Tribimaximal Mixing From Small Groups

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The Standard Model

Gauge group:

$$\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$$

Particle content:

Q	$(\mathbf{3},\mathbf{2})_{1/3}$	L	$(\mathbf{1},\mathbf{2})_{-1}$	H	$(\mathbf{1},\mathbf{2})_{1}$
$\bar{u}$	$(\overline{\mathbf{3}},\mathbf{1})_{-4/3}$	$\bar{e}$	$(\mathbf{1},\mathbf{1})_{2}$	$\bar{H}$	$(\mathbf{1},\mathbf{2})_{-1}$
$\bar{d}$	$(\overline{\mathbf{3}},\mathbf{1})_{2/3}$	$\bar{\nu}$	$(\mathbf{1},\mathbf{1})_{0}$		

Why Are We Not Happy With the Standard Model?

(i) Too many free parameters

Gauge sector: 3 couplings g', g, g_3	3
Quark sector: 6 masses, 3 mixing angles, 1 CP phase	10
Lepton sector: 6 masses, 3 mixing angles and 1-3 phases	10
Higgs sector: Quartic coupling λ and vev v	2
$\boldsymbol{\theta}$ parameter of QCD	1
	26

Why Are We Not Happy With the Standard Model?

(ii) Structure of gauge symmetry

$$\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \ \overset{?}{\subset}\ \mathrm{SU}(5) \ \overset{?}{\subset}\ \mathrm{SO}(10) \ \overset{?}{\subset}\ \mathrm{E}_6 \ \overset{?}{\subset}\ \mathrm{E}_8$$

Why 3 different coupling constants g', g, g_3?

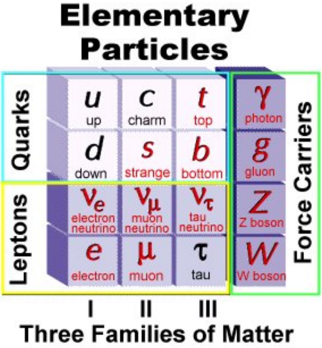
(iii) Structure of family multiplets

$$\underset{Q}{(\mathbf{3},\mathbf{2})_{1/3}} + \underset{\bar{u}}{(\bar{\mathbf{3}},\mathbf{1})_{-4/3}} + \underset{\bar{e}}{(\mathbf{1},\mathbf{1})_{-2}} + \underset{\bar{d}}{(\bar{\mathbf{3}},\mathbf{1})_{2/3}} + \underset{L}{(\mathbf{1},\mathbf{2})_{-1}} + \underset{\bar{\nu}}{(\mathbf{1},\mathbf{1})_{0}} \overset{?}{=} \mathbf{16}$$

Why Are We Not Happy With the Standard Model?

(iv) Repetition of Families

Why is the pattern for 1 generation replicated 3 times?



Why Are We Not Happy With the Standard Model?

(v) Mass Hierarchies and Yukawa Textures

up-quark mass $\sim 2 \times 10^{-3}$ GeV $\leftrightarrow$ top-quark mass ~ 172.3 GeV
Yukawa coupling of top ~ 1, but why are the other quarks so light?

Minimal mixing in quark sector

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \simeq \begin{pmatrix} 0.97 & 0.22 & 0.00 \\ 0.22 & 0.97 & 0.04 \\ 0.00 & 0.04 & 0.99 \end{pmatrix}$$

Why Are We Not Happy With the Standard Model?

(vi) Light neutrinos and texture of Yukawa couplings

Why are neutrinos so light?

$$\Delta m_\nu^2 \sim 10^{-2} - 10^{-5}\ \text{eV}, \quad \sum m_\nu < 2\ \text{eV}$$

Maximal mixing in lepton sector

$$U_{\text{PMNS}} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \simeq \begin{pmatrix} 0.8 & 0.5 & 0.0 \\ \text{-}0.4 & 0.6 & 0.7 \\ 0.4 & \text{-}0.6 & 0.7 \end{pmatrix}$$

Why Are We Not Happy With the Standard Model?

➢ And many other problems/shortcomings:

Hierarchy problem, dark matter, dark energy,
quantum gravity, baryon asymmetry, charge quantization, . . .

➢ Our work addresses the questions:

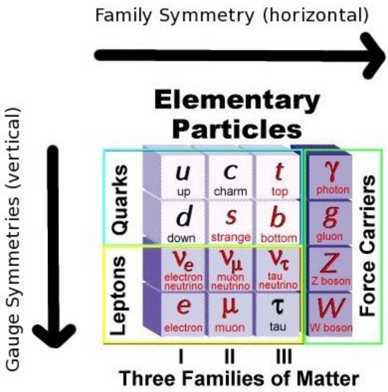
- Number of parameters in the SM $\rightarrow$ (i)
- Repetition of families $\rightarrow$ (iv)
- Light neutrinos and form of U_{PMNS} $\rightarrow$ (vi)

➢ There are cross-connections to:

- Mass hierarchies and form of U_{CKM} $\rightarrow$ (v)
- Grand Unification (see-saw scale) $\rightarrow$ (ii) and (iii)
- Baryon asymmetry (leptogenesis)

Horizontal Symmetries

➢ Introduce relations between families of quarks and leptons



Neutrino Mixing Matrix

What we know about the mixing angles ...

$$U_{\mathrm{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -c_{23}s_{12}-s_{13}s_{23}c_{12}e^{i\delta} & c_{23}c_{12}-s_{13}s_{23}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12}-s_{13}c_{23}c_{12}e^{i\delta} & -s_{23}c_{12}-s_{13}c_{23}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & e^{-i\delta}s_{13} \\ 0 & 1 & 0 \\ -e^{i\delta}s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

T. Schwetz, M. A. Tortola, and J. W. F. Valle, "Three-flavour neutrino oscillation update," *New J. Phys.* **10** (2008) 113011, 0808.2016.

Angle	1σ	2σ	3σ
θ_{12}	$32.46^\circ - 34.82^\circ$	$31.31^\circ - 36.27^\circ$	$30.00^\circ - 37.47^\circ$
θ_{23}	$41.55^\circ - 49.02^\circ$	$38.65^\circ - 52.54^\circ$	$36.87^\circ - 54.94^\circ$
θ_{13}	$0.00^\circ - 9.28^\circ$	$0.00^\circ - 11.54^\circ$	$0.00^\circ - 13.69^\circ$

Harrison-Perkins-Scott Matrix

Presently our best guess ...

P. F. Harrison, D. H. Perkins, and W. G. Scott, "Tri-bimaximal mixing and the neutrino oscillation data," *Phys. Lett.* **B530** (2002) 167, hep-ph/0202074.

$$U_{\mathrm{HPS}} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

Suggestive of an underlying symmetry ...

Some groups that have been considered in the literature:

Review: G. Altarelli and F. Feruglio, "Discrete Flavor Symmetries and Models of Neutrino Mixing," 1002.0211.

$S_3,\ D_4,\ D_7,\ A_4,\ A_5,\ \widetilde{T},\ S_4,\ (C_3 \times C_3) \rtimes_\varphi C_3,\ C_7 \rtimes_\varphi C_3,\ \mathrm{PSL}_2(7)$

$\rightsquigarrow$ As a paradigm, we will consider a model with $A_4 \times C_3$ symmetry and then generalize it to other symmetry groups

Altarelli-Feruglio Model Revisited

G. Altarelli and F. Feruglio, "Tri-Bimaximal Neutrino Mixing, A4 and the Modular Symmetry," *Nucl. Phys.* **B741** (2006) 215–235, hep-ph/0512103.

❶ Symmetries of the model

$$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \times \mathrm{U}(1)_R \times A_4 \times C_3$$

❷ Particle content and charges

Field	$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$	$\mathrm{U}(1)_R$	A_4	C_3	$A_4 \times C_3$
L	$(\mathbf{2},\text{-}1)$	1	$\mathbf{3}$	ω	$\mathbf{3'}$
e	$(\mathbf{1},\ 2)$	1	$\mathbf{1}$	ω^2	$\mathbf{1'}$
μ	$(\mathbf{1},\ 2)$	1	$\mathbf{1''}$	ω^2	$\mathbf{1^{(8)}}$
τ	$(\mathbf{1},\ 2)$	1	$\mathbf{1'}$	ω^2	$\mathbf{1^{(5)}}$
h_u	$(\mathbf{2},\ 1)$	0	$\mathbf{1}$	1	$\mathbf{1}$
h_d	$(\mathbf{2},\text{-}1)$	0	$\mathbf{1}$	1	$\mathbf{1}$
φ_T	$(\mathbf{1},\ 0)$	0	$\mathbf{3}$	1	$\mathbf{3}$
φ_S	$(\mathbf{1},\ 0)$	0	$\mathbf{3}$	ω	$\mathbf{3'}$
ξ	$(\mathbf{1},\ 0)$	0	$\mathbf{1}$	ω	$\mathbf{1''}$

❸ Breaking the family symmetry

$$\varphi_T = (v_T, v_T, v_T), \quad \varphi_S = (v_S, 0, 0), \quad \xi = v_\xi,$$

Group Information from GAP

The GAP Group, "GAP – Groups, Algorithms, and Programming, Version 4.4.12.", http://www.gap-system.org

"GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory."

```
group := SmallGroup(36,11);;
Display(StructureDescription(group));
chartab := Irr(group);;
Display(chartab);
SizesConjugacyClasses(CharacterTable(group));
LoadPackage("repsn");;
for i in [1..Size(chartab)] do
  rep := IrreducibleAffordingRepresentation(chartab[i]);
  for el in Elements(group) do
    Display(el^rep);
  od;
od;
```

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➢ Specify the group that we will work with

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➢ The "human readable" name of the group

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➢ The character table

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➢ Dimensions of the conjugacy classes

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  od;
od;
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➢ The matrices for the representations

The Character Table of $\mathbf{A_4} \times \mathbf{C_3}$

	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}	K_{11}	K_{12}
$\mathbf{1}$	1	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1'}$	1	1	ω^2	1	1	ω^2	ω	ω^2	ω^2	ω	ω	ω
$\mathbf{1''}$	1	1	ω	1	1	ω	ω^2	ω	ω	ω^2	ω^2	ω^2
$\mathbf{1'''}$	1	ω^2	1	1	ω	ω^2	1	1	ω	ω^2	1	ω
$\mathbf{1^{(4)}}$	1	ω	1	1	ω^2	ω	1	1	ω^2	ω	1	ω^2
$\mathbf{1^{(5)}}$	1	ω^2	ω^2	1	ω	ω	ω	ω^2	1	1	ω	ω^2
$\mathbf{1^{(6)}}$	1	ω	ω	1	ω^2	ω^2	ω^2	ω	1	1	ω^2	ω
$\mathbf{1^{(7)}}$	1	ω^2	ω	1	ω	1	ω^2	ω	ω^2	ω	ω^2	1
$\mathbf{1^{(8)}}$	1	ω	ω^2	1	ω^2	1	ω	ω^2	ω	ω^2	ω	1
$\mathbf{3}$	3	0	3	-1	0	0	3	-1	0	0	-1	0
$\mathbf{3'}$	3	0	3ω	-1	0	0	$3\omega^2$	ω	0	0	$1+\omega$	0
$\mathbf{3''}$	3	0	$3\omega^2$	-1	0	0	3ω	$1+\omega$	0	0	ω	0

$\omega = e^{2\pi i/3}$ is the primitive third root of unity

Decomposition of Tensor Products

From the character table and the dimensions of the conjugacy classes:

$\mathbf{1}\otimes\mathbf{1}=\mathbf{1}$
$\mathbf{1}\otimes\mathbf{1^{(5)}}=\mathbf{1^{(5)}}$
$\mathbf{1}\otimes\mathbf{3'}=\mathbf{3'}$
$\mathbf{1'}\otimes\mathbf{1^{(4)}}=\mathbf{1^{(8)}}$
$\mathbf{1'}\otimes\mathbf{3}=\mathbf{3''}$
$\mathbf{1''}\otimes\mathbf{1^{(4)}}=\mathbf{1^{(6)}}$
$\mathbf{1''}\otimes\mathbf{3}=\mathbf{3'}$
$\mathbf{1'''}\otimes\mathbf{1^{(5)}}=\mathbf{1^{(8)}}$
$\mathbf{1'''}\otimes\mathbf{3'}=\mathbf{3'}$
$\mathbf{1^{(4)}}\otimes\mathbf{1^{(7)}}=\mathbf{1''}$
$\mathbf{1^{(5)}}\otimes\mathbf{1^{(5)}}=\mathbf{1^{(6)}}$
$\mathbf{1^{(5)}}\otimes\mathbf{3'}=\mathbf{3}$
$\mathbf{1^{(6)}}\otimes\mathbf{3}=\mathbf{3'}$
$\mathbf{1^{(7)}}\otimes\mathbf{3}=\mathbf{3'}$
$\mathbf{1^{(8)}}\otimes\mathbf{3'}=\mathbf{3}$

$\mathbf{1}\otimes\mathbf{1'}=\mathbf{1'}$
$\mathbf{1}\otimes\mathbf{1^{(6)}}=\mathbf{1^{(6)}}$
$\mathbf{1}\otimes\mathbf{3''}=\mathbf{3''}$
$\mathbf{1'}\otimes\mathbf{1^{(5)}}=\mathbf{1^{(7)}}$
$\mathbf{1'}\otimes\mathbf{3'}=\mathbf{3}$
$\mathbf{1''}\otimes\mathbf{1^{(5)}}=\mathbf{1'''}$
$\mathbf{1''}\otimes\mathbf{3'}=\mathbf{3''}$
$\mathbf{1'''}\otimes\mathbf{1^{(6)}}=\mathbf{1''}$
$\mathbf{1'''}\otimes\mathbf{3''}=\mathbf{3''}$
$\mathbf{1^{(4)}}\otimes\mathbf{1^{(8)}}=\mathbf{1^{(5)}}$
$\mathbf{1^{(5)}}\otimes\mathbf{1^{(6)}}=\mathbf{1}$
$\mathbf{1^{(5)}}\otimes\mathbf{3''}=\mathbf{3'}$
$\mathbf{1^{(6)}}\otimes\mathbf{3'}=\mathbf{3''}$
$\mathbf{1^{(7)}}\otimes\mathbf{3'}=\mathbf{3''}$
$\mathbf{1^{(8)}}\otimes\mathbf{3''}=\mathbf{3'}$

$\mathbf{1}\otimes\mathbf{1''}=\mathbf{1''}$
$\mathbf{1}\otimes\mathbf{1^{(7)}}=\mathbf{1^{(7)}}$
$\mathbf{1'}\otimes\mathbf{1'}=\mathbf{1''}$
$\mathbf{1'}\otimes\mathbf{1^{(6)}}=\mathbf{1^{(4)}}$
$\mathbf{1'}\otimes\mathbf{3''}=\mathbf{3'}$
$\mathbf{1''}\otimes\mathbf{1^{(6)}}=\mathbf{1^{(8)}}$
$\mathbf{1''}\otimes\mathbf{3''}=\mathbf{3}$
$\mathbf{1'''}\otimes\mathbf{1^{(7)}}=\mathbf{1^{(6)}}$
$\mathbf{1^{(4)}}\otimes\mathbf{1^{(4)}}=\mathbf{1'''}$
$\mathbf{1^{(4)}}\otimes\mathbf{3}=\mathbf{3}$
$\mathbf{1^{(5)}}\otimes\mathbf{1^{(7)}}=\mathbf{1^{(4)}}$
$\mathbf{1^{(6)}}\otimes\mathbf{1^{(6)}}=\mathbf{1^{(5)}}$
$\mathbf{1^{(6)}}\otimes\mathbf{3''}=\mathbf{3}$
$\mathbf{1^{(7)}}\otimes\mathbf{3''}=\mathbf{3}$

$\mathbf{1}\otimes\mathbf{1'''}=\mathbf{1'''}$
$\mathbf{1}\otimes\mathbf{1^{(8)}}=\mathbf{1^{(8)}}$
$\mathbf{1'}\otimes\mathbf{1''}=\mathbf{1}$
$\mathbf{1'}\otimes\mathbf{1^{(7)}}=\mathbf{1'''}$
$\mathbf{1''}\otimes\mathbf{1''}=\mathbf{1'}$
$\mathbf{1''}\otimes\mathbf{1^{(7)}}=\mathbf{1^{(5)}}$
$\mathbf{1'''}\otimes\mathbf{1'''}=\mathbf{1^{(4)}}$
$\mathbf{1'''}\otimes\mathbf{1^{(8)}}=\mathbf{1'}$
$\mathbf{1^{(4)}}\otimes\mathbf{1^{(5)}}=\mathbf{1'}$
$\mathbf{1^{(4)}}\otimes\mathbf{3'}=\mathbf{3'}$
$\mathbf{1^{(5)}}\otimes\mathbf{1^{(8)}}=\mathbf{1''}$
$\mathbf{1^{(6)}}\otimes\mathbf{1^{(7)}}=\mathbf{1'}$
$\mathbf{1^{(7)}}\otimes\mathbf{1^{(7)}}=\mathbf{1^{(8)}}$
$\mathbf{1^{(8)}}\otimes\mathbf{1^{(8)}}=\mathbf{1^{(7)}}$

$\mathbf{1}\otimes\mathbf{1^{(4)}}=\mathbf{1^{(4)}}$
$\mathbf{1}\otimes\mathbf{3}=\mathbf{3}$
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$\mathbf{1'}\otimes\mathbf{1^{(8)}}=\mathbf{1^{(6)}}$
$\mathbf{1''}\otimes\mathbf{1'''}=\mathbf{1^{(7)}}$
$\mathbf{1''}\otimes\mathbf{1^{(8)}}=\mathbf{1^{(4)}}$
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$\mathbf{1'''}\otimes\mathbf{3}=\mathbf{3}$
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$\mathbf{1^{(4)}}\otimes\mathbf{3''}=\mathbf{3''}$
$\mathbf{1^{(5)}}\otimes\mathbf{3}=\mathbf{3''}$
$\mathbf{1^{(6)}}\otimes\mathbf{1^{(8)}}=\mathbf{1'''}$
$\mathbf{1^{(7)}}\otimes\mathbf{1^{(8)}}=\mathbf{1}$
$\mathbf{1^{(8)}}\otimes\mathbf{3}=\mathbf{3''}$

$\mathbf{3}\otimes\mathbf{3}=\mathbf{1}+\mathbf{1'''}+\mathbf{1^{(4)}}+2\otimes\mathbf{3}$
$\mathbf{3}\otimes\mathbf{3''}=\mathbf{1'}+\mathbf{1^{(5)}}+\mathbf{1^{(8)}}+2\otimes\mathbf{3''}$
$\mathbf{3'}\otimes\mathbf{3''}=\mathbf{1}+\mathbf{1'''}+\mathbf{1^{(4)}}+2\otimes\mathbf{3}$

$\mathbf{3}\otimes\mathbf{3'}=\mathbf{1''}+\mathbf{1^{(6)}}+\mathbf{1^{(7)}}+2\otimes\mathbf{3'}$
$\mathbf{3'}\otimes\mathbf{3'}=\mathbf{1'}+\mathbf{1^{(5)}}+\mathbf{1^{(8)}}+2\otimes\mathbf{3''}$
$\mathbf{3''}\otimes\mathbf{3''}=\mathbf{1''}+\mathbf{1^{(6)}}+\mathbf{1^{(7)}}+2\otimes\mathbf{3'}$

Decomposition of Tensor Products

From the character table and the dimensions of the conjugacy classes:

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$\mathbf{1}' \otimes \mathbf{1}^{(4)} = \mathbf{1}^{(8)}$
$\mathbf{1}' \otimes \mathbf{3} = \mathbf{3}''$
$\mathbf{1}'' \otimes \mathbf{1}^{(4)} = \mathbf{1}^{(6)}$
$\mathbf{1}'' \otimes \mathbf{3} = \mathbf{3}'$
$\mathbf{1}''' \otimes \mathbf{1}^{(5)} = \mathbf{1}^{(8)}$
$\mathbf{1}''' \otimes \mathbf{3}' = \mathbf{3}'$
$\mathbf{1}^{(4)} \otimes \mathbf{1}^{(7)} = \mathbf{1}''$
$\mathbf{1}^{(5)} \otimes \mathbf{1}^{(5)} = \mathbf{1}^{(6)}$
$\mathbf{1}^{(5)} \otimes \mathbf{3}' = \mathbf{3}$
$\mathbf{1}^{(6)} \otimes \mathbf{3} = \mathbf{3}'$
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$\mathbf{1} \otimes \mathbf{1}^{(6)} = \mathbf{1}^{(6)}$
$\mathbf{1} \otimes \mathbf{3}'' = \mathbf{3}''$
$\mathbf{1}' \otimes \mathbf{1}^{(5)} = \mathbf{1}^{(7)}$
$\mathbf{1}' \otimes \mathbf{3}' = \mathbf{3}$
$\mathbf{1}'' \otimes \mathbf{1}^{(5)} = \mathbf{1}'''$
$\mathbf{1}'' \otimes \mathbf{3}' = \mathbf{3}''$
$\mathbf{1}''' \otimes \mathbf{1}^{(6)} = \mathbf{1}''$
$\mathbf{1}''' \otimes \mathbf{3}'' = \mathbf{3}''$
$\mathbf{1}^{(4)} \otimes \mathbf{1}^{(8)} = \mathbf{1}^{(5)}$
$\mathbf{1}^{(5)} \otimes \mathbf{1}^{(6)} = \mathbf{1}$
$\mathbf{1}^{(5)} \otimes \mathbf{3}'' = \mathbf{3}'$
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$\mathbf{1}^{(8)} \otimes \mathbf{3}'' = \mathbf{3}'$

$\mathbf{1} \otimes \mathbf{1}'' = \mathbf{1}''$
$\mathbf{1} \otimes \mathbf{1}^{(7)} = \mathbf{1}^{(7)}$
$\mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}''$
$\mathbf{1}' \otimes \mathbf{1}^{(6)} = \mathbf{1}^{(4)}$
$\mathbf{1}' \otimes \mathbf{3}'' = \mathbf{3}'$
$\mathbf{1}'' \otimes \mathbf{1}^{(6)} = \mathbf{1}^{(8)}$
$\mathbf{1}'' \otimes \mathbf{3}'' = \mathbf{3}$
$\mathbf{1}''' \otimes \mathbf{1}^{(7)} = \mathbf{1}^{(6)}$
$\mathbf{1}^{(4)} \otimes \mathbf{1}^{(4)} = \mathbf{1}'''$
$\mathbf{1}^{(4)} \otimes \mathbf{3} = \mathbf{3}$
$\mathbf{1}^{(5)} \otimes \mathbf{1}^{(7)} = \mathbf{1}^{(4)}$
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$\mathbf{1}^{(6)} \otimes \mathbf{1}^{(8)} = \mathbf{1}'''$
$\mathbf{1}^{(7)} \otimes \mathbf{1}^{(8)} = \mathbf{1}$
$\mathbf{1}^{(8)} \otimes \mathbf{3} = \mathbf{3}''$

$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} + \mathbf{1}''' + \mathbf{1}^{(4)} + 2 \otimes \mathbf{3}$
$\mathbf{3} \otimes \mathbf{3}'' = \mathbf{1}' + \mathbf{1}^{(5)} + \mathbf{1}^{(8)} + 2 \otimes \mathbf{3}''$
$\mathbf{3}' \otimes \mathbf{3}'' = \mathbf{1} + \mathbf{1}''' + \mathbf{1}^{(4)} + 2 \otimes \mathbf{3}$

$\mathbf{3} \otimes \mathbf{3}' = \mathbf{1}'' + \mathbf{1}^{(6)} + \mathbf{1}^{(7)} + 2 \otimes \mathbf{3}'$
$\mathbf{3}' \otimes \mathbf{3}' = \mathbf{1}' + \mathbf{1}^{(5)} + \mathbf{1}^{(8)} + 2 \otimes \mathbf{3}''$
$\mathbf{3}'' \otimes \mathbf{3}'' = \mathbf{1}'' + \mathbf{1}^{(6)} + \mathbf{1}^{(7)} + 2 \otimes \mathbf{3}'$

Invariant Lagrangian

- Terms that are invariant, have 2 leptons and mass dimension ≤ 6:

$$L\,L\,h_u\,h_u\,\varphi_S\,, \quad L\,L\,h_u\,h_u\,\xi, \quad L\,e\,h_d\,\varphi_T\,, \quad L\,\mu\,h_d\,\varphi_T\,, \quad L\,\tau\,h_d\,\varphi_T$$

Invariant Lagrangian

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$$L\,L\,h_u\,h_u\,\varphi_S, \quad L\,L\,h_u\,h_u\,\xi, \quad L\,e\,h_d\,\varphi_T, \quad L\,\mu\,h_d\,\varphi_T, \quad L\,\tau\,h_d\,\varphi_T$$

$$\mathbf{3'}\otimes\mathbf{3'}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{3'}=\left(\mathbf{1'}+\mathbf{1^{(5)}}+\mathbf{1^{(8)}}+2\times\mathbf{3''}\right)\otimes\mathbf{3'}=2\times\mathbf{1}+2\times\mathbf{1'''}+2\times\mathbf{1^{(4)}}+7\times\mathbf{3}$$

Invariant Lagrangian

- Terms that are invariant, have 2 leptons and mass dimension ≤ 6:

$$L\,L\,h_u\,h_u\,\varphi_S, \quad L\,L\,h_u\,h_u\,\xi, \quad L\,e\,h_d\,\varphi_T, \quad L\,\mu\,h_d\,\varphi_T, \quad L\,\tau\,h_d\,\varphi_T$$

$$\mathbf{3'}\otimes\mathbf{3'}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{3'}=\left(\mathbf{1'}+\mathbf{1^{(5)}}+\mathbf{1^{(8)}}+2\times\mathbf{3''}\right)\otimes\mathbf{3'}=2\times\mathbf{1}+2\times\mathbf{1'''}+2\times\mathbf{1^{(4)}}+7\times\mathbf{3}$$

- Contract family indices:

$$\tfrac{1}{\sqrt{3}}L_2\,L_3\,h_u\,h_u\,\varphi_{S,1}+\tfrac{1}{\sqrt{3}}L_3\,L_1\,h_u\,h_u\,\varphi_{S,2}+\tfrac{1}{\sqrt{3}}L_1\,L_2\,h_u\,h_u\,\varphi_{S,3}+\tfrac{1}{\sqrt{3}}L_1\,L_1\,h_u\,h_u\,\xi$$
$$+\tfrac{1}{\sqrt{3}}L_2\,L_2\,h_u\,h_u\,\xi+\tfrac{1}{\sqrt{3}}L_3\,L_3\,h_u\,h_u\,\xi$$

Invariant Lagrangian

- Terms that are invariant, have 2 leptons and mass dimension ≤ 6:

$$L\,L\,h_u\,h_u\,\varphi_S,\quad L\,L\,h_u\,h_u\,\xi,\quad L\,e\,h_d\,\varphi_T,\quad L\,\mu\,h_d\,\varphi_T,\quad L\,\tau\,h_d\,\varphi_T$$

$$\mathbf{3'}\otimes\mathbf{3'}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{3'}=\left(\mathbf{1'}+\mathbf{1^{(5)}}+\mathbf{1^{(8)}}+2\times\mathbf{3''}\right)\otimes\mathbf{3'}=2\times\mathbf{1}+2\times\mathbf{1'''}+2\times\mathbf{1^{(4)}}+7\times\mathbf{3}$$

- Contract family indices:

$$\tfrac{1}{\sqrt{3}}L_2\,L_3\,h_u\,h_u\,\varphi_{S,1}+\tfrac{1}{\sqrt{3}}L_3\,L_1\,h_u\,h_u\,\varphi_{S,2}+\tfrac{1}{\sqrt{3}}L_1\,L_2\,h_u\,h_u\,\varphi_{S,3}+\tfrac{1}{\sqrt{3}}L_1\,L_1\,h_u\,h_u\,\xi$$
$$+\tfrac{1}{\sqrt{3}}L_2\,L_2\,h_u\,h_u\,\xi+\tfrac{1}{\sqrt{3}}L_3\,L_3\,h_u\,h_u\,\xi$$

- Contract $\mathrm{SU}(2)$ indices and substitute *vevs* $\langle\varphi_S\rangle=(v_S,0,0)$, etc:

$$\tfrac{1}{\sqrt{3}}L_2^{(1)}\,L_3^{(1)}\,v_u\,v_u\,v_S+\tfrac{1}{\sqrt{3}}L_1^{(1)}\,L_1^{(1)}\,v_u\,v_u\,v_\xi+\tfrac{1}{\sqrt{3}}L_2^{(1)}\,L_2^{(1)}\,v_u\,v_u\,v_\xi+\tfrac{1}{\sqrt{3}}L_3^{(1)}\,L_3^{(1)}\,v_u\,v_u\,v_\xi$$

Finally, the Mixing Angles

- Mass matrices

$$M_{\ell^+} = \begin{array}{c} \\ L_1^{(2)} \\ L_2^{(2)} \\ L_3^{(2)} \end{array} \begin{array}{c} \begin{array}{ccc} e & \mu & \tau \end{array} \\ \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{pmatrix} \end{array}, \qquad M_\nu = \begin{array}{c} \\ L_1^{(1)} \\ L_2^{(1)} \\ L_3^{(1)} \end{array} \begin{array}{c} \begin{array}{ccc} L_1^{(1)} & L_2^{(1)} & L_3^{(1)} \end{array} \\ \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \end{array}$$

Finally, the Mixing Angles

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$$M_{\ell^+} = \begin{array}{c} \\ L_1^{(2)} \\ L_2^{(2)} \\ L_3^{(2)} \end{array} \begin{array}{c} \begin{array}{ccc} e & \mu & \tau \end{array} \\ \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{pmatrix} \end{array}, \qquad M_\nu = \begin{array}{c} \\ L_1^{(1)} \\ L_2^{(1)} \\ L_3^{(1)} \end{array} \begin{array}{c} \begin{array}{ccc} L_1^{(1)} & L_2^{(1)} & L_3^{(1)} \end{array} \\ \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \end{array}$$

- Singular value decomposition: $\hat{M}_{\ell^+} = D_L M_{\ell^+} D_R^\dagger, \quad \hat{M}_\nu = U_L M_\nu U_R^\dagger$

$$D_\mathrm{L} = \begin{pmatrix} -0.5774+i\,0.0000 & -0.5774+i\,0.0000 & -0.5774+i\,0.0000 \\ 0.5738-i\,0.0636 & -0.2319+i\,0.5287 & -0.3420-i\,0.4652 \\ 0.5731-i\,0.0702 & -0.3474-i\,0.4612 & -0.2257+i\,0.5314 \end{pmatrix}, \quad U_\mathrm{L} = \begin{pmatrix} 0.0000 & -0.7071 & -0.7071 \\ -1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.7071 & -0.7071 \end{pmatrix}$$

Finally, the Mixing Angles

- Mass matrices

$$M_{\ell^+} = \begin{array}{c} \\ L_1^{(2)} \\ L_2^{(2)} \\ L_3^{(2)} \end{array} \begin{array}{c} \begin{array}{ccc} e & \mu & \tau \end{array} \\ \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{pmatrix} \end{array}, \qquad M_\nu = \begin{array}{c} \\ L_1^{(1)} \\ L_2^{(1)} \\ L_3^{(1)} \end{array} \begin{array}{c} \begin{array}{ccc} L_1^{(1)} & L_2^{(1)} & L_3^{(1)} \end{array} \\ \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \end{array}$$

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- Neutrino mixing matrix: $U_{\mathrm{PMNS}} = D_{\mathrm{L}} U_{\mathrm{L}}^\dagger$ (needs rephasing)

$$U_{\mathrm{PMNS}} = \begin{pmatrix} 0.8165+i\,0.0000 & 0.5774+i\,0.0000 & 0.0000+i\,0.0000 \\ 0.4058-i\,0.0449 & -0.5738+i\,0.0636 & 0.0778+i\,0.7028 \\ 0.4052-i\,0.0497 & -0.5731+i\,0.0702 & -0.0860-i\,0.7019 \end{pmatrix}$$

- Mixing angles: $\theta_{12} = 35.26, \quad \theta_{23} = 45.00, \quad \theta_{13} = 0.00$ Tribimaximal ✔

Finally, the Mixing Angles

- Mass matrices

$$M_{\ell^+} = \begin{array}{c} \\ L_1^{(2)} \\ L_2^{(2)} \\ L_3^{(2)} \end{array} \begin{array}{c} \begin{array}{ccc} e & \mu & \tau \end{array} \\ \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{pmatrix} \end{array}, \qquad M_\nu = \begin{array}{c} \\ L_1^{(1)} \\ L_2^{(1)} \\ L_3^{(1)} \end{array} \begin{array}{c} \begin{array}{ccc} L_1^{(1)} & L_2^{(1)} & L_3^{(1)} \end{array} \\ \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \end{array}$$

- Singular value decomposition: $\hat{M}_{\ell^+} = D_L M_{\ell^+} D_R^\dagger, \quad \hat{M}_\nu = U_L M_\nu U_R^\dagger$

$$D_{\mathrm{L}} = \begin{pmatrix} -0.5774+i\,0.0000 & -0.5774+i\,0.0000 & -0.5774+i\,0.0000 \\ 0.5738-i\,0.0636 & -0.2319+i\,0.5287 & -0.3420-i\,0.4652 \\ 0.5731-i\,0.0702 & -0.3474-i\,0.4612 & -0.2257+i\,0.5314 \end{pmatrix}, \quad U_{\mathrm{L}} = \begin{pmatrix} 0.0000 & -0.7071 & -0.7071 \\ -1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.7071 & -0.7071 \end{pmatrix}$$

- Neutrino mixing matrix: $U_{\mathrm{PMNS}} = D_{\mathrm{L}} U_{\mathrm{L}}^\dagger$ (needs rephasing)

$$|U_{\mathrm{PMNS}}| = \begin{pmatrix} 0.8165 & 0.5774 & 0.0000 \\ 0.4082 & 0.5774 & 0.7071 \\ 0.4082 & 0.5774 & 0.7071 \end{pmatrix}$$

- Mixing angles: $\theta_{12} = 35.26, \quad \theta_{23} = 45.00, \quad \theta_{13} = 0.00$ Tribimaximal ✔

Finally, the Mixing Angles

➢ Mass matrices

$$M_{\ell^+} = \begin{array}{c} \\ L_1^{(2)} \\ L_2^{(2)} \\ L_3^{(2)} \end{array} \begin{array}{c} \begin{array}{ccc} e & \mu & \tau \end{array} \\ \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{pmatrix} \end{array}, \qquad M_\nu = \begin{array}{c} \\ L_1^{(1)} \\ L_2^{(1)} \\ L_3^{(1)} \end{array} \begin{array}{c} \begin{array}{ccc} L_1^{(1)} & L_2^{(1)} & L_3^{(1)} \end{array} \\ \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \end{array}$$

➢ Singular value decomposition: $\hat{M}_{\ell^+} = D_L M_{\ell^+} D_R^\dagger, \quad \hat{M}_\nu = U_L M_\nu U_R^\dagger$

$$D_{\mathrm{L}} = \begin{pmatrix} -0.5774+i\,0.0000 & -0.5774+i\,0.0000 & -0.5774+i\,0.0000 \\ 0.5738-i\,0.0636 & -0.2319+i\,0.5287 & -0.3420-i\,0.4652 \\ 0.5731-i\,0.0702 & -0.3474-i\,0.4612 & -0.2257+i\,0.5314 \end{pmatrix}, \quad U_{\mathrm{L}} = \begin{pmatrix} 0.0000 & -0.7071 & -0.7071 \\ -1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.7071 & -0.7071 \end{pmatrix}$$

➢ Neutrino mixing matrix: $U_{\mathrm{PMNS}} = D_{\mathrm{L}} U_{\mathrm{L}}^\dagger$ (needs rephasing)

$$|U_{\mathrm{PMNS}}| = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

➢ Mixing angles: $\theta_{12} = 35.26, \;\; \theta_{23} = 45.00, \;\; \theta_{13} = 0.00$ Tribimaximal ✔

What is the Bottom Line?

- We used the Altarelli-Feruglio model only as a paradigm
- The analysis is completely independent of the family symmetry
- GAP gives us all the relevant information about the group
- Complexity of group is hereby irrelevant
- We use Python to interact w/GAP and do symbolic manipulations
- From symmetry to lagrangian to mixing angles takes less than 1 second per model (checking all vacua!)

Where Do We Go From Here?

1. Generalize family symmetry:

$$A_4 \times C_3 \quad \rightarrow \quad \text{1048 groups of order} \leq 100 \quad \rightarrow \quad \text{90 w/3-dim irreps}$$

2. Keep same particle content:

$$L, \quad e, \quad \mu, \quad \tau, \quad h_u, \quad h_d, \quad \varphi_T, \quad \varphi_S, \quad \xi$$

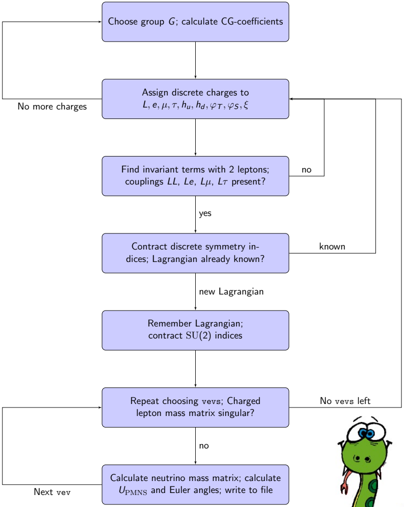
3. Generalize family charge assignments:

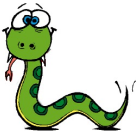
$$(L, e, \mu, \tau, h_u, h_d, \varphi_T, \varphi_S, \xi) \rightarrow (\mathbf{3}', \mathbf{1}', \mathbf{1}^{(8)}, \mathbf{1}^{(5)}, \mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{3}', \mathbf{1}'')$$
$$\rightarrow (*, \; *, \; *, \; *, \; *, \; *, \; *, \; *, \; *)$$

4. Generalize symmetry breaking patterns:

$$\langle \varphi_T \rangle = (*, *, *), \quad \langle \varphi_S \rangle = (*, *, *), \quad \langle \xi \rangle = (*, *, *)$$

Scanning for the Models





- Consider $A_4 \times C_3$
- 14,594,580 different family charge assignments (particles w/same gauge/R-symmetry charges are considered identical)
- Computer takes 17 hours (3 GHz Intel Xeon) $\rightsquigarrow$ Distribute job to 17 machines and get results in 1 hour
- 39,900 different Lagrangians
- Plus results for 75 more groups!

Results for $\mathbf{A_4} \times \mathbf{C_3}$

➢ We consider 2 models equivalent, if their Lagrangians are the same after contracting the family indices, but before the vevs are substituted

➢ In this sense, we have 39,900 inequivalent models/Lagrangians

➢ 22,932 models have non-singular charged lepton and neutrino mass matrices:

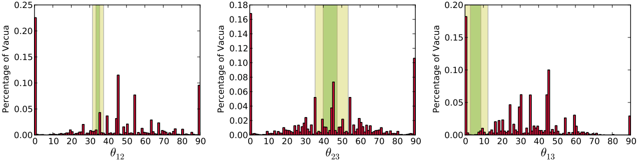
$$\hat{M}_{\ell^+} = D_L M_{\ell^+} D_R^\dagger, \qquad \hat{M}_\nu = U_L M_\nu U_R^\dagger, \qquad U_{\mathrm{PMNS}} \equiv D_{\mathrm{L}} U_{\mathrm{L}}^\dagger$$

➢ 4,481 consistent w/experiment at 3σ level (19.5%)

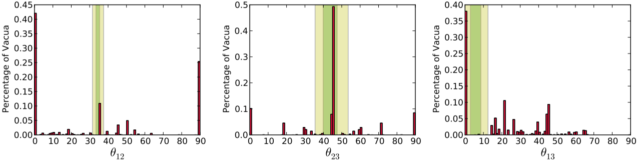
➢ 4,233 are tribimaximal (18.5%)

➢ Probably largest set of viable neutrino models ever constructed!

Most Likely Mixing Angles

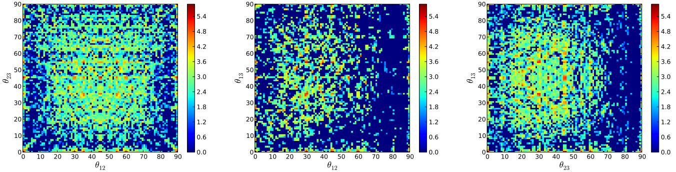


(a) Number of models that give θ_{ij} with no constraints on the other 2 angles. Each histogram has 15,992,118 entries.

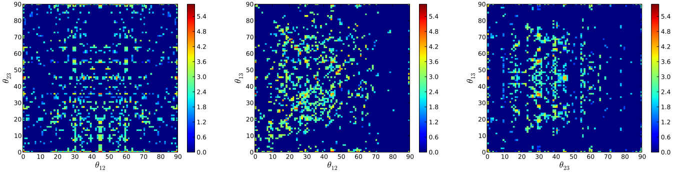


(b) Number of models that give θ_{ij} with the other 2 angles restricted to their 3σ interval. The histograms have 838,289, 148,886 and 225,844 entries, respectively.

Correlation Between Pairs of Mixing Angles

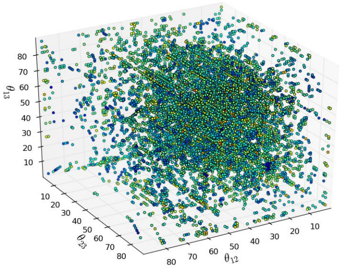


(c) Number of models that give θ_{ij} and θ_{mn} with no constraint on the remaining angle. Each histogram has 15,768,810 entries.

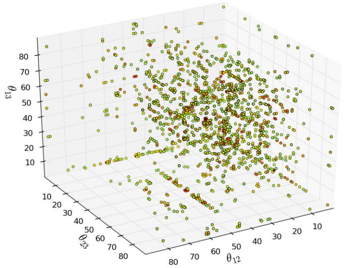


(d) Number of models that give θ_{ij} and θ_{mn} with the remaining angle restricted to its 3σ interval. The histograms have 2,591,752, 4,060,640 and 1,214,874 entries, respectively.

Correlation Between All Mixing Angles



(e) The 12,230 bins that are ≥ 1.

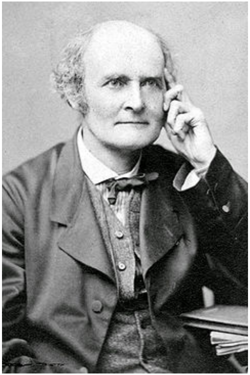


(f) The 1,586 bins that are ≥ 1000.

Are We Looking Under the Lamppost?

Small Groups

Arthur Cayley (1821-1895) is the first to systematically construct groups; in 1854, he determined all groups of order 4 and 6 . . .



Small Groups

The first few of the 1048 groups of order ≤ 100

✔ $= \mathrm{U}(n)$ and ✔ $= \mathrm{SU}(n)$ for $n = 2, 3$

GAP ID	Group	**3**	$\mathrm{U}(3)$	$\mathrm{U}(2)$	$\mathrm{U}(2)\times\mathrm{U}(1)$	A_4
$[1,1]$	1	✗	✗	✗	✗	✗
$[2,1]$	C_2	✗	✗	✗	✗	✗
$[3,1]$	C_3	✗	✗	✗	✗	✗
$[4,1]$	C_4	✗	✗	✗	✗	✗
$[4,2]$	$C_2 \times C_2$	✗	✗	✗	✗	✗
$[5,1]$	C_5	✗	✗	✗	✗	✗
$[6,1]$	S_3	✗	✔	✔	✔	✗
$[6,2]$	C_6	✗	✗	✗	✗	✗
$[7,1]$	C_7	✗	✗	✗	✗	✗
$[8,1]$	C_8	✗	✗	✗	✗	✗

Small Groups

The first few of the 1048 groups of order ≤ 100
✔ = $\mathrm{U}(n)$ and ✔ = $\mathrm{SU}(n)$ for $n = 2, 3$

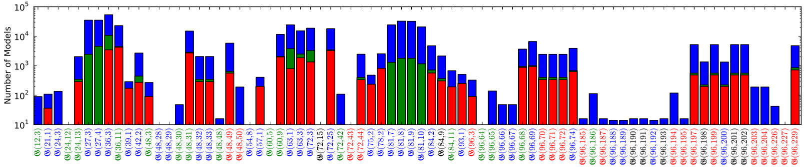
GAP ID	Group	**3**	U(3)	U(2)	U(2)×U(1)	A_4
[8, 2]	$C_4 \times C_2$	✗	✗	✗	✗	✗
[8, 3]	D_4	✗	✔ (SU)	✔ (U)	✔ (U)	✗
[8, 4]	Q_8	✗	✔ (SU)	✔ (SU)	✔ (SU)	✗
[8, 5]	$C_2 \times C_2 \times C_2$	✗	✗	✗	✗	✗
[9, 1]	C_9	✗	✗	✗	✗	✗
[9, 2]	$C_3 \times C_3$	✗	✗	✗	✗	✗
[10, 1]	D_5	✗	✔ (SU)	✔ (U)	✔ (U)	✗
[10, 2]	C_{10}	✗	✗	✗	✗	✗
[11, 1]	C_{11}	✗	✗	✗	✗	✗
[12, 1]	$C_3 \rtimes_\varphi C_4$	✗	✔ (SU)	✔ (SU)	✔ (SU)	✗

Small Groups

The first few of the 1048 groups of order ≤ 100
✔ $= \mathrm{U}(n)$ and ✔ $= \mathrm{SU}(n)$ for $n = 2, 3$

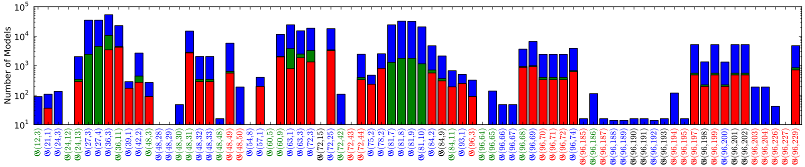
GAP ID	Group	**3**	U(3)	U(2)	U(2)×U(1)	A_4
[12, 2]	C_{12}	✗	✗	✗	✗	✗
[12, 3]	A_4	✔	✔	✗	✗	✔
[12, 4]	D_6	✗	✔	✔	✔	✗
[12, 5]	$C_6 \times C_2$	✗	✗	✗	✗	✗
[13, 1]	C_{13}	✗	✗	✗	✗	✗
[14, 1]	D_7	✗	✔	✔	✔	✗
[14, 2]	C_{14}	✗	✗	✗	✗	✗
[15, 1]	C_{15}	✗	✗	✗	✗	✗
[16, 1]	C_{16}	✗	✗	✗	✗	✗
[16, 2]	$C_4 \times C_4$	✗	✗	✗	✗	✗

All Models



➢ 76 of 90 groups can be scanned in less than 60 days

All Models



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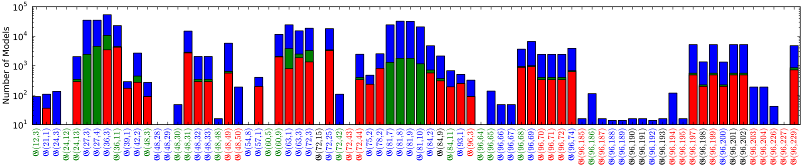
All Models



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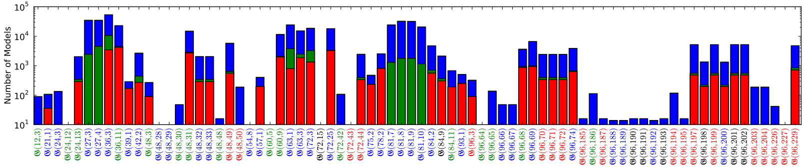

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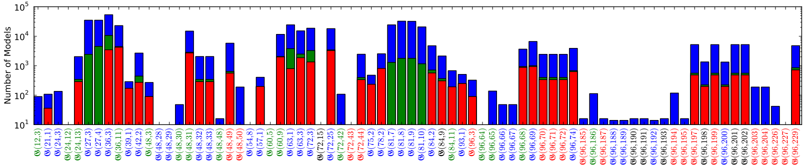

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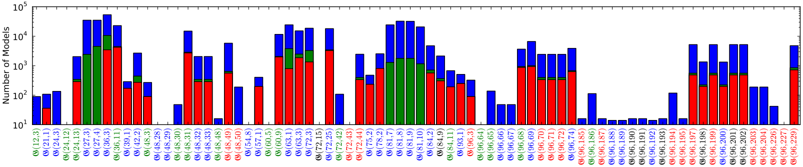

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Conclusions

- Constructed thousands of new models of tribimaximal mixing
  - 18.5% of all $A_4 \times \mathbb{Z}_3$ models are TBM. Encouraging!
  - Prediction for θ_{13}: If A_4 and $\theta_{13} \lesssim 12^\circ \rightsquigarrow \theta_{13} = 0^\circ$
  - Prediction for δ: 0°
  - Correlations between mixing angles: Fix two, predict the third

- Constructed specific models
  - $\theta_{13} \neq 0$ possible: $\theta_{12} \simeq 34^\circ$, $\theta_{23} \simeq 41^\circ$ and $\theta_{13} \simeq 5^\circ$
  - Altarelli-Feruglio model works with $\mathbb{Z}_2$: $A_4 \times \mathbb{Z}_2 \simeq \Sigma(24)$
  - TBM possible for T_7, $\Sigma(24)$, T_{13}, T_{14}, $\Delta(48)$, T_{19}, ...

- Is A_4 special? Are TBM and A_4 connected?
  - 50% of the 76 groups we scanned can accomodate TBM
  - Metacyclic group T_{13} has larger fraction of TBM models
  - Smallest group w/TBM is T_7

- `GAP` as a new tool for model builders