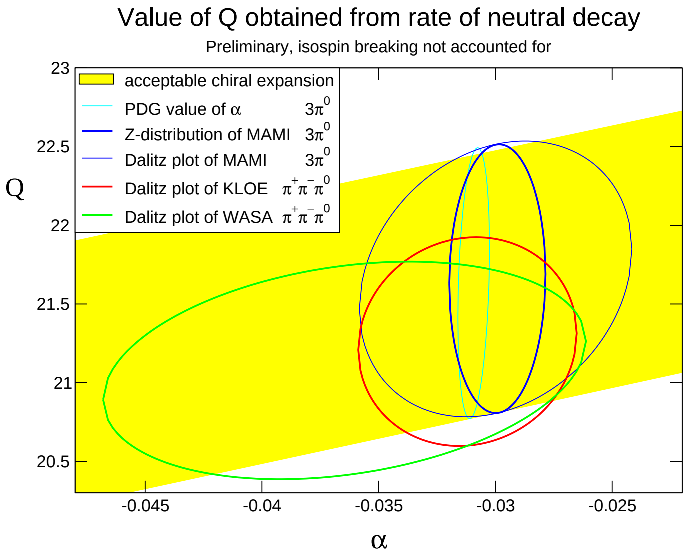

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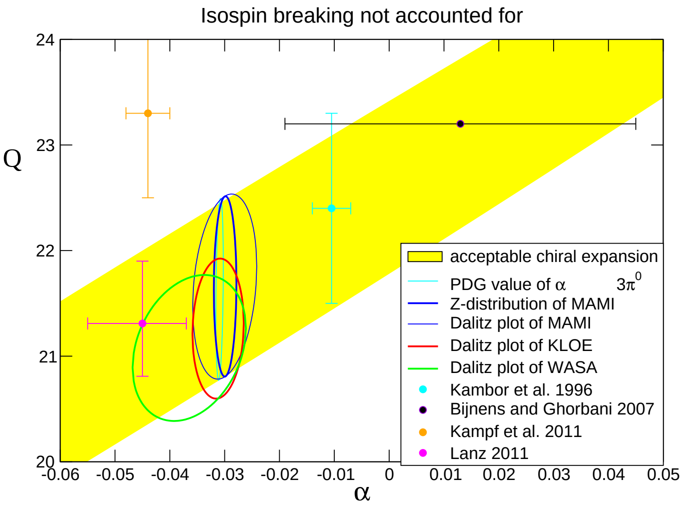

Isospin breaking

Dispersive calculation performed in the isospin limit:

$$M_\pi = M_{\pi^+} \qquad e = 0$$

- we correct for $M_{\pi^0} \neq M_{\pi^+}$ by "stretching" $s, t, u \Rightarrow$ boundaries of isospin-symmetric phase space = boundaries of physical phase space
- physical thresholds inside the phase space can also be mimicked "by hand"
- analysis of Ditsche, Kubis, Meissner (09) used as guidance and check. Same for Gullström, Kupsc and Rusetsky (09)
- $e \neq 0$ effects partly corrected for in the data analysis for the rest we rely on one-loop ChPT – formulae given by Ditsche, Kubis, Meissner (09)
- $\Rightarrow$ to be completed

Isospin breaking

Dispersive calculation performed in the isospin limit:

$$M_\pi = M_{\pi^+} \qquad e = 0$$

- NREFT approach (Schneider, Kubis, Ditsche (11)): systematic method to take into account isospin breaking
- matching between dispersive representation and NREFT in the isospin limit $\Rightarrow$ determine NREFT isospin-conserving parameters
- switch on isospin breaking and fit the data
- for the future

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Quark mass ratios from CHPT

A new dispersive analysis of $\eta \to 3\pi$
- Isospin breaking

Summary and Outlook

Summary

- Quark masses are fundamental and yet unexplained parameters of the standard model
- I have reviewed the determination based on
  - lattice
  - chiral perturbation theory
- I have discussed the extraction of the quark mass ratio Q from $\eta \to 3\pi$ decays based on dispersion relations
- a combination of
  - high precision lattice determinations in the isospin limit
  - Q from $\eta \to 3\pi$

  is at present the best method to determine m_u and m_d